

Fractional powers of the Laplacian and the geometry of phase transition models

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Notion of Γ -convergence:

Let X metric space, $F_\epsilon : X \rightarrow [0, +\infty]$. $F_\epsilon \xrightarrow{\Gamma} F$ as $\epsilon \rightarrow 0$ if both:

- *Lower bound inequality:* $\forall u \in X, \forall (u_\epsilon)$ such that $u_\epsilon \rightarrow u$, then

$$\liminf_{\epsilon \rightarrow 0} F_\epsilon(u_\epsilon) \geq F(u)$$

- *Upper bound inequality:* $\forall u \in X, \exists (u_\epsilon)$ such that $u_\epsilon \rightarrow u$ and

$$\limsup_{\epsilon \rightarrow 0} F_\epsilon(u_\epsilon) = F(u)$$

Aim: Minimize F .

Note: If (u_ϵ) minimizes F_ϵ , every cluster point of (u_ϵ) minimizes F .

Strategy: Let $\epsilon \rightarrow 0$, (u_ϵ) such that $F_\epsilon[u_\epsilon]$ bounded. To prove: (u_ϵ) precompact in X .

Phase transitions:

Container: $\Omega \subset \mathbb{R}^3$, let $0 < V < \text{vol}(\Omega)$

Configurations: $u : \Omega \rightarrow \{0, 1\}$, $\int_{\Omega} u = V$, $u \in BV(\Omega, \{0, 1\})$

Interface: S_u = set of discontinuities of u

Energy: $F(u) := \sigma \mathcal{H}^2(S_u)$, $\sigma = \text{cst}$

Equilibrium: $\min F(u)$ among all possible configurations

Cahn-Hilliard model:

Configurations: $u : \Omega \rightarrow [0, 1]$, $\int_{\Omega} u = V$, $u \in W^{1,2}(\Omega)$

Double well potential: $W(z) = 0 \Leftrightarrow z = 0, 1$

Energy: $F_{\epsilon}(u) = \epsilon \int_{\Omega} |\nabla u|^2 + \frac{1}{\epsilon} \int_{\Omega} W(u)$

Theorem (Modica-Mortola) Let $\sigma := 2 \int_0^1 \sqrt{W(t)} dt$. Then

$F_{\epsilon} \xrightarrow{\Gamma} F$ + compactness

Two phase fluids: (Alberti-Bouchitte-Seppecher)

Double well potentials: $W(x) = 0 \Leftrightarrow x = \alpha, \beta$, $V(x) = 0 \Leftrightarrow x = \alpha', \beta'$

Functional: $F_\epsilon(u) = \epsilon \int_\Omega |\nabla u|^2 + \frac{1}{\epsilon} \int_\Omega W(u) + \lambda_\epsilon \int_{\partial\Omega} V(Tu)$

Capilarity model with line tension effect:

$$\phi(u, v) = \sigma \mathcal{H}^2(S_u) + \int_{\partial\Omega} |H(Tu) - H(v)| + c \mathcal{H}^1(Sv)$$

$$F(u) := \inf \{ \phi(u, v) : v \in BV(\partial\Omega) \}$$

Theorem: Let H primitive of $2\sqrt{W}$, $\sigma = |H(\beta) - H(\alpha)|$, $c = (\beta' - \alpha')K/\pi$, where $\lim_{\epsilon \rightarrow 0} \epsilon \log \lambda_\epsilon = K$, then

$$F_\epsilon \xrightarrow{\Gamma} F \quad + \quad \text{compactness}$$

Fractional Laplacian:

$$(-\Delta)^s f(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x - \xi|^{n+2s}} d\xi, \quad s \in (0, 1)$$

Pseudo-differential operator: $\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s} \widehat{f}(\xi)$

Poisson Kernel (upper half-space):

$$P(x, y) = c_{n,a} \frac{y^{1-a}}{(|x|^2 + |y|^2)^{\frac{n+1-a}{2}}}, \quad s = \frac{1-a}{2}$$

Extension problem:

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ u = f, & \text{at } y = 0 \end{cases} \rightarrow (-\nabla)^s f = - \lim_{y \rightarrow 0} y^a \partial_y u$$

Mean curvature motion:

Numerical algorithm (Bence-Merriman-Osher): Let $C_0 \subset \mathbb{R}^n$ compact.

$$\text{Solve } \begin{cases} u_t - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = \chi_{C_0}, & \text{at } t = 0 \end{cases}$$

Fix $\epsilon > 0$ small. Consider $C_\epsilon := \{x \in \mathbb{R}^n : u(x, t) \geq 1/2\}$. Reinitiate.

Theorem (Evans, etc): Suitable scaling $\Rightarrow \partial\Omega_t$ moves by mean curvature flow.

Heuristic argument:

Same result if we use Poisson kernel for fractional Laplacian, $s > 1/2$.
Non-local operators if $s < 1/2$.

Application: Dislocation dynamics for crystals (Da Lio-Forcadel-Monneau)

Problem:

- Fix $s > 1/2 \Leftrightarrow a < 0$

- Study Gamma-convergence + compactness of:

$$F_\epsilon(u) = \epsilon^{1-a} \int_\Omega |\nabla u|^2 \text{dist}(x, \partial\Omega)^a + \frac{1}{\epsilon^{1-a}} \int_\Omega W(u) \text{dist}(x, \partial\Omega)^{-a} + \lambda_\epsilon \int_{\partial\Omega} V(Tu)$$

- Correct scaling of λ_ϵ ?
- Conjecture: same behavior as $s = 1/2$.
- Two ingredients needed in proof.

Result: Let $E \subset \mathbb{R}$ interval. Consider the functional

$$G_\epsilon(v, E) = \epsilon^{1-a} \iint_{E^2} \frac{|v(x') - v(x)|^2}{|x' - x|^{1+2s}} dx' dx + K_\epsilon \int_E V(v) dx$$

such that

$$\lim_{\epsilon \rightarrow 0} \epsilon^{1-a} \log(K_\epsilon) = K$$

and the functional

$$G(v, E) := C_0 \#(S_v)$$

Then

$$G_\epsilon \xrightarrow{\Gamma} G \quad + \quad \text{compactness}$$

where

$$C_0 := \frac{K(\beta' - \alpha')^2}{s},$$