A Hamiltonian Numerical Scheme for Large Scale Geophysical Fluid Systems

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Outline

1. Introduction & Motivation
2. Hamiltonian parcel formulation for shallow water equations (SWE)
3. Hamiltonian particle-mesh method
4. Numerical example
5. Generalization to adiabatic atmosphere
6. Conclusions & Outlook

Given:

In the limit of no forcing and dissipation, our climate system is governed by conservation laws:

$$\frac{\partial T}{\partial t} + \boldsymbol{\nabla} \cdot F = 0$$

- T = mass, momentum, energy and vorticity (barotropic fluid); and F its associated flux,
- system can be rewritten in Hamiltonian way,
- it has a special phase space structure.

However:

Climate system is weakly dissipative.

So why bother about Hamiltonian systems?

Hypothesis:

Numerical methods that reproduce the correct flow structure in limit of no forcing and dissipation provide better climate predictions.

Arguments??

+ Promising results for symplectic time integration for ensemble of dissipatively perturbed low-order models (e.g., Cotter & Reich, 2003).

+ Since that symplectic time integrators recover the limiting behavior correctly, they allow for accurate long time integrations of such models (Hairer et al., 2002).

– But: does this also hold for ensemble averaged flow?

Our starting point: no forcing and friction

Tasks:

- Find Hamiltonian description of fluid system.
- Find Hamiltonian discretization.

Point:

No general method for Eulerian Hamiltonian discretization.

→ We will use discretization based on the Hamiltonian
parcel formulation for the fluid system (Bokhove & Oliver, 2006).

Intermezzo: Shallow Water Equations:

Eulerian description

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + f \boldsymbol{v}^{\perp} + g \boldsymbol{\nabla} h = 0$$

$$\frac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot (h \boldsymbol{v}) = 0$$

- $\boldsymbol{v} = (u, v)$, $\boldsymbol{v}^{\perp} = (-v, u)$
- $h(x, y, t) \rightarrow$ height of the fluid free surface
- $f \rightarrow$ Coriolis parameter: effect of earth rotation

Hamiltonian description of fluid system:

The parcel formulation

Explained for SWE:

1. Follow a specific parcel in the fluid, and label it by its initial position $\boldsymbol{A} = (A, B)$.
2. Associated flow map:
   $\boldsymbol{\chi}^t$ maps this fluid parcel to its later Eulerian position $\boldsymbol{X}(t) \equiv \boldsymbol{\chi}^t(\boldsymbol{A})$, having velocity $\boldsymbol{V}(t) \equiv \dot{\boldsymbol{\chi}}^t(\boldsymbol{A})$.

3. Parcel's motion given by

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{\nabla}_{\boldsymbol{V}} H = V \,, \tag{1a}$$

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -f\,\boldsymbol{\nabla}_{\boldsymbol{V}}^{\perp} H - \boldsymbol{\nabla}_{\boldsymbol{X}} H = -f\,\boldsymbol{V}^{\perp} - g\boldsymbol{\nabla}_{\boldsymbol{X}} h \,, \tag{1b}$$

with $(\boldsymbol{\nabla}_{\boldsymbol{X}}, \boldsymbol{\nabla}_{\boldsymbol{V}}) \equiv (\frac{\partial}{\partial \boldsymbol{X}}, \frac{\partial}{\partial \boldsymbol{V}})$.

The parcel's Hamiltonian reads

$$H(\boldsymbol{X}, \boldsymbol{V}, t) = \frac{1}{2}\boldsymbol{V}^2 + g\,h(\boldsymbol{X}, t). \tag{2}$$

Parcel's Hamiltonian:

$$H(\boldsymbol{X}, \boldsymbol{V}, t) = \frac{1}{2}\boldsymbol{V}^2 + g\,h(\boldsymbol{X}, t).$$

Crux here:

Potential h regarded as **Eulerian** function evaluated at parcel position.
→ The parcel formulation is partially Lagrangian and partially Eulerian.

Write $h(x, y, t)\big|_{(x,y)=\boldsymbol{X}} \equiv h(\boldsymbol{X}, t)$.

What's new??

4. Procedure 1-3 repeated for *all* parcels that constitute the fluid.
5. Density of specific parcel $\boldsymbol{A}$ follows directly from

$$h(\boldsymbol{X},t) = \int h(\boldsymbol{a},0)\,\delta((\boldsymbol{X}) - \boldsymbol{\chi}^t(\boldsymbol{a}))\,\mathrm{d}\boldsymbol{a}. \tag{3}$$

Indeed: integrand nonzero ⇔ $\boldsymbol{\chi}^t(\boldsymbol{a}) = \boldsymbol{\chi}^t(\boldsymbol{A}) \equiv \boldsymbol{X}$.

⇒ Have found truly new Hamiltonian continuum description!

Hamiltonian discretization:

The Hamiltonian Particle-Mesh Method (HPM)

(Frank et al., 2002)

- Based on the parcel formulation, so that $h(\boldsymbol{X}, t)$ is computed from density distribution on the grid.
- Exact conservation of fluid volume.
- Mass conservation law preserved.
- Satisfies a Kelvin circulation theorem for the vorticity.
- Includes a symplectic time integrator assuring preservation of phase space structure.
- Asymptotic conservation of energy.

The algorithm:

i) Fluid decomposed in N particles.

ii) **Lagrangian step**:
Apply symplectic time integrator to move the particles.

iii) **Eulerian steps**:

a. Fixed grid (x_i, y_j).

b. Redistribution of particles induces new density field.
Density on grid found from discretization of (3).

c. Interpolate grid values $h_{ij}(t)$ back to $\bar{h}(x, y, t)$ using appropriate interpolation function.

Numerical example: Burger's profile

Assume

- 1-dimensional flow, $f = 0$.
- $u(x,t) + 2\sqrt{g\,h(x,t)} = K$, with K constant.

Then the SWE reduces to Burgers' equation

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial x} = 0$$

for $q(x,t) = K - 3\sqrt{g\,h(x,t)}$.

→ Analytical solution implicitly defined by $q(x,t) = q_0(x - q(x,t)\,t)$.
→ Wave breaking after certain time.

Movie

Input

- Initial profile: $q_0 = \sin(x)$ with $K = 3$ on domain $[0, 2\pi]$.
  → Wave breaking at $t = 1$. Take 200 time steps.
- 100 gridpoints, 2 particles per cell initially.
- Second order symplectic time integrator.
- Smoothing operator to avoid non-smooth interpolation of grid values h_{ij}.

Output

- Nearly second order spatial convergence.
- Exponentially small drift in total energy.

Generalization to adiabatic atmosphere

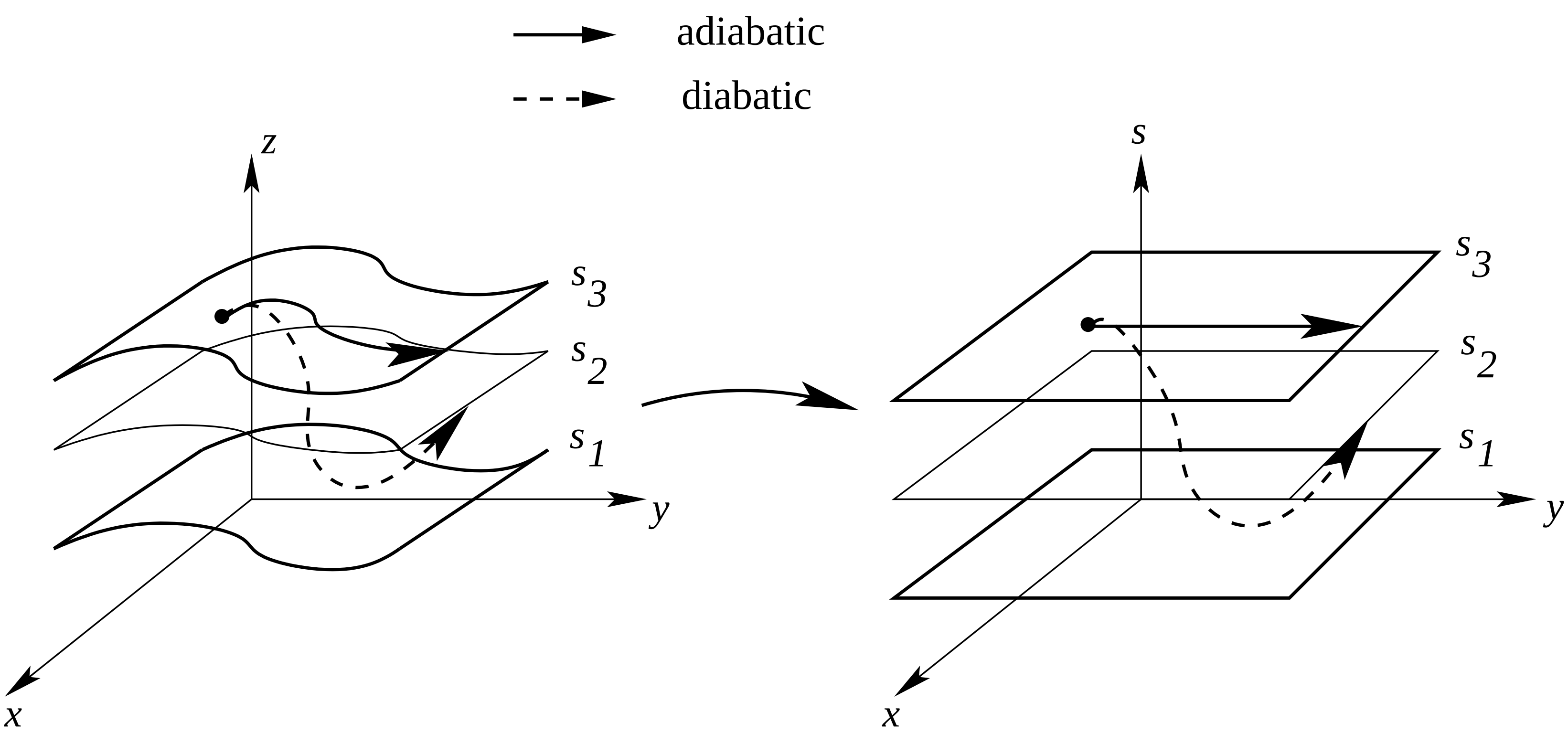
Model assumptions:

1) fluid on rotating plane,
2) hydrostatic balance,
3) reversible thermodynamics ⇔ entropy conserved per fluid parcel,
4) statically stable atmosphere.

Denote specific entropy by s.
→ statically stable conditions if $\frac{\partial s}{\partial z} > 0$.

3) + 4) allow for **isentropic** frame of reference: $(x, y, z) \to (x, y, s)$.

- Since entropy is materially conserved, the flow becomes horizontal within this frame: $\boldsymbol{v} = (u, v, 0)^T$, $\boldsymbol{\nabla} \equiv (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0)^T$.



The governing equations:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + f\boldsymbol{v}^{\perp} + \boldsymbol{\nabla} M = 0\,, \tag{4a}$$

$$M = g\,Z + c_p\,T\,, \tag{4b}$$

$$-\rho\,g\,\frac{\partial Z}{\partial s} = \frac{\partial p}{\partial s}\,, \tag{4c}$$

$$\frac{\partial \sigma}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\sigma + \sigma\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\ \ , \tag{4d}$$

$$\sigma = \rho\,\frac{\partial Z}{\partial s}\,, \tag{4e}$$

$$p = -\frac{\partial U}{\partial(\rho^{-1})}\,, \qquad T = \frac{\partial U}{\partial s}. \tag{4f}$$

- M is the Montgomery potential,
- σ is the pseudodensity,
- U is the internal energy per unit mass.

Steps:

- Eliminate ρ and p from system.
- To get M, we need to calculate Z from:

$$\sigma = -\frac{p_{00}}{g}\frac{\partial}{\partial s}\left(\left(\frac{\sigma}{\rho_{00}\frac{\partial Z}{\partial s}}\right)^{\frac{c_p}{c_v}} e^{\frac{s-s_{00}}{c_v}}\right). \tag{5}$$

This is 1D nonlinear elliptic boundary value problem in s.

Parcel formulation for adiabatic model

Changes w.r.t. parcel formulation SWE:

- $h \to M$ in momentum equation and Hamiltonian,
- $\rho \to \sigma$ in continuity equation,
- New equation (5) for the coupling σ to M.

Conclusion:

In isentropic coordinates, the adiabatic atmospheric flow forms a continuum stack of 2D SWE type of flows coupled via a 1D elliptic inversion.

Outlook

- We have very recently found a Hamiltonian discretization for the adiabatic atmosphere (for simplified boundary conditions), based on HPM and a Finite Element discretization in the s-direction.

  Still need to:

  - *Implement* the discretization for the adiabatic atmosphere.
  - Extend to more *general* boundary conditions.
  - Find way to account for *unresolved* but important dynamics, like gravity waves.
  - Extend to a *spherical* model.
- **Forcing & friction**.

Questions ??

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