A MASS TRANSPORT APPROACH FOR THE RELATIVISTIC HEAT EQUATION

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The relativistic heat equation

We are interested in the Cauchy problem for the relativistic heat equation

$$\partial_t \rho = \text{div } (\rho \frac{\nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}) = \text{div } (\rho \frac{\nabla \log \rho}{\sqrt{1 + |\nabla \log \rho|^2}}). \qquad (1)$$

introduced to impose an upperbound for the propagation velocity.

(Ref Brenier (01), Rosenau (92), Mihalas-Mihalas (84))

Assumptions $\rho(t,x)$ with $t \in [0,T]$ and $x \in \Omega$, bounded domain of $\mathbf{R}^d$ and

$$0 < m \leq \rho_0 \leq M \text{ and } \int_\Omega \rho dx = 1.$$

This problem has been independently adressed in

Andreu, Caselles, Mazon (ref Non Linear Analysis 2005 and JEMS 2005)

Mass Transport Strategy

The aim of our work is to implement a different point of view following the ideas of Jordan, Kinderleherer, Otto SIAM J.Math. Anal.(98)

$\longrightarrow$ construction of solutions of general equation of the type

$$\partial_t \rho = \text{div} \, (\rho \nabla c^*(\nabla(F'(\rho))))$$

where F is a convex Entropy and c is a convex cost function

c^* it's Legendre transform i.e. $c^*(x) = \sup_{y \in \mathbf{R}^d} x \cdot y - c(y)$.

$\longrightarrow$ time discrete scheme involving a double minimization process.

Time discrete scheme

Let $P(\Omega)$ be the set of probability measures on Ω,

$\rho_0 \in P(\Omega)$ given, find $\rho^h(t,x) \in P(\Omega)$ defined by

$$\begin{cases} \rho^h(0,x) = \rho_0(x) \\ \\ \rho^h(t,x) = \rho_i^h(x) \quad \text{for } t \in]ih;(i+1)h] \quad h \text{ being the time step} \end{cases} \tag{2}$$

where

$$\rho_i^h(x) = \text{argmin}_{\rho\in P(\Omega)}\left(\int_\Omega F(\rho(y))dy + h \inf_{\gamma\in\Gamma_i^h(\rho_{i-1}^h,\rho)} \int_{\Omega\times\Omega} c\left(\frac{x-y}{h}\right)d\gamma(x,y)\right),$$

$\Gamma_i^h(\rho_{i-1}^h,\rho)$ is the set of transport plans between ρ_{i-1}^h and ρ.

$\longrightarrow$ Generalisation of discrete gradient flow to general convex cost function

Ref Ambrosio Gigli Savaré Lectures in math ETH (2005),

Villani Graduate studies in Math AMS (2003)

Previous results

- Jordan-Kinderlehrer-Otto SIAM Journ Math Anal (98)

$c(z) = \frac{|z|^2}{2}$ and $F(\rho) = \beta^{-1}\rho\log\rho - V\rho$, V given.

$\longrightarrow$ Linear Fokker Plank equation

$$\partial_t \rho = \mathrm{div}\ (\nabla V \rho) + \beta^{-1}\Delta\rho.$$

- Otto Preprint (96) $c(z) = \frac{|z|^q}{q}$ and $F(\rho) = \frac{n\rho^m}{m(m-1)}$ where $m = n + \frac{p-2}{p-1}$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq 2$

$\longrightarrow$ Doubly degenerate equation

$$\partial_t \rho = \mathrm{div}\ (|\nabla \rho^n|^{p-2}\nabla\rho^n)$$

- Agueh Adv Diff Equ (05) $\beta|z|^q \leq c(z) \leq \alpha(|z|^q + 1)$ and F convex + displacement convexity

$\longrightarrow$ A large set of equation

$$\partial_t \rho = \mathrm{div}\ (\rho\nabla c^*(\nabla(F'(\rho))))$$

Cost Function and Entropy for the relativistic heat equation

The relativistic heat equation does not belong to those set of equations since

$$c(z) = \begin{cases} 1 - \sqrt{1 - |z|^2} \text{ if } |z| \leq 1 \\ \infty \text{ elsewhere } . \end{cases}$$

and $F(\rho) = \rho \log \rho - \rho$.

This cost function is strictly convex and discontinuous ($c(z) = \infty$ if $|z| > 1$)

$\Longleftrightarrow$ the velocity ∇c^* is bounded

$\Longrightarrow$ characteristic property of a relativistic phenomenum.

General assumptions on the cost and on the Entropy

This present work will in fact apply for any

- Cost function

$$c(z) = \begin{cases} \tilde{c}(|z|) \geq 0 \text{ if } |z| \leq 1 \\ \\ \infty \text{ elsewhere} \end{cases}$$

where $\tilde{c}$ is strictly convex, $C^0[0,1] \cup C^2([0,1[)$,

with $|\nabla c(z)| \to \infty$ when $|z| \to 1$.

- Entropy function $F \in C^2(\mathbf{R})$ satisfying $\frac{F(\lambda)}{\lambda} \to \infty$ when $\lambda \to \infty$

and $\lambda^d F(\lambda^{-d})$ is convex $\to$ displacement convexity assumption,

see McCann Adv. Math. (97)

Formal argument I

Step 1 Find γ_i^h for each time intervall $[ih;(i+1)h]$, its second marginal ρ_i^h and S_i^h, the associated optimal map i.e. $\gamma_i^h = \delta(x - S_i^h(y))$.

The existence of ρ_i^h is ensured by the double minimization process.

Problem 1 Definition of the map

Step 2 Derive the Euler-Lagrange equations (dervation / ρ and then ∇)

$$\nabla(F'(\rho(y))) = \nabla c(\frac{S_i^h(y) - y}{h}) \implies \frac{S_i^h(y) - y}{h} = \nabla c^*(\nabla(F'(\rho(y))))$$

since $\nabla c^*(\nabla c(z)) = z$.

Problem 2 We prove $\{|S_i^h(y) - y| \leq h\}$ but if $|S_i^h(y) - y| = h$,

then $\nabla c(\frac{S_i^h(y) - y}{h})$ is not defined.

Formal argument II

Step 3 Obtain an approximate time discrete equation for ρ^h by $\times \rho \nabla \phi$

$$\frac{\rho_i^h - \rho_{i-1}^h}{h} = \operatorname{div}(\rho_i^h \nabla c^* (\nabla (F'(\rho_i^h))) + \text{Correction terms in O(h).}$$

Step 4 It remains to pass to the limit when the time step h goes to zero

$$\Longrightarrow \quad \partial_t \rho = \operatorname{div}(\rho \nabla c^* (\nabla (F'(\rho))).$$

Problem 3 We want to use a monotonicity argument (Minty-Browder) to identify the limit but lack of regularity.

Construction of the optimal map (drop the h)

The classical result of Gangbo McCann CRAS (95) can not be applied here. Indeed it is based on the Kantorovich duality:

$$\int_{\Omega\times\Omega} c(x-y)d\gamma_{opt}(x,y) = \int_\Omega \phi(x)d\rho_0(x) + \int_\Omega \psi(y)d\rho_1(y)$$

where ϕ is the c-transform of ψ, i.e.

$$\phi(x) = \inf_y (c(x-y) - \psi(y))$$

When $c \in C^{1,\alpha}$, ψ is Lipschitz and then differentiable which means that when $(x,y) \in \mathrm{supp}\gamma_{opt}$, we have

$$\nabla c(x-y) = -\nabla\psi(y) \text{ and then } x = y + \nabla c^*(-\nabla\psi(y))$$

In our case, ψ is no more Lipschitz, we have a problem when $|x-y| = 1$.

The strategy consists in introducing a mollified problem.

The mollified problem

We introduce the Yoshida mollification of the convex function c

Ref Brezis Operateur maximaux monotones et semi-groupes de contraction dans les espaces de Hilbert (73) by a

Convexification of c^*

$$c^{\epsilon *}(x) = c^*(x) + \frac{\epsilon}{2}|x|^2$$

$\Longrightarrow$ Mollification of c

$$c^{\epsilon}(x) = \inf_{z \in \mathbf{R}^d} (c(x-z) + \frac{|z|^2}{2\varepsilon})$$

and we define $\gamma_i^{\epsilon h}$, the optimal transport plan between ρ_{i-1}^{h} and $\rho_i^{\epsilon h}$ obtained when the minimization involves the mollified cost function c^{ϵ}.

Kantorovich duality and optimal map for the mollified problem

$$\int_{\Omega\times\Omega} c^\epsilon(x-y)d\gamma^\epsilon(x,y) = \int_\Omega \phi^\epsilon(x)d\rho_0(x) + \int_\Omega \psi^\epsilon(y)d\rho^\epsilon(y)$$

where the potential function satisfies

$$\phi^\epsilon(x) = \inf_{y\in\mathbf{R}^d}(c^\epsilon(x-y) - \psi^\epsilon(y)).$$

Gangbo-McCann $\Rightarrow$ existence of a map $\quad S^\epsilon(y) = y + \nabla c^{\epsilon*}(-\nabla\psi^\epsilon(y))$

Agueh $\Rightarrow$ Euler-Lagrange eq. $\quad S^\epsilon(y) = y + \nabla c^{\epsilon*}(\nabla(F'(\rho^\epsilon(y))))$

and then $\quad \psi^\epsilon(y) = -F'(\rho^\epsilon(y))$ and $\rho^\epsilon(y)$ is bounded in $W^{1,1}(\Omega)$

since we have the entropy-information inequality

$$\int \rho^\epsilon \nabla c^{\epsilon*}(\nabla(F'(\rho^\epsilon)))\cdot\nabla(F'(\rho^\epsilon))dy \le \int [F(\rho_0(y)) - F(\rho^\epsilon(y))]dy.$$

Limiting process to obtain the optimal map

1- the limit $\lim_{\epsilon \longrightarrow 0} \gamma_i^{\epsilon h} = \gamma_i^h$ and
supp $\gamma_i^h \subset \{(x,y)$ such that $|x-y| \leq h\} \cap Z_i^h$ where $\gamma_i^h(Z_i^h) = 0$.

2- $\lim_{\epsilon \longrightarrow 0} \rho_i^{\epsilon h} = \rho_i^h \in BV(\Omega) \Longrightarrow$ approximatively differentiable

$\Longrightarrow$ avoid a.e. the undetermination of the map.

Ref Ambrosio Gigli Savaré Lectures in math ETH (2005),

3- Kantorovich duality

$$\int_{\Omega \times \Omega} hc(\frac{x-y}{h}) d\gamma_i^h(x,y) = \int_\Omega \phi_i^h(x) d\rho_{i-1}^h(x) + \int_\Omega \psi_i^h(y) d\rho_i^h(y).$$

with $\phi_i^h(x) = \inf(hc(\frac{x-y}{h}) - \psi_i^h(y))$ and $\psi_i^h(y)) = -F'(\rho_i^h)$

$\Longrightarrow$ Optimal map + Euler-Lagrange equation

$$S_i^h(y) = y + h\nabla c^*(-\nabla \psi_i^h(y)) = y + h\nabla c^*(\nabla(F'(\rho_i^h(y)))).$$

From the time discrete equation to the continuous equation

By multiplying the Euler-Lagrange equation by $\rho_i^h \nabla \phi$, we obtain the approximate time discrete equation

$$\frac{\rho_i^h - \rho_{i-1}^h}{h} = \text{div}(\rho_i^h \nabla c^*(\nabla(F'(\rho_i^h))) + \text{Correction terms in O(h)}.$$

and when h goes to zero, we obtain the continuous equation

$$\partial_t \rho = \text{div } (\rho A)$$

where ρ is the $L^1([0,T] \times \Omega)$ limit of ρ^h

and A is the $\text{w}^* L^\infty(\Omega)$ limit of $\nabla c^*(\nabla(F'(\rho^h))$.

It remains to identify the limit A

Main Result

Theorem

(i) **Support of the optimal measure: Finite speed of propagation**

$$\text{supp}\ \gamma_i^h \subset \{(x,y) \quad | \quad \frac{|x-y|}{h} < 1\} \cup Z_i^h \text{ with } \gamma_i^h(Z_i^h) = 0.$$

(ii) **Euler-Lagrange equation: a discrete scheme**

$$\gamma_i^h(x,y) = \delta(x - S_i^h(y)) \text{ with } S_i^h(y) = y + h\nabla c^*(\nabla(F'(\rho_i^h(y)))).$$

(iii) **Convergence of the measure** ρ^h. Up to a subsequence,

$$\rho^h \longrightarrow \overline{\rho} \text{ in } L^1([0,T] \times \Omega) \text{ and w-}L^1_w([0,T], BV(\Omega))$$

$$\text{and } \overline{\rho} \in L^\infty([0,T] \times \Omega) \cap L^1_w([0,T], BV(\Omega)).$$

(iv) **Limiting equation** $\overline{\rho}$ is a solution in the sense of distribution to

$$\partial_t \overline{\rho} = \text{div}(\overline{\rho}\nabla c^*(\nabla(F'(\overline{\rho})))).$$