

Pressureless Euler/Euler-Poisson systems via adhesion dynamics and scalar conservation laws

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Problem type:

Let $\alpha, \beta \in \mathbb{R}$ and consider the system

$$\left\{ \begin{array}{ll} \partial_t \rho + \partial_x(\rho v) = 0 & \text{in } \mathbb{R} \times (0, T), \\ \partial_t(\rho v) + \partial_x(\rho v^2) = \rho(\alpha \partial_x \Phi + \beta) & \text{in } \mathbb{R} \times (0, T), \\ \partial_{xx}^2 \Phi = \rho & \text{in } \mathbb{R}. \end{array} \right. \quad (\text{sys})$$

- If $\alpha = \beta = 0$, then (sys) describes the pressureless Euler system in spatial dimension one.
- If $\alpha = \pm 1, \beta = 0$, then (sys) describes the pressureless attractive/repulsive Euler-Poisson system with zero background charge.

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IVP for Euler/Euler-Poisson

- Global existence via adhesion dynamics (“sticky particles”; idea due to **Zeldovich**):
Brenier & Grenier (ρ_0 compactly supported, v_0 continuous; scalar conservation laws approach); **E, Rykov & Sinai** (ρ_0 either discrete or absolutely continuous w.r.t. Lebesgue measure, v_0 continuous of sublinear growth); **Grenier**.
- Global existence via “classical” means:
Bouchut & James (v_0 continuous and bounded; duality solutions, uniqueness); **Boudin** (regular initial data); **Cheng, Li & Zhang**;
Huang & Wang (v_0 may be discontinuous but essentially bounded) etc.
- Uniqueness (necessary and sufficient conditions; [HW] class of solutions): **Huang & Wang**.

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Notation and basic facts:

- $\mathcal{P}_2(\mathbb{R})$ (*Wasserstein space*) the set of all Borel probabilities on \mathbb{R} with finite second-order moments;
- $\mathcal{T}_\mu \mathcal{P}_2(\mathbb{R}) = L^2(\mu)$ the tangent space at μ ;
- W_p the p -Wasserstein metric;
- M the right-continuous cumulative distribution function of the Borel probability μ ;
- $N := M^{-1}$ (*generalized inverse*) is the optimal map such that $N_{\#} \chi_{(0,1)} = \mu$;
- $|u'| (t) := \lim_{h \rightarrow 0^+} \frac{d(u_t, u_{t+h})}{h}$ the *metric derivative* of a curve $t \rightarrow u_t$ in a metric space;
- $AC^2(0, T; \mathcal{P}_2(\mathbb{R}))$ the set of all *absolutely continuous curves* in $\mathcal{P}_2(\mathbb{R})$ with *metric derivatives* in $L^2(0, T)$.

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“Generic” initial conditions:

- (H1) *The initial distribution of mass $\rho_0 \in \mathcal{P}_2(\mathbb{R})$;*
- (H2) *There exists $0 \leq \Lambda < +\infty$ such that*

$$v_0 \in L^2(\rho_0) \cap C(\mathbb{R}) \text{ and } |v_0(x)| \leq \Lambda(1 + x^2) \text{ for all } x \in \mathbb{R}.$$

Theorem 0.1. *The initial-value problem for (sys) admits a global weak solution in the sense of distributions if (H1), (H2) hold.*

Theorem 0.2. *If $\alpha = 0 = \beta$, then the “sticky particles” solution satisfies the Oleinik entropy condition, i.e. for a.e. $t > 0$*

$$v(t, x_2) - v(t, x_1) \leq (x_2 - x_1)/t \text{ for } \rho(t, \cdot) - \text{ a.e. } x_1 \leq x_2.$$

Corollary 0.3. *If $\alpha = 0 = \beta$ and v_0 is bounded, then the “sticky particles” solution is the unique solution in the [HW] class.*

From a scalar conservation law to (sys) (1)

Let $f := v_0 \circ N_0$, $F(m) := \int_0^m f(\omega) d\omega$, and assume

$$\partial_t M + \partial_x [\tilde{F}(t, M)] = 0 \text{ in } \mathcal{D}'([0, T] \times \mathbb{R}), \quad (\text{s.c.l.})$$

for some time-linear perturbation $\tilde{F}(t, \cdot) := F + t\Psi$, $\Psi \in C^1[0, 1]$.

Also, assume $F \circ M \in BV([0, T] \times \mathbb{R})$ and

$$\nabla[\tilde{F}(t, M)] = (g + t\psi)\nabla M + \mathbf{V},$$

for some $g \in L^1(|\nabla M|)$ and the vector field $\mathbf{V} := (\Psi \circ M, 0)$. Here,

$$\psi(t, x) := \int_0^1 \Psi'((1-s)M(t, x-) + sM(t, x)) ds.$$

From a scalar conservation law to (sys) (2)

Note that

$$\partial_t M + v \partial_x M = 0,$$

where $v(t, \cdot) := g(t, \cdot) + t\psi(t, \cdot)$ is defined $\partial_x M(t, \cdot) =: \rho(t, \cdot)$ -a.e.

Then,

$$\begin{aligned} \partial_t(\rho v) &= \partial_t [\partial_x [\tilde{F}(t, M)]] = \partial_x [\partial_t [\tilde{F}(t, M)]] \\ &= \partial_x [v \partial_t M + \Psi(M)] = \partial_x (-v^2 \rho) + \psi \rho. \end{aligned}$$

In our case, $\Psi(m) = \alpha m^2 / 2 + \beta m$ and

$$\psi(t, x) = \beta + \alpha [M(t, x) - \frac{1}{2} \rho(t, \{x\})].$$

We take $\Phi(t, x) = \int_{-\infty}^x M(t, y) dy$ to conclude.

Adhesion dynamics: discrete level (1)

Let m_i be n masses initially located at $-\infty < x_1 < \dots < x_n < +\infty$ and moving with initial velocities v_i . The trajectory of the i^{th} particle before collision is given by

$$x_i(t) = x_i + tv_i + \frac{t^2}{2}a_n^i,$$

where

$$a_n(m) = a_n^i := \alpha \left(\sum_{j=1}^i m_j - \frac{1}{2}m_i \right) + \beta$$

whenever $M_0^n(x_i -) \leq m < M_0^n(x_i)$ (where $m_0 = m_{n+1} = 0$).

Recall that M_0^n is the right-continuous cumulative distribution

function of $\rho_0^n := \sum_{i=1}^n m_i \delta_{x_i}$.

Adhesion dynamics: discrete level (2)

At collision, the *particles stick together* and the initial velocity of the newly formed particle is given by the *conservation of momentum*, i.e. if the masses m_j , $i \leq j \leq k$ collide at time $t_0 > 0$, then

$$v_i(t_0+) = \frac{\sum_{j=i}^k m_j v_j(t_0-)}{\sum_{j=i}^k m_j} .$$

Proposition 0.4. *The function $M_n(t, x) := \sum_{j=1}^n m_j H(x - x_j(t))$ is the entropy solution of*

$$\partial_t M + \partial_x [\tilde{F}_n(t, M)] = 0, \quad M(0, \cdot) = M_0^n \quad (\text{s.c.l.n})$$

for an appropriate \tilde{F}_n .

Adhesion dynamics: discrete level (3)

Take $\tilde{F}_n : [0, T] \times [0, 1] \rightarrow \mathbb{R}$ the function given by

$$\tilde{F}_n(t, m) = \int_0^m [f_n(\omega) + ta_n(\omega)] d\omega,$$

where $N_0^n \# \chi_{(0,1)} = \rho_0^n$ optimally and $f_n := v_0 \circ N_0^n$.

According to the Strong Law of Large Numbers and de la Vallée-Poussin Lemma, we may take initial approximating measures

of the form $\rho_0^n := \frac{1}{n} \sum_{i=1}^n \delta_{x_i^{(n)}}$ such that $W_2(\rho_0, \rho_0^n) \rightarrow 0$ and

$$\int_{\mathbb{R}} \zeta(v_0^2) d\rho_0^n \rightarrow \int_{\mathbb{R}} \zeta(v_0^2) d\rho_0, \quad \int_{\mathbb{R}} \zeta(x^2) d\rho_0^n \rightarrow \int_{\mathbb{R}} \zeta(x^2) d\rho_0 < +\infty,$$

where ζ is some nonnegative, convex, increasing function

$\zeta \in C^1([0, +\infty))$ satisfying $\zeta(0) = 0$ and $\lim_{t \rightarrow +\infty} \frac{\zeta(t)}{t} = +\infty$.

Convergence to the continuous problem

Proposition 0.5. *Let $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ and consider a sequence of discrete probabilities ρ_0^n as above. Then there exists a Borel function $M : [0, \infty) \times \mathbb{R} \rightarrow [0, 1]$ such that for any given $T > 0$,*

$$\max_{0 \leq t \leq T} \left\{ W_2 (\partial_x M_n(t, \cdot), \partial_x M(t, \cdot)), \|M_n(t, \cdot) - M(t, \cdot)\|_{L^1(\mathbb{R})} \right\} \rightarrow 0.$$

Moreover, M is the entropy solution of the problem (s.c.l.).

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Time-regularity and energy dissipation

Proposition 0.6. *The solution path $t \rightarrow \rho(t, \cdot)$ satisfies*

(i) *For any $0 < T < +\infty$, we have*

$$W_2(\rho(t, \cdot), \rho(s, \cdot)) \leq C_T |t - s| \quad \text{for all } t, s \in [0, T].$$

(ii) *The energy is nonincreasing, i.e.*

$$\int_{\mathbb{R}} |v(t, x)|^2 d\rho(t, x) \leq \int_{\mathbb{R}} |v_0(x)|^2 d\rho_0(x) \quad \text{for all } t \geq 0.$$

Entropy condition; uniqueness (1)

Proposition 0.7. (*Gangbo, Nguyen, T.*) Consider $\sigma \in AC^2(0, T; \mathcal{P}_2(\mathbb{R}))$. Let v be the velocity associated to σ and $N(t, \cdot) : (0, 1) \rightarrow \mathbb{R}$ be the monotone nondecreasing map such that $N(t, \cdot) \# \chi_{(0,1)} = \sigma(t, \cdot)$. For each t , modifying $N(t, \cdot)$ on a countable subset of $(0, 1)$ if necessary, we may assume without loss of generality that $N(t, \cdot)$ is right-continuous. Then, $N \in H^1(0, T; L^2(0, 1))$ and

$$\dot{N}(t, x) = v(t, N(t, x))$$

for \mathcal{L}^2 -almost every $(t, x) \in (0, T) \times (0, 1)$.

Theorem 0.8. For Lebesgue almost all $t \in (0, T)$ we have

$$v(t, x_2) - v(t, x_1) \leq \frac{1}{t}(x_2 - x_1) \text{ for } \rho(t, \cdot) - a.e. \ x_1 \leq x_2. \quad (\text{ent})$$

Entropy condition; uniqueness (2)

Proposition 0.9. *The solution constructed above satisfies, for all $\varphi \in C_b(\mathbb{R})$,*

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}} v(t, x) \varphi(x) d\rho(t, x) = \int_{\mathbb{R}} v_0(x) \varphi(x) d\rho_0(x)$$

and

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}} v^2(t, x) \varphi(x) d\rho(t, x) = \int_{\mathbb{R}} v_0^2(x) \varphi(x) d\rho_0(x). \quad (\text{w.c.e.})$$

- *Remark:* (ent) and (w.c.e.) give uniqueness in the [HW] class when v_0 is bounded!

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Things to do

- Uniqueness without the boundedness assumption on the initial velocity;
- Restrict to “generic” initial conditions;
- Euler-Poisson with background (uniform) charge;
- Higher dimensions.

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