Chaotic response of the 2D Semigeostrophic equations to gentle periodic perturbations

Dorian Goldman[1] Robert J. McCann[2]

[1] University of Toronto
[2] University of Toronto

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Outline

1. The 2D Semigeostrophic (SG) equations.
2. Exact solutions in the case our domain is an elliptical tank.
3. Perturbations of the domain eccentricity.
4. How do we conclude the existence of chaotic dynamics?
5. Conclusion: When do these solutions evolve chaotically?

2D Incompressible Euler equations

Recall the $2D$ incompressible Euler equations on a bounded domain $\Omega \subset \mathbb{R}^2$ with Coriolois paramater $f_0 = 1$ can be written as,

$$\frac{D\mathbf{u}}{Dt} + J \cdot \mathbf{u} = -\nabla\phi \text{ where } J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

Here the *advection operator* is defined as

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla. \tag{3}$$

Approximations to 2D Incompressible Euler

Making a first order approximation to (1) we ignore the acceleration terms and get the geostrophic velocity

$$\mathbf{u_g} := J\nabla\phi. \tag{4}$$

This yields

$$\frac{D\mathbf{u_g}}{Dt} + J\mathbf{u} = -\nabla\phi \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{6}$$

Equations (5)–(6) are know as the *2D Semigeostrophic equations*.

Note that (6) allows one to write $\mathbf{u} = J\nabla\psi$ for a stream function $\psi : \mathbb{R}^2 \to \mathbb{R}$.

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Dual formulation of the 2D SG equations

McCann and Oberman (2004) make the *quadratic ansatz* that ψ and ϕ can be written as

$$\psi(t,\mathbf{x}) = \mathbf{x}^T \cdot \Psi(t) \cdot \mathbf{x} + \psi(t) \cdot \mathbf{x} + \psi(t)$$

$$\phi(t,\mathbf{x}) = \mathbf{x}^T \cdot \Phi(t) \cdot \mathbf{x}.$$

Hoskins introduced the change of variables $\mathbf{X} = \mathbf{x} + \nabla\phi$ called dual variables. Defining the *geopotential*

$$P(t,\mathbf{x}) = \frac{1}{2}\mathbf{x} \cdot \mathbf{x} + \phi,$$

note $\mathbf{X} = \nabla P(t,\mathbf{x})$.
Assuming $P(t,\cdot)$ is convex we define the *Legendre transform*,

$$R(t,\mathbf{X}) := P^*(t,\mathbf{X}) = \sup_{y\in\Omega} \mathbf{X} \cdot \mathbf{y} - P(t,\mathbf{y}), \tag{7}$$

so $\mathbf{x} = \nabla R(t,\mathbf{X})$.

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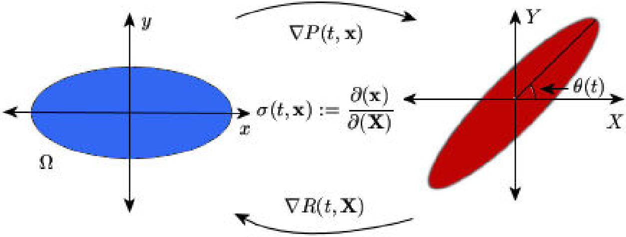
so $\mathbf{x} = \nabla R(t, \mathbf{X})$.
We now seek to model the SG evolution in dual variables

Evolution in dual variables

$\sigma(t, \mathbf{X}) =$ *inverse potential vorticity.*



Conclusion:
The *SG* dynamics are constrained to lie on a finite dimensional submanifold of the full dynamics goverened by the evolution of a *dual inverse potential vortex patch*.

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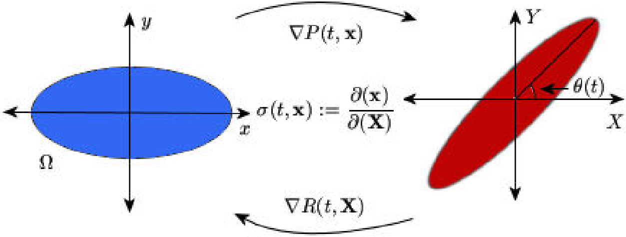

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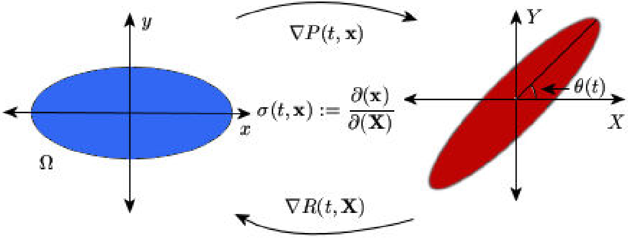
Exact solutions for fluid restricted to an elliptical tank

The equations have a canonical Hamiltonian structure and can be written as,

$$\frac{d}{dt}(r,\theta) = J\nabla H_{SG}^{\sigma,s}(r,\theta), \tag{10}$$

$$\frac{H_{SG}^{\sigma,s}}{2}(r,\theta) = \sigma^2 s + r - \sigma\left(2 + 2rs + 2\cos\theta\sqrt{(r^2-1)(s^2-1)}\right)^{1/2}, \tag{11}$$



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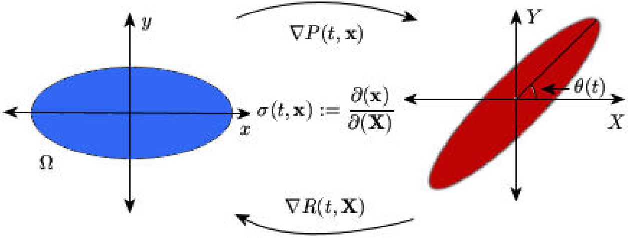

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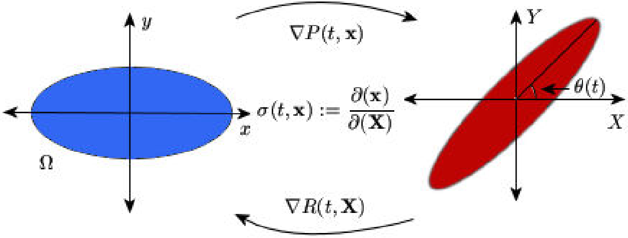
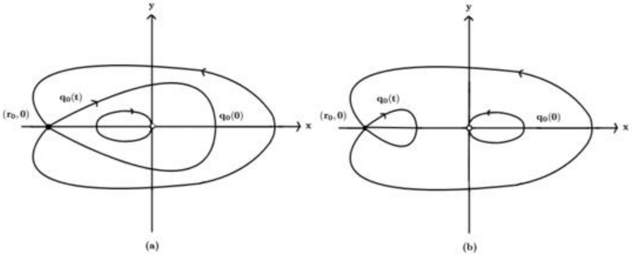


$s = \frac{1}{2}(a_b + 1/a_b)$ $\qquad$ $r = \frac{1}{2}(a_r + 1/a_r)$

Existence of homoclinic orbits in low potential vorticity regime

For **small** values of **potential vorticity**, $1/\sigma$, there is a *homoclinic saddle connection* connecting a *fixed point* to itself.



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Perturbation of physical domain eccentricity

We consider a time periodic perturbation of the physical domain eccentricity in our Hamiltonian:

$$\frac{d}{dt}(r,\theta) = J\nabla H_{SG}^{\sigma,s}(r,\theta) + \epsilon\cos(kt+kt_0)J\nabla\frac{\partial H_{SG}^{\sigma,s}}{\partial s} + O(\epsilon^2).$$

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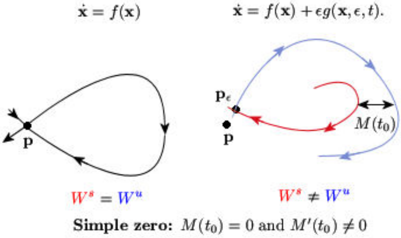
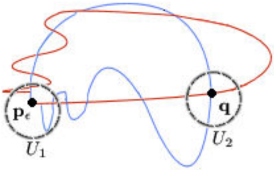


Existence of chaotic dynamics in the perturbed solutions

After the perturbation our resulting phase space may appear as,



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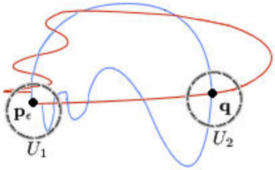
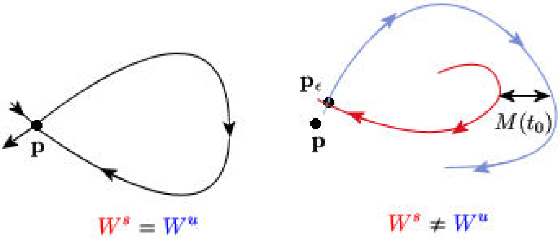


$$W^s \neq W^u$$

To find chaotic dynamics in the phase space:
Search for transversal intersections of W^s and W^u.

Measuring the seperation of W^s and W^u

$\dot{\mathbf{x}} = f(\mathbf{x})$ $\qquad$ $\dot{\mathbf{x}} = f(\mathbf{x}) + \epsilon g(\mathbf{x}, \epsilon, t).$



Simple zero: $M(t_0) = 0$ and $M'(t_0) \neq 0$

Melnikov (1963) concluded that,

$$M(t_0) = \int_{-\infty}^{\infty} J\nabla H_{SG}^{\sigma,s}(\mathbf{q_0}(\mathbf{t})) \wedge J\nabla \frac{\partial H_{SG}^{\sigma,s}(\mathbf{q_0}(\mathbf{t}))}{\partial s}\bigg|_{s_0} \cos(kt + kt_0)dt.$$

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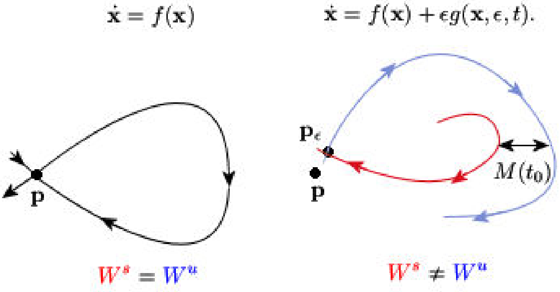

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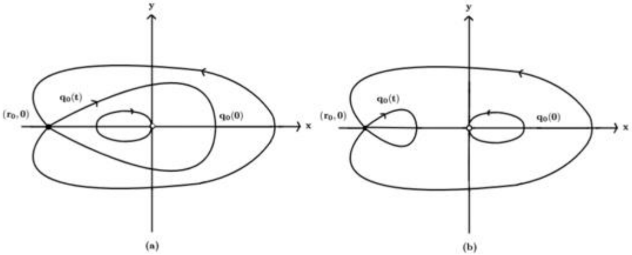
Evaluating the Melnikov integral



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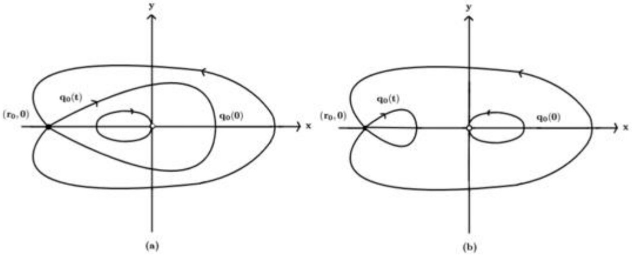

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The existence of simple zeros in the Melnikov integral

$$M(t_0) = \sin kt_0 \int_{-\infty}^{\infty} f(t)\sin(kt)dt$$

$$= \sin kt_0 \hat{f}(k).$$

When does $M(t_0)$ have simple zeros?

- Precisely when $\hat{f}(k) \neq 0$.

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Measuring the separation of the perturbed manifolds

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Assume $\exists t \in \mathbb{R}$ such that $f(t) \neq 0$. Then,

- If $f(t) \in L^1(\mathbb{R})$ THEN $\exists [k_1, k_2] \subset \mathbb{R}$ s.t $\hat{f}(k) \neq 0$
- If $f(t)$ decays exponentially THEN $\hat{f}(k)$ is analytic and hence *vanishes at no more than countably many points*

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For these values of k, we conclude the existence of chaotic dynamics in these solutions to the 2D SG equations when our physical domain eccentricity is perturbed periodically.

⋆ **The method of proof is quite general and relies only on**

1. The symmetry of the Hamiltonian $H(r,\theta) = H(r,-\theta)$
2. The existence of a homoclinic fixed point.
3. Local estimates for decay of vector fields around homoclinic fixed points.

Eg. A similar calculation confirms the chaotic response of the $3D$ Quasigeostrophic equations to gentle periodic shearing.

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3. Local estimates for decay of vector fields around homoclinic fixed points.

Eg. A similar calculation confirms the chaotic response of a set of solutions to the $3D$ Quasigeostrophic equations (studied by Meacham, Morrison, Flierl (1997) to gentle periodic shearing).

Please see my poster for further details and summary.
Thank you!