Hamiltonian Reductions Relating To Optimal Mass Transport

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(Joint Work with Boris Khesin)

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- Tangent space $T_\phi Diff$ at $\phi \in Diff$ is given by

$$\{X \circ \phi : M \to TM | X \text{ is a vector field}\}.$$

Symmetry Group

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- $SDiff$ acts on the diffeomorphism group

$$\Phi_g(\phi) = \phi \circ g, \quad g \in SDiff, \quad \phi \in Diff.$$

Moment Map

- The pushforward of the action Φ_g defined an action on $TDiff$ which is given by

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- **The corresponding moment map $\mu : TDiff \to SVect^*$ is given by**

$$\mu(X \circ \phi)(\xi) = \int_M < X, \phi_* \xi > \phi_* \mu, \quad \xi \in SVect.$$

Burgers Equation as Hamiltonian System

- **Define a Riemannian metric on the diffeomorphism group $Diff$:**

$$< X_1 \circ \phi, X_2 \circ \phi >^{Diff} = \int_M < X_1 \circ \phi, X_2 \circ \phi > \mu.$$

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- **If $H^{Diff}(X \circ \phi) = \frac{1}{2} < X \circ \phi, X \circ \phi >^{Diff}$ is the corresponding kinetic energy Hamiltonian and and ϕ_t is the corresponding geodesic, then it satisfies the Burgers equation:**

$$\partial_t \phi = X_t \circ \phi_t, \quad \partial_t X + \nabla_{X_t} X_t = 0.$$

Solution of Burgers Equation

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- **If X_t is the corresponding velocity field of the geodesic ϕ_t, then $X_t \circ \phi_t$ is a trajectory of the Hamiltonian flow of the kinetic energy.**

Conservation of Momentum

Moment Map: $\mu(X \circ \phi)(\xi) = \int_M < X, \phi_* \xi > \phi_* \mu.$**,**
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- **Conservation of Momentum: For each fixed** $z \in SVect$, **the Hamiltonian flow of** H^{Diff} **preserve the level set** $\mu^{-1}(z)$.
- **Consider** $z = 0$, **the solutions of the Burgers equation with gradient initial velocity field** $\phi_t = \exp(t\nabla f)$ **satisfies** $\partial_t \phi_t = \nabla f_t \circ \phi_t$, **where** $f_t \in C^\infty(M)$ **for each** t.

Other Hamiltonian Systems

- **Let** $H : TM \longrightarrow \mathbb{R}$ **and let** H^{Diff} **be the Hamiltonian on** $TDiff$

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- **Conservation of Momentum: Let** Φ_t^H **be the Hamiltonian flow of** H **on** TM **and let** $\pi : TM \to M$ **be the tangent bundle projection. Set** $\phi_t = \pi \circ \Phi_t^H \circ \nabla f$**, then** $\Phi_t^H = \nabla f_t \circ \phi_t$**, where** f_t **are functions on the manifold** M**.**

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- **Method of Characteristics: f_t above satisfies the Hamilton-Jacobi equation $\partial_t f_t + H(\nabla f_t) = 0$.**

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- **Marsden Weinstein quotient:**

$$\mu^{-1}(0)/SDiff = T(Diff/SDiff) \cong T\mathcal{W}.$$

Reduction of Dynamics(cont')

- **The kinetic energy** $H^{Diff}(X \circ \phi) = \int_M < X_1 \circ \phi, X_2 \circ \phi > \mu$ **is invariant under the** $SDiff$**-action on** $TDiff$**. It induces a kinetic energy Hamiltonian** $H^{\mathcal{W}}$ **on** $T\mathcal{W}$**.**

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- **By conservation of momentum, the Hamiltonian flow of H^{Diff} restricts to a flow on the the level $\mu^{-1}(0)$. This flow can also be reduced to a flow on the Marsden-Weinstein quotient $T\mathcal{W}$.**

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- **Reduction of Dynamics: The Hamiltonian flow of $H^{\mathcal{W}}$ is the reduced flow described above.**

- **The geodesics on $\mathcal{W}$ corresponding to the Hamiltonian $H^{\mathcal{W}}$ are $\exp(t\nabla f)_*\mu$.**

Relations with Optimal Transport

- **Optimal Transport: Let d be the Riemannian distance function. Find $\phi : M \to M$ which minimizes**

$$\int_M d^2(x, \phi(x))\mu$$

among all maps satisfying $\phi_* \mu = \nu$.

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- **(Otto 00') The curve $\exp(t\nabla f)_*\mu$ are geodesics on the Wasserstein space.**

Relations with Optimal Transport(cont')

- **Let $L : TM \to \mathbb{R}$ be a Lagrangian and let $c : M \times M \to \mathbb{R}$ be the cost function defined by**

$$c(x,y) = \inf_{\{\gamma:[0,1]\to M \,|\, \gamma(0)=x, \gamma(1)=y\}} \int_0^1 L(\gamma(t), \dot{\gamma}(t))dt.$$

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- **We can see this as a consequence of Hamiltonian reduction.**