Market completeness with options

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Motivation

Classical picture: (S_t) is the stock price in its natural filtration $(\mathcal{F}_t)$

$\exists$ unique EMM $\mathbb{Q} \sim \mathbb{P} \iff$ No-arbitrage and completeness,

i.e. for any $\mathcal{F}_T$ measurable H, $\exists(\alpha_t)$ $H = \mathbb{E}^{\mathbb{Q}}H + \int_0^T \alpha_t dS_t$.

Extended picture: filtration generated by a multi-dimensional process

$\exists$ multiple EMM $\mathbb{Q} \sim \mathbb{P} \iff$ No-arbitrage but incompleteness

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assume no-arbitrage
$\implies$ work under an EMM
model only stock and use $A_t^i = \mathbb{E}[A_T^i | \mathcal{F}_t]$
investigate completeness: for H find (α_t)
$H = \mathbb{E}H + \int_0^T \alpha_t dA_t$
Bajeux-Besnainou & Rochet, Romano & Touzi, Davis

Setup

SETUP:
The filtration $(\mathcal{F}_t) = (\mathcal{F}^{\xi}_t)$ is generated by some factor process (ξ_t) in $\mathbb{R}^d$.
Typically $\xi^1_t = S_t$ is the stock price process.

We observe prices $A_t = (A^1_t, \dots, A^d_t)$ of d assets. $A^1_t = S_t$ is the stock price, and A^i is an option with maturity T_i and a payoff $h_i(S_{T_i})$.

We assume there is *no-arbitrage* in the market and work in the risk-neutral measure, so that discounted prices are martingales, i.e. $\tilde{A}_t = \mathrm{e}^{-rt} A_t$ is an $(\mathcal{F}_t)$-martingale for $t \le T \le \min_i T_i$.

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We want to devise a necessary and sufficient criterion under which market on $[0, T]$ is complete, i.e. $\forall (\mathcal{F}_T)-$measurable H, $\exists (\alpha_t)$ $H = \mathbb{E}H + \int_0^T \alpha_t d\tilde{A}_t$.

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TOOLS:

We assume (ξ_t) is a solution to an SDE driven by d independent Brownian motions and Poisson processes $\rightsquigarrow$ *Feynman-Kac and std Markovian techniques*.

Continuous setup - intuition

(Solely for the ease of presentation) *assume* $r = 0$.

(ξ_t) is the unique strong solution to

$$d\xi_t = m(\xi_t)dt + \sigma(\xi_t)dW_t,\ t \leq T,$$

where W_t is a Brownian motion in $\mathbb{R}^d$.

Consider payoffs $A^i_T = h_i(\xi_T)$. Markov property entails that assets' prices at time t satisfy $A^i_t = \mathbb{E}[h_i(\xi_T)|\mathcal{F}_t] = v_i(t, \xi_t)$ with $v_i(t,x) = \mathbb{E}_x[h_i(\xi_{T-t})]$.

Martingale property and Itô's formula then imply $dA_t = G(t, \xi_t)\sigma(\xi_t)dW_t$ where $G(t,x) = \left(\frac{\partial v_i(t,x)}{\partial x_j}\right)_{1\leq i,j\leq d}$

Consider any $\mathcal{F}_T$-measurable H, $\mathbb{E}H^2 < \infty$. Then $\exists\ (\chi_t)$ such that $H = \mathbb{E}H + \int_0^T \chi_t dW_t$.

If we can define $\alpha_t = \chi_t \sigma^{-1}(\xi_t)G^{-1}(t, \xi_t)$ then $H = \mathbb{E}H + \int_0^T \alpha_t dA_t$.

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Continuous setup - sufficient condition

Assume m, σ are Lipschitz continuous, $\sigma(x)\sigma(x)^{\mathrm{T}}$ uniformly positive definite and h_i have linear growth.

Theorem (Romano & Touzi '97, Davis '04)

Suppose that the matrix

$$G(t,x) = \left(\frac{\partial v_i(t,x)}{\partial x_j}\right)_{1\le i,j\le d} = \begin{pmatrix} \nabla v_1(t,x) \\ \vdots \\ \nabla v_d(t,x) \end{pmatrix}$$

is non-singular for all $(t,x) \in [0,T) \times \mathbb{R}^d$. *Then we have a complete market model.*

Romano-Touzi proved non-singularity in a stochastic vol model ($d = 2$) for European calls and puts (under some additional ass.).

Continuous setup - necessary and sufficient condition

Assume $\sigma\sigma^{\mathrm{T}}$ positive definite and v_i are C^2 functions on $[0,T) \times \mathbb{R}^d$ and let $G(t,x) = \left(\frac{\partial v_i(t,x)}{\partial x_j}\right)_{1\le i,j\le d}$. Define $\mathcal{S} = \left(\det G(t,x)\right)^{-1}\left(\{0\}\right) \subset [0,T) \times \mathbb{R}^d$.

Theorem

The market is complete, i.e. $\forall \mathcal{F}_T$*-measurable* H, $\mathbb{E}H^2 < \infty$, $\exists(\alpha_t)$ *s.t.* $H = \mathbb{E}H + \sum_{i=1}^d \int_0^T \alpha_t^i dA_t^i$, *if and only if* $\int_0^T \mathbf{1}_{(t,\xi_t)\in\mathcal{S}} dt = 0$ *a.s.*

Proof:

$\Leftarrow$ For $H = \mathbb{E}H + \int_0^T \chi_t d\xi_t$ we define $\alpha_t = \chi_t G^{-1}(t,\xi_t)\mathbf{1}_{(t,\xi_t)\notin\mathcal{S}}$ and $H = \mathbb{E}H + \int_0^T \alpha_t dA_t$.

$\Rightarrow$ With measurable selection theorem we can choose $\beta \in ker(G)$ and define a non-zero claim $H = \int_0^T \mathbf{1}_{(t,\xi_t)\in\mathcal{S}}\beta(t,\xi_t)dt$ which is orthogonal to all stochastic integrals w.r.t. (A_t).

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Remarks about the class of assets

- Our assets need to be linearly independent: matrix G is obviously degenerate if there exists $z \in \mathbb{R}^d$, $\sum_{i=1}^d z_i h_i(\xi_T) = 0$ a.s.
- We can use options with different maturities $T_i \geq T$.
- We could use variance swap and the market is complete iff it is complete when we use an option paying $h(\xi_T) = \log(S_T/S_0)$. Indeed, for $A_t^1 = S_t$ and $X_t = \log(S_t/S_0)$ we have

$$v_{VS}(t,\xi_t) = \mathbb{E}[\langle X\rangle_T|\mathcal{F}_t] = 2\int_0^t \frac{dS_u}{S_u} - 2\mathbb{E}[X_T|\mathcal{F}_t] = 2\int_0^t \frac{dS_u}{S_u} - 2v_{log}(t,\xi_t),$$

  and $\nabla v_{VS}(t,x) = 2\frac{\nabla v_1(t,x)}{v_1(t,x)} - 2\nabla v_{log}(t,x)$. In consequence singularities of G remain the same when we change variance swap for log contract in our assets.
- Same story for other path dependent options.

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Our criterion and the resulting PDE question

Recall the criterion: $\int_0^T \mathbf{1}_{(t,\xi_t)\in\mathcal{S}} dt = 0$ a.s.
Consider the case $d = 2$, $A_t^1 = S_t$ so that $v_1(t,x) = x_1$. We have $\det G(t,x) = 0 \Leftrightarrow \frac{\partial v_2(t,x)}{\partial x_2} = 0$, where $v_2(t,\xi_t) = \mathbb{E}[h_2(\xi_T)|\mathcal{F}_t]$.
Function v_2 solves

$$\frac{\partial v_i}{\partial t} + \mathcal{G}v_2 - rv_2 = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^2, \quad v_2(T,x) = h_2(x),$$

where $\mathcal{G}$ is the generator of ξ.

Question: under what conditions on $\mathcal{G}$ and h_2 is the set $\mathcal{S} = \{(t,x) : \frac{\partial v_2(t,x)}{\partial x_2} = 0\}$ of Lebesgue measure zero?

Naturally, then we are interested in the multi-dimensional version of the result!

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Brownian example

Suppose $m(x) = 0$ and $\sigma(x) = \sigma$ is constant, so that $\xi_t = \sigma W_t$ is a (correlated) Brownian motion in $\mathbb{R}^d$ and take $h_i \geq 0$ with at most polynomial growth. The functions v_i are given explicitly as

$$v_i(t,x) = \mathbb{E}_x[h_i(\xi_{T-t})] = \frac{\sqrt{\det(Q)}}{\sqrt{T-t}(2\pi)^{d/2}} \int_{\mathbb{R}^d} h_i(y) \mathrm{e}^{-\frac{Q(y-x)\cdot(y-x)}{2(T-t)}} dy,$$

where $Q = (\sigma^{-1})^{\mathrm{T}} \sigma^{-1}$. We can compute derivatives matrix

$$G(t,x) = \frac{1}{T-t} \Big(\mathbb{E}_x[h_i(\xi_{T-t})(\xi^j_{T-t} - x_j)] \Big)_{i,j \leq d} \cdot Q.$$

and conclude that $\det(G)$ is an analytic function on $[0,T) \times \mathbb{R}^d$ and $\mathcal{S}$ is either equal to $[0,T) \times \mathbb{R}^d$ or *is of Lebesgue measure zero and the market is complete.*

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Discontinuous setup

(ξ_t) is the unique strong solution to

$$d\xi_t = m(\xi_{t-})dt + \sigma(\xi_{t-})dW_t + \gamma(\xi_{t-})d\tilde{N}_t,\ t \leq T,$$

where W_t is a Brownian motion in $\mathbb{R}^d$ and $\tilde{N}_t = (N_t^1 - \lambda_1 t, \ldots, N^k - \lambda_k t)^{\mathrm{T}}$ is a vector of k independent compensated Poisson processes.

We have $(d+k)$ assets with payoffs $A_T^i = h_i(\xi_T)$. Markov property entails that $A_t^i = \mathbb{E}[h_i(\xi_T)|\mathcal{F}_t] = v_i(t, \xi_t)$ with $v_i(t,x) = \mathbb{E}_x[h_i(\xi_{T-t})]$.

Martingale property (*no-arbitrage*) and Itô's formula then imply

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where $G(t,x) = \left(\frac{\partial v_i(t,x)}{\partial x_j}\right)$ and e_l is the l^{th} base vector.

Discontinuous setup

(ξ_t) is the unique strong solution to

$$d\xi_t = m(\xi_{t-})dt + \sigma(\xi_{t-})dW_t + \gamma(\xi_{t-})d\tilde{N}_t,\ t \le T,$$

where W_t is a Brownian motion in $\mathbb{R}^d$ and $\tilde{N}_t = (N_t^1 - \lambda_1 t, \dots, N^k - \lambda_k t)^{\mathrm{T}}$ is a vector of k independent compensated Poisson processes.

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Discontinuous setup - market completeness theorem

In the setup above, assume $\sigma(x)\sigma^{\mathrm{T}}(x)$ is positive definite and v_i are C^2 functions.
Define $\mathcal{S} = \big(\det M(t,x)\big)^{-1}(\{0\}) \subset [0,T) \times \mathbb{R}^{d+k}$.

Theorem

The market is complete, i.e. $\forall \mathcal{F}_T^{(w,N)}$*-measurable* H, $\mathbb{E}H^2 < \infty$, $\exists(\alpha_t)$ *s.t.* $H = \mathbb{E}H + \sum_{i=1}^{d+k} \int_0^T \alpha_t^i dA_t^i$, *if and only if* $\int_0^T \mathbf{1}_{(t,\xi_t)\in\mathcal{S}} dt = 0$ *a.s.*

Note: the criterion is the same as t is the predictable compensator of both W_t and $\tilde{N}_t$.

Discontinuous setup - market completeness criterion

We have $\mathcal{S} = \big(\det M(t,x)\big)^{-1}\big(\{0\}\big) \subset [0,T) \times \mathbb{R}^{d+k}$ but also $\mathcal{S} = \big(\det(G(t,x), M^d(t,x))\big)^{-1}\big(\{0\}\big)$ where

$$(G(t,x), M^d(t,x)) = \left(\left(\frac{\partial v_i(t,x)}{\partial x_j} \right)_{i,j} \Big(A^i(t, x + \gamma(x)e_l) - A^i(t,x) \Big)_{i,l} \right)_{\substack{1 \le i \le d+k \\ 1 \le j \le d \\ 1 \le l \le k}}$$

Functions v_i solve now a PIDE and we ask about the measure of $\mathcal{S}$.

However, matrix (G, M^d) combines two parts of different nature and it seems harder to investigate its singularities.

It turns out that choosing our assets well we can sometimes reduce the problem to singularities of G.

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A toy example

Consider a simple example:

$$dS_t = S_{t-}d(\sigma W_t + c(N_t - \lambda t)), \quad c > -1.$$

The variance swap pays $V_T = [X]_T$, where $X_T = \log(S_T/S_0)$. Using Itô, we have

$$X_T = \sigma W_T - c\lambda T + \frac{1}{2}\sigma^2 T + \log(1+c)N_T.$$

Thus $V_T = [X]_T = \sigma^2 T + \log^2(1+c)N_T$ and variance swap price at time t is given by $V_t = \mathbb{E}[V_T|\mathcal{F}_t] = \log^2(1+c)(N_t - \lambda t) + T(\sigma^2 - \lambda \log^2(1+c))$. In consequence,

$$d\begin{pmatrix} S_t \\ V_t \end{pmatrix} = \begin{pmatrix} \sigma S_{t-} & cS_{t-} \\ 0 & \log^2(1+c) \end{pmatrix} d\begin{pmatrix} W_t \\ N_t - \lambda t \end{pmatrix}$$

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Stochastic volatility with jump model

Consider a stochastic volatility model with jumps in the stock price (driven by a single Poisson process), i.e. $d = 2$, $k = 1$, $\xi_t = (S_t, Y_t)$ with

$$\begin{cases} dS_t = S_{t-}\sigma(\xi_{t-})dW_t + cS_{t-}d(N_t - \lambda t), \\ dY_t = m(Y_t)dt + \eta(Y_t)dw_t^2, \ (S_0, Y_0) = x, \end{cases}$$

under suitable assumptions. Consider a *triplet of assets* with payoffs $A_T = (S_T, X_T, [X]_T)$, where $X_T = \log(S_T)$. Market completeness is not affected if we change the last asset's payoff to $V_T := [X]_T + 2\int_0^T \frac{dS_t}{S_{t-}} - 2X_T$.

Proposition

Trading in assets $\overline{A}_t$ with payoffs $\overline{A}_T = (S_T, h_2(S_T), V_T)$ completes the market if and only if $\int_0^T \mathbf{1}_{\frac{\partial v_2}{\partial x_2}(t,\xi_{t-})=0} dt = 0$ a.s., where $v_2(x,t) = \mathbb{E}_x[h_2(S_{T-t})]$.

Unlike in the continuous setup, variance swap has an advantage over European options in simplifying market completeness question.

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In consequence

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Summary (and questions)

- Assuming the market admits no-arbitrage, traded assets can be used to show completeness.
- Take as many assets as we have independent factors spanning the filtration, the market is complete under non-degeneracy of the matrix governing evolution of prices.
- However, requiring strick non-degeneracy is too-strong. Necessary and sufficient condition is that the process does not spend any time in the set of degenerate points a.s.
- When considering models with jumps, the choice of assets can notably simplify the condition for completeness. In particular, variance swaps prove very useful.
- Completeness criterion leads to a question about regularity of solutions to PDE or PIDE, which seems open.
- We considered completeness relative to $\mathcal{F}_T^{\xi}$ which implies completeness relative to $\mathcal{F}_T^{\xi^1} = \mathcal{F}_T^{S}$. Can the former be in fact weaker?

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