Utility-Based Methods for Derivative Asset Pricing

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Further Developments in Quantitative Finance
9 July 2007

Structure of Talk

- Dual representations
- Marginal Utility Based Prices
- Marginal Utility Based Price Processes
- (Utility Indifference Prices)

The Market Model

- Filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.
- Security market with $d + 1$ assets – one numeraire asset and d risky assets.
- Semimartingale price process $S = (S^i)_{1 \leq i \leq d}$ on time interval $[0, T]$, representing the discounted prices of the d risky assets.

The gains from trading with a strategy $H \in L(S)$ correspond to the stochastic integral $H \cdot S$ where

$$(H \cdot S)_t = \int_0^t H_u \mathrm{d}S_u.$$

Consider first the case of a utility function $U : (0, \infty) \to \mathbb{R}$

Primal problem:

$$u(\mathscr{E}) := \sup_{H \in \mathscr{H}} \mathrm{E}_{\mathbb{P}} \left[U\big(\mathscr{E} + (H \cdot S)_T\big) \right]$$

where $\mathscr{H}$ denotes the cone of admissible strategies and $\mathscr{E}$ is a bounded random endowment.

Dual representations

With CSW in mind, can we obtain a dual representation of the form

$$u(\mathscr{E}) = \min_{\mu \in \mathscr{D}} \left\{ \mathrm{E}_{\mathbb{P}} \left[V\left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}} \right) \right] + \mu(\mathscr{E}) \right\}?$$

Here,

- $\mathscr{D} \subseteq ba$ is the polar cone to the cone of bounded, zero cost contingent claims:

$$\mathscr{C} := \{X \in L^\infty : X \leq (H \cdot S)_T \text{ some } H \in \mathscr{H}\}$$
$$\mathscr{D} := \{\mu \in ba : \mu(X) \leq 0 \text{ all } X \in \mathscr{C}\} \subseteq ba^+.$$

- $V(y) = \sup_{x \geq 0}\{U(x) - xy\}$ convex conjugate to U;
- μ^r (resp. μ^s) is the σ-additive (resp. singular) part of μ;
- $\mu(\mathscr{E}) := \int_\Omega \mathscr{E} \mathrm{d}\mu$.

Fenchel Inequality

$H \in \mathscr{H}$, $\mu \in \mathscr{D}$, $\mathscr{E} + (H \cdot S)_T \geq 0 \implies$

$$\begin{aligned} \mathrm{E}_{\mathbb{P}}\left[U\big(\mathscr{E} + (H \cdot S)_T\big)\right] &\leq \mathrm{E}_{\mathbb{P}}\left[V\left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}}\right)\right] + \mu^r(\mathscr{E} + (H \cdot S)_T) \\ &\leq \mathrm{E}_{\mathbb{P}}\left[V\left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}}\right)\right] + \mu(\mathscr{E}) + \mu\big((H \cdot S)_T\big) \\ &\leq \mathrm{E}_{\mathbb{P}}\left[V\left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}}\right)\right] + \mu(\mathscr{E}). \end{aligned}$$

Taking sup of the LHS and inf of the RHS gives

$$u(\mathscr{E}) \leq \inf_{\mu \in \mathscr{D}} \left\{ \mathrm{E}_{\mathbb{P}}\left[V\left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}}\right)\right] + \mu(\mathscr{E}) \right\} =: v(\mathscr{E}),$$

so the question of a dual representation is really whether there is no duality gap.

Abstract version

$$\begin{aligned}\mathbb{U}_{\mathscr{E}}(X) &:= \mathrm{E}_{\mathbb{P}}\left[U(\mathscr{E}+X)\right]\\ \mathbb{L}(X,\mu) &:= \mathbb{U}_{\mathscr{E}}(X) - \mu(X)\\ \mathbb{V}_{\mathscr{E}}(\mu) &:= \sup_{X\in L^\infty} \mathbb{L}(X,\mu).\end{aligned}$$

$$\sup_{X\in\mathscr{C}} \mathbb{U}_{\mathscr{E}}(X) \le \sup_{X\in L^\infty} \inf_{\mu\in\mathscr{D}} \mathbb{L}(X,\mu) \le \inf_{\mu\in\mathscr{D}} \sup_{X\in L^\infty} \mathbb{L}(X,\mu) = \inf_{\mu\in\mathscr{D}} \mathbb{V}_{\mathscr{E}}(\mu).$$

Same question: When equal?

In fact,

$$\sup_{X\in\mathscr{C}} \mathbb{U}_{\mathscr{E}}(X) \le u(\mathscr{E}) \le v(\mathscr{E}) \le \inf_{\mu\in\mathscr{D}} \mathbb{V}_{\mathscr{E}}(\mu),$$

so answering second question would answer first.

Apply Lagrange Duality Theorem: Endow L^∞ with positive cone $P := -\mathscr{C}$. Positive cone $P^\oplus$ of ba is $\mathscr{D}$. Two technical conditions needed:

1. P should have non-empty interior – $1 \in P$;
2. $\sup_{X\in\mathscr{C}} \mathbb{U}_{\mathscr{E}}(X) < \infty$.

Then

$$\sup_{X\in\mathscr{C}} \mathbb{U}_{\mathscr{E}}(X) = \inf_{\mu\in\mathscr{D}} \mathbb{V}_{\mathscr{E}}(\mu) = \mathbb{V}_{\mathscr{E}}(\widehat{\mu}),$$

and the set of minimisers, $\mathscr{D}(\mathscr{E})$, is nonempty.

Strict convexity of $V \implies \widehat{\mu}^r$ and $\widehat{\mu}(\mathscr{E})$ are constant for all $\widehat{\mu} \in \mathscr{D}(\mathscr{E})$.

Theorem

Assuming $u(\mathscr{E}) < U(\infty)$*,*

$$\sup_{H \in \mathscr{H}} \mathrm{E}_{\mathbb{P}} \left[U\big(\mathscr{E} + (H \cdot S)_T\big) \right] = \min_{\mu \in \mathscr{D} \setminus \{0\}} \left\{ \mathrm{E}_{\mathbb{P}} \left[V \left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}} \right) \right] + \mu(\mathscr{E}) \right\}.$$

Note that the result does not require RAE or a growth condition on V (c.f. KS99), or even $\tilde{v}(y) < \infty$ for all $y > 0$ (c.f. KS03), where

$$\tilde{v}(y) := \inf_{\mathbb{Q} \in \mathcal{M}^a} \mathrm{E}_{\mathbb{P}} \left[V \left(y \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right) \right].$$

What about $U : \mathbb{R} \to \mathbb{R}$?

Assuming $u(\mathscr{E}) < U(\infty)$, the set $\mathcal{M}_V^a$ of abs continuous local* martingale measures with finite entropy is non-empty and

$$u(\mathscr{E}) = \sup_{H \in \mathscr{H}} \mathrm{E}_{\mathbb{P}} \left[U\big(\mathscr{E} + (H \cdot S)_T\big) \right]$$

$$= \min_{\mu \in \mathrm{Cone}(\mathcal{M}_V^a) \setminus \{0\}} \left\{ \mathrm{E}_{\mathbb{P}} \left[V \left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathbb{P}} \right) \right] + \mu(\mathscr{E}) \right\} =: v(\mathscr{E}).$$

The minimiser $\widehat{\mu}$ exists and is unique.

Again, one does not need any RAE condition or growth condition on V.

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*locally bounded case

Marginal Utility Based Prices

Definition (Hugonnier Kramkov Schachermayer '05)

$p \in \mathbb{R}$ is called a MUBP for a contingent claim B if

$$\sup_{H \in \mathscr{H}} \mathrm{E}_{\mathbb{P}} \left[U\big(x + (H \cdot S)_T\big) \right] = \sup_{\substack{H \in \mathscr{H} \\ q \in \mathbb{R}}} \mathrm{E}_{\mathbb{P}} \left[U\big(x + (H \cdot S)_T + q(B - p)\big) \right].$$

In other words,

$$u(x) = \sup_{q \in \mathbb{R}} u(x + q(B - p)).$$

Let

$$\mathcal{X}(x) := \{X \geq 0 : X = x + (H \cdot S)_T \text{ for some } H \in \mathscr{H}\}$$
$$\mathcal{Y} := \{Y \geq 0 : Y_0 = 1 \text{ and } XY \text{ is a supermartingale } \forall X \in \mathcal{X}(1)\}.$$

$$u(x) := \sup_{X \in \mathcal{X}(x)} \mathrm{E}_{\mathbb{P}}\left[U(X_T)\right]$$
$$\tilde{v}(y) := \inf_{Y \in \mathcal{Y}} \mathrm{E}_{\mathbb{P}}\left[V(yY_T)\right].$$

Let $\widehat{X}(x), \widehat{Y}(y)$ denote the optimal solutions to $u(x)$ and $\tilde{v}(y)$ respectively.

Theorem (Hugonnier Kramkov Schachermayer '05)

Suppose that $\tilde{v}(y) < \infty$ for all $y > 0$. Let $\tilde{X} \in \mathcal{X}(1)$ be maximal admissible.

1. *If $\widehat{Y}(y)\tilde{X}$ is a uniformly integrable martingale, and if $|B| < c\tilde{X}_T$ then $p = \mathrm{E}_{\mathbb{P}}\left[\widehat{Y}_T B\right]$.*
2. *If $\widehat{Y}(y)\tilde{X}$ fails to be a uniformly integrable martingale then one can find a contingent claim B, $0 \leq B \leq \tilde{X}_T$ which admits an interval of MUBP's.*

Corollary (HKS05)

Suppose that $\tilde{v}(y) < \infty$ *for all* $y > 0$. *Define* $y = u'(x)$*, and let* $\widehat{Y}(y)$ *be the unique optimal solution to* $\tilde{v}(y)$. *Then the following are equivalent:*

1. $\widehat{Y}(y)$ *is the density process of an equivalent local martingale measure* $\widehat{\mathbb{Q}}$*;*
2. *Every* $B \in L^{\infty}$ *admits a unique MUBP* $p = \mathrm{E}_{\widehat{\mathbb{Q}}}[B]$.

Let $\widehat{\mathscr{D}}(x) = \{\mu/\mu(\Omega) : \mu \in \mathscr{D}(x)\}$ denote the normalised optimal measures in the dual problem.

Theorem

The following are equivalent:

1. $p \in \mathbb{R}$ *is a MUBP of* B*;*
2. $p = \widehat{\mathbb{Q}}(B)$*, where* $\widehat{\mathbb{Q}} \in \widehat{\mathscr{D}}(x)$.

N.B. $\mathbb{Q} \in ba$!

Proof: (1) $\Longrightarrow$ (2): Recall the definition of a MUBP. The LHS admits a dual representation

$$u(x) = \inf_{\mu \in \mathscr{D}} \left\{ \mathrm{E}_{\mathbb{P}} \left[V \left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}} \right) \right] + \mu(x) \right\}. \qquad (*)$$

Setting $\mathscr{E} = x + q(B - p)$, the RHS admits a dual representation

$$\sup_{q \in \mathbb{R}} u(x+q(B-p)) = \sup_{q \in \mathbb{R}} \inf_{\mu \in \mathscr{D}} \left\{ \mathrm{E}_{\mathbb{P}} \left[V \left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}} \right) \right] + \mu(x + q(B - p)) \right\}. \qquad (**)$$

Therefore

$$\begin{aligned}
(*) = (**) &\geq \sup_{q \in [-1,1]} \inf_{\mu \in \mathscr{D}} \left\{ \mathrm{E}_{\mathbb{P}} \left[V \left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}} \right) \right] + \mu(x) + q\mu(B - p)) \right\} \\
&\overset{\text{minimax}}{=} \inf_{\mu \in \mathscr{D}} \sup_{q \in [-1,1]} \left\{ \mathrm{E}_{\mathbb{P}} \left[V \left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}} \right) \right] + \mu(x) + q\mu(B - p)) \right\} \\
&= \inf_{\mu \in \mathscr{D}} \left\{ \mathrm{E}_{\mathbb{P}} \left[V \left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}} \right) \right] + \mu(x) + |\mu(B - p)|) \right\}.
\end{aligned}$$

Therefore, we must have $\widehat{\mu}(B-p)=0$ for some $\widehat{\mu}\in\mathscr{D}(x)$. Normalising gives

$$p=\widehat{\mathbb{Q}}(B).$$

(1) $\implies$ (2): Suppose that $p=\widehat{\mathbb{Q}}(B)$ where $\widehat{\mathbb{Q}}=\widehat{\mu}/\widehat{\mu}(\Omega)$ for some $\widehat{\mu}\in\mathscr{D}(x)$. Then for any $H\in\mathscr{H}$ and $q\in\mathbb{R}$

$$\begin{aligned}
&\mathrm{E}_{\mathbb{P}}\left[U(x+(H\cdot S)_T+q(B-p))\right]\\
&\qquad\leq\left\{\mathrm{E}_{\mathbb{P}}\left[V\left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathbb{P}}\right)\right]+\widehat{\mu}(x+q(B-p))\right\}\\
&\qquad=v(x)+q\widehat{\mu}(\Omega).(\widehat{\mathbb{Q}}(B)-p)=u(x).
\end{aligned}$$

Therefore p is a MUBP of B. □

Corollary

- *MUBPs exist for all* $B \in L^\infty$*;*
- *MUBP is unique if and only if* $\widehat{\mathbb{Q}}^s(B)$ *is constant as* $\widehat{\mathbb{Q}}$ *ranges over* $\widehat{\mathscr{D}}(x)$.

Alternative simple proof of one of HKS's results:

Corollary

Suppose that the solution to the primal problem, $\widehat{X} := x + (\widehat{H} \cdot S)_T$ *exists. The MUBP is unique provided there exists* $c > 0$ *such that* $|B| \leq c\widehat{X}_T$.

Proof: By the previous corollary, it's enough to show that $\mathbb{Q}^s(|B|) = 0$ for all $\mathbb{Q} \in \widehat{D}(x)$. This will, in turn, follow if we show that $\mathbb{Q}^s(\widehat{X}_T) = 0$. This must hold for the Fenchel inequality to be sharp. □

What about the case $U : \mathbb{R} \to \mathbb{R}$?

HKS have already commented that this is trivial. Indeed $\mathscr{D}(x) = \{\widehat{\mu}\}$ where $\widehat{\mu} \in \mathcal{M}_V^a$, so $p = \mathrm{E}_{\widehat{\mathbb{Q}}}[B]$ is unique.

What about a dynamic version of MUBP's – a Marginal Utility Based Price **Process**?

Marginal Utility-Based Price Processes

Consider an investor $(U, \mathscr{E})$ and a financial market represented by a d-dimensional locally bounded semimartingale S.

Notation:

$$u(\mathscr{E}) \longrightarrow u_{\mathscr{E}}(S) = \sup_{H \in \mathscr{H}} \mathrm{E}_{\mathbb{P}} \left[U\big(\mathscr{E} + (H \cdot S)_T\big) \right],$$

$$v(\mathscr{E}) \longrightarrow v_{\mathscr{E}}(S) = \min_{\mu \in \mathrm{Cone}(\mathcal{M}_V^a(S)) \setminus \{0\}} \left\{ \mathrm{E}_{\mathbb{P}} \left[V\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathbb{P}} \right) \right] + \mu(\mathscr{E}) \right\}.$$

Let $\widehat{\mu}(S)$ denote the optimal measure, and $\widehat{\mathbb{Q}}(S)$ denote the normalisation of $\widehat{\mu}(S)$.

Let S' be a d'-dimensional locally bounded semimartingale which represents the price of some new financial assets (zero supply) which are to be introduced into the market.

Definition

We shall say that a stochastic process S' is a Marginal Utility Based Price Process (MUBPP) with respect to $(U, \mathscr{E})$ and S if $u_{\mathscr{E}}(S, S') = u_{\mathscr{E}}(S)$.

Interpretation: New asset is in zero total supply, therefore zero total demand.

Theorem

In the case $U : \mathbb{R} \to \mathbb{R}$, S' is a MUBPP if and only if S' is a $\widehat{\mathbb{Q}}(S)$ local martingale.

Proof:

$$
\begin{aligned}
u_{\mathscr{E}}(S, S') = u_{\mathscr{E}}(S) \\
&\iff v_{\mathscr{E}}(S, S') = v_{\mathscr{E}}(S) \\
&\iff \min_{\mu \in \mathrm{Cone}(\mathcal{M}_V^a(S,S'))} \left\{ \dots \right\} = \min_{\mu \in \mathrm{Cone}(\mathcal{M}_V^a(S))} \left\{ \dots \right\} \\
&\iff \widehat{\mu}(S) \in \mathrm{Cone}(\mathcal{M}_V^a(S, S')) \\
&\iff \widehat{\mathbb{Q}}(S) \in \mathcal{M}_V^a(S, S') \\
&\iff S' \text{ is a } \widehat{\mathbb{Q}}(S)\text{-local martingale} \qquad \square
\end{aligned}
$$

What about $U : (0, \infty) \to \mathbb{R}$?

Definition

Let $\mathbb{Q} \in ba_+$. A cádlág semimartingale S is called a $\mathbb{Q}$-σ-martingale if $\mathbb{Q}\big((H \cdot S)_T\big) \leq 0$ for each $H \in \mathscr{H}$.

Proposition

Let $S = (S_t)_{t \in [0,T]}$ be a semimartingale, and let $\mathbb{Q} \ll \mathbb{P}$. Then S is a $\mathbb{Q}$-σ-martingale if it is a σ-martingale w.r.t. the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in [0,T]}, \mathbb{Q})$. If S is locally bounded, the converse holds as well.

Proposition

S' is a MUBPP if and only if S' is a $\mathbb{Q}(S)$-σ-martingale for some $\mathbb{Q}(S) \in \widehat{\mathscr{D}}(S)$.

Utility Indifference Pricing

Definition

The utility indifferent purchase price, $p = p(B; U, \mathscr{E})$, of B is defined implicitly as the solution to the equation

$$u(\mathscr{E} + B - p) = u(\mathscr{E}).$$

Proposition (Indifference Prices)

There exists a unique solution for the utility indifferent purchase price, $p(B) = p$. Moreover,

1. *(Range of prices)*

$$\inf_{\mathbb{Q}\in\mathcal{M}^a(S)} \mathrm{E}_{\mathbb{Q}}[B] \leq p(B) \leq \mathrm{E}_{\widehat{\mathbb{Q}}}[B];$$

2. *(Translation invariance) For $c \in \mathbb{R}$ we have $p(B + c) = p(B) + c$;*

3. *(Pricing replicable claims) If $B = (H \cdot S)_T$ for some $H \in \mathscr{H}$ then $p(B) = 0$;*

3. (Monotonicity) If $B \leq C$ then $p(B) \leq p(C)$;

4. (Concavity) Given contingent claims B_1, B_2 and $\lambda \in [0, 1]$,

$$p_{\mathscr{E}}\big(\lambda B_1 + (1-\lambda)B_2\big) \geq \lambda p(B_1) + (1-\lambda)p(B_2);$$

5. (Pricing via entropic penalty)

$$p(B) = \inf_{\mathbb{Q} \in \mathcal{M}^a(S)} \{ \mathrm{E}_{\mathbb{Q}}[B] + \alpha(\mathbb{Q}) \},$$

where the penalty functional, or dual Orlicz norm, $\alpha : \mathcal{M}^a(S) \to [0, \infty)$ is defined by

$$\alpha(\mathbb{Q}) := \inf_{y>0} \frac{1}{y} \{ \mathbb{V}_{\mathscr{E}}(y\mathbb{Q}) - v(\mathscr{E}) \} ;$$

6. (Strong Continuity) If $(B_n)_{n\in\mathbb{N}}$ is a sequence of contingent claims such that

$$\sup_{\mathbb{Q}\in\mathcal{M}^a(S)} \mathrm{E}_{\mathbb{Q}}\left[B_n - B\right] \to 0 \text{ and } \inf_{\mathbb{Q}\in\mathcal{M}^a(S)} \mathrm{E}_{\mathbb{Q}}\left[B_n - B\right] \to 0$$

as $n \to \infty$ then $p(B_n) \to p(B)$;

6. (Fatou property) If $(B_n)_{n\geq 0}$ is a sequence of contingent claims then

$$p(\limsup_n B_n) \geq \limsup_n p(B_n);$$

7. (Continuity from above) If $(B_n)_{n\in\mathbb{N}}$ is a sequence of contingent claims such that $B_n \searrow B$ $\mathbb{P}$-a.s. then $p(B_n) \searrow p(B)$.

Definition

For $\beta > 0$, the *average utility indifferent purchase price* for β units of the contingent claim B is defined by

$$p(B, \beta) := \frac{p(\beta B)}{\beta}.$$

Proposition (Volume Asymptotics)

$p(B, \beta)$ is a continuous, non-increasing function of β. Moreover,

1. $\inf_{\mathbb{Q} \in \mathcal{M}_V^a(S)} \mathrm{E}_{\mathbb{Q}}[B] \leq p(B, \beta) \leq \mathrm{E}_{\widehat{\mathbb{Q}}}[B]$;
2. $\lim_{\beta \to \infty} p(B, \beta) = \inf_{\mathbb{Q} \in \mathcal{M}_V^a(S)} \mathrm{E}_{\mathbb{Q}}[B]$;
3. $\lim_{\beta \to 0} p(B, \beta) = \mathrm{E}_{\widehat{\mathbb{Q}}}[B]$.

Conclusion

- Simple proof of dual representation for utility maximisation problem without the need for RAE/growth conditions;
- Characterisation of Marginal Utility Based Prices;
- Theory of Marginal Utility-Based Price Processes;
- Asymptotic average indifference prices coincide with MUBP for $U : \mathbb{R} \to \mathbb{R}$ as volume tends to zero.