Dispersive PDEs meet discontinuities
Retreat for Women in in Applied Mathematics

Beatrice Pelloni

Heriot Watt University & Maxwell Institute - Edinburgh

in collaboration with mentors, colleagues, and students

About me - CV in a nutshell

PhD - Yale 1996

Integrable or near-integrable wave models - solitary wave analysis and simulation

started in 1988 but interrupted, had children, then slowly finished.

Marie Curie Fellowship - 1997

more work on wave models, with emphasis on modelling realistic boundary conditions - development of the Unified Transform

This was the turning point for my career. It took me to Imperial College. I stayed on for a PDRA position until 2001

2001-2016

Lecturer, then Reader (2005), then Professor (2012) and HoD at the University of Reading

since 2016

Professor at Heriot-Watt - also Head of School 2016-2022
Also currently Deputy Director at ICMS

Highlights of my work

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2. **Large-scale atmospheric flow models** - analysis via optimal transport with a non-standard cost, numerics via semi-discrete optimal transport.
3. Mathematical study of the phenomenon of **dispersive quantisation**, aka Talbot effect, in nonlocal/aperiodic systems.

A 2-point boundary value problem - *work with mentors*

Stokes/Airy equation

$$u_t = u_{xxx}, \quad u(x,0) = u_0(x), \quad u(0,t) = u(1,t) = u_x(0,t) = 0.$$

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Solution: complex contour integral

$$\begin{aligned} 2\pi u(x,t) &= \int_{\mathbb{R}} \mathrm{e}^{-i\lambda^3 t + i\lambda x} \hat{u}_0(\lambda) d\lambda + \int_{\partial D^+} \mathrm{e}^{-i\lambda^3 t + i\lambda x} \frac{\tilde{F}(\lambda)}{\Delta(\lambda)} d\lambda \\ &\quad + \int_{\partial D^-} \mathrm{e}^{-i\lambda^3 t + i\lambda(x-1)} \frac{\tilde{G}(\lambda)}{\Delta(\lambda)} d\lambda \end{aligned}$$

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This integral is not equivalent to a series representation

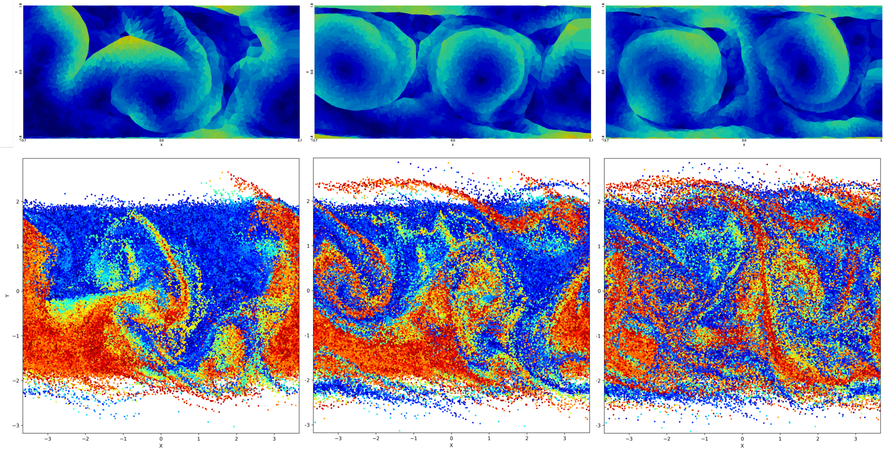
$$\tilde{F} = \hat{u}_0(\lambda)\mathrm{e}^{i\lambda} + \omega \hat{u}_0(\omega\lambda)\mathrm{e}^{i\omega\lambda} + \omega^2 \hat{u}_0(\omega^2\lambda)\mathrm{e}^{i\omega^2\lambda}, \quad \tilde{G} = ...$$

$$\omega = \mathrm{e}^{2\pi i/3}, \quad \partial D^{\pm} = \text{lines where } \omega \text{ is purely imaginary},$$

$$\Delta(\lambda) = \left(\mathrm{e}^{i\lambda} + \omega \mathrm{e}^{i\omega\lambda} + \omega^2 \mathrm{e}^{i\omega^2\lambda} \right).$$

A semigeostrophic cyclone - *work with PhD students*

$t \approx 16$ Days $\quad t \approx 20$ Days $\quad t \approx 25$ Days

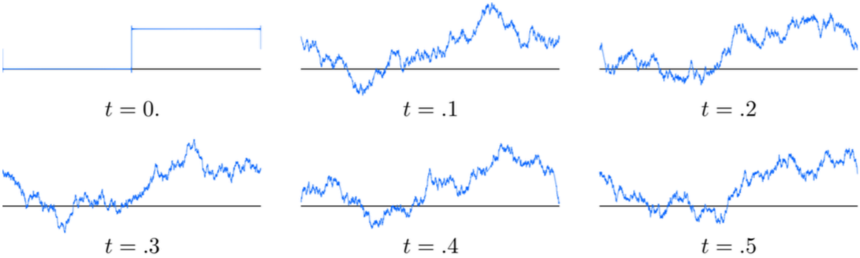


Velocity and temperature of a twin cyclone

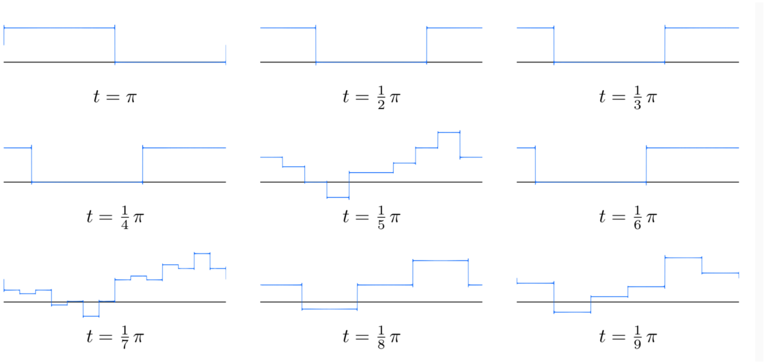
Using a semi-discrete optimal transport scheme, 36k points

Dispersive revivals - *work with colleagues*

In a seminar talk, Peter Olver showed these pictures for the solution at various time of the 2π-periodic Airy equation $u_t = u_{xxx}$ starting from a step function:



...and then the solution at special values of the time



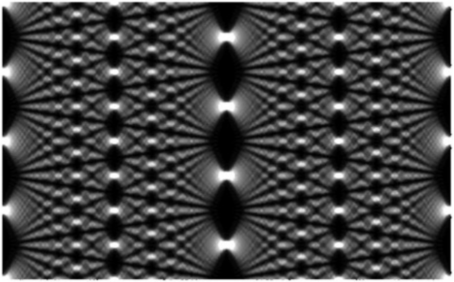
Rational times

A step back: Talbot and his optical effect

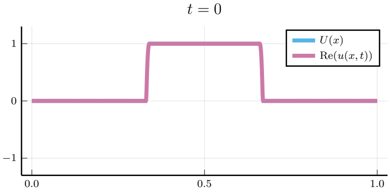
Talbot observations in 1835:

Observing light passing through a diffraction grating, he observed that, at each **rational** *multiple of a fixed distance, the diffraction pattern appears to reproduce a finite number of copies of the grating pattern.*

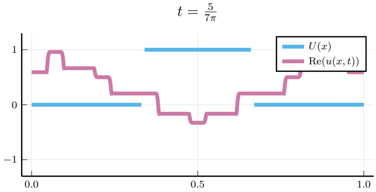
Talbot effect: the *self-imaging* of a diffraction grating. At regular distances from the grating, the light diffracted through it forms a nearly perfect image of the grating itself.



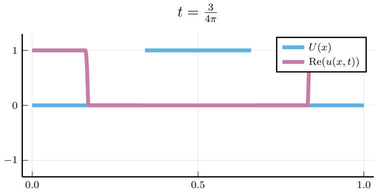
Free-space Schrödinger on $\mathbb{T}$ - step initial condition on $[0, 1]$



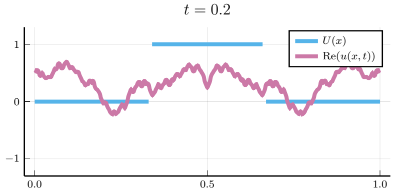
(a) Step initial condition



(b) Solution at $t = 5/7\pi$

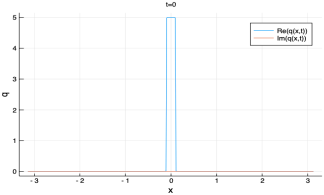


(c) Solution at $t = 3/4\pi$

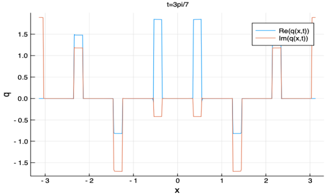


(d) Solution at irrational time $t = 0.2$

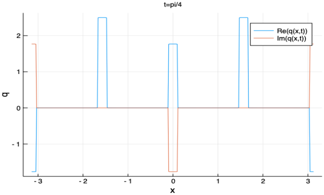
Free-space Schrödinger on $\mathbb{T}$ - box initial profile



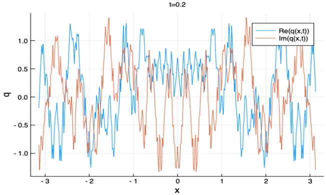
(a) Narrow box initial condition



(b) Solution at $t = 3\pi/7$



(c) Solution at $t = \pi/4$



(d) Solution at $t = 0.2$

The periodic, free-space Schrödinger equation

$$\partial_t u(x,t) = i\partial_x^2 u(x,t) \qquad x \in \mathbb{T}, \quad t > 0$$
$$u(x,0) = u_0(x) \qquad x \in \mathbb{T}.$$

Theorem

Let $u_0 \in \mathrm{BV}(\mathbb{T})$. Then:

(a)

$$u\Big(x, 2\pi\frac{p}{q}\Big) = \frac{1}{q}\sum_{k,m=0}^{q-1} \mathrm{e}^{2\pi i\frac{km}{q}}\mathrm{e}^{2\pi i\frac{p}{q}m^2} u_0\Big(x - 2\pi\frac{k}{q}\Big)$$

for co-prime $p, q \in \mathbb{N}$ (u_0 is *revived* if $t \in 2\pi\mathbb{Q}$);

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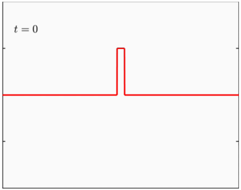
(b) **but...** $Re(u)$, $Im(u)$ are continuous in x for $t \notin 2\pi\mathbb{Q}$ (continuous also in t if u_0 is continuous);

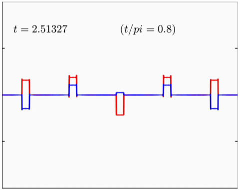
(c) if $u_0 \notin H^s(\mathbb{T})$, $s > \frac{1}{2}$, for almost all $t > 0$ the graph of both $Re(u)$ and $Im(u)$ has fractal dim $= \frac{3}{2}$ (*fractalisation*).

True also for the nonlinear PDE- NLS: $iu_t + u_{xx} + |u|^2 u = 0$

Theorem by Erdogan-Tzirakis stating a *weak revival property*



(a)



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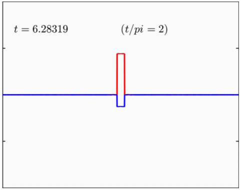
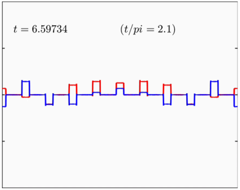


(c)



(d)

In summary: Periodic revivals

Periodic revivals: the solution of a linear dispersive periodic problem, at times equal to rational multiples of (*a constant depending on*) the period, is a finite linear combination of translated and reflected copies of the initial profile

[1] M. B. Erdoğan, N. Tzirakis, *Dispersive PDEs*, (Cambridge University Press, 2016)

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Theorem[1] Consider the dispersive PDE

$$\partial_t u(x,t) = iP(-i\partial_x)u(x,t), \quad u(x,0) = u_0(x), \quad x \in \mathbb{T}$$

$P(k)$ a polynomial with integer coefficients.

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At $t = 2\pi \frac{p}{q}$, the solution admits the representation

$$u(x, 2\pi\frac{p}{q}) = \frac{1}{q}\sum_{k=0}^{q-1} G_{p,q}(k) u_0(x - 2\pi\frac{k}{q}),$$

$$G_{p,q}(k) = \sum_{m=0}^{q-1} e^{-2\pi i P(m)\frac{p}{q}} e^{2\pi i m \frac{k}{q}}.$$

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Proof sketch

The proof is based on three elementary properties:

- Modularity is preserved by polynomial with integer coefficients: $k \equiv_q h \mod q \Longrightarrow P(k) \equiv_q P(h)$:

$$u(x, \frac{2\pi p}{q}) = \frac{1}{2\pi} \sum_{m=0}^{q-1} e^{-iP(m)\frac{2\pi p}{q}} \underbrace{\sum_{\substack{j \in \mathbb{Z} \\ j \underset{q}{\equiv} m}} \langle u_0, e^{ij(\cdot)} \rangle e^{ijx}}_{T}$$

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- The sum of all n-th roots of unity satisfies

$$\frac{1}{q} \sum_{k=0}^{q-1} e^{2\pi i (m-j)\frac{k}{q}} = \begin{cases} 1 & j \underset{q}{\equiv} m \\ 0 & j \underset{q}{\not\equiv} m \end{cases} \Rightarrow T = \frac{1}{q} \sum_{k=0}^{q-1} e^{2\pi i m \frac{k}{q}} \underbrace{\sum_{j \in \mathbb{Z}} \left\langle u_0, e^{ij(\cdot)} \right\rangle e^{ij(x - 2\pi \frac{k}{q})}}_{2\pi u_0(x - 2\pi \frac{k}{q})}$$

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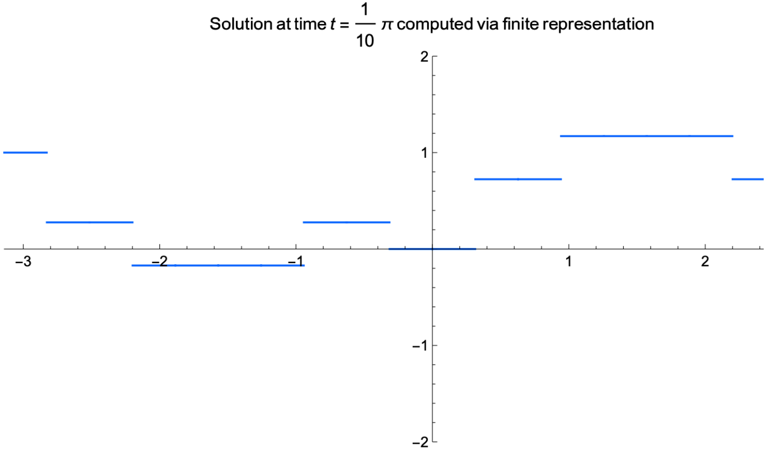
- Translation in x corresponds to multiplication of the Fourier coefficients by an exponential.

Finite sum solution of Airy - step initial condition



In summary: Fractalisation

Fractalisation: At *irrational* times (hence a.e. in $t > 0$) the solution, starting from a BV (*hence possibly discontinuous*) initial profile, is a continuous function of x whose graph has fractal dimension > 1 - ($\frac{3}{2}$ for Schrödinger, in $[\frac{5}{4}, \frac{7}{4}]$ for Airy).

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[2] H. L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, (American Mathematical Soc., 1994)

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Hence **the solution has better regularity properties at irrational than at rational times**.

The *proof* is based on number theoretical results[2] and on regularity estimates in Besov spaces[3].

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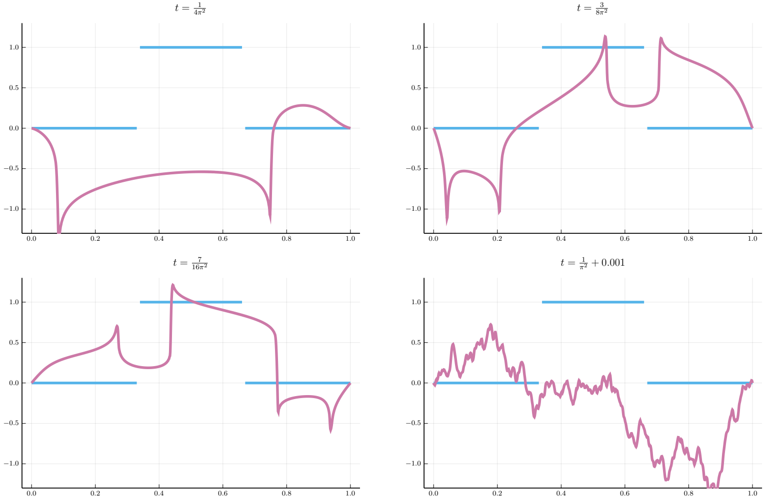
What about the periodicity? Airy with $u(0,t) = u(1,t) = 0$, $u_x(0,t) = u_x(1,t)$



blue: *initial condition* - magenta: *exact solution*

What is going on?

The answer is hidden in the spectral asymptotics of the spatial operator and the interaction with the periodic Hilbert transform[4]

[4] *L. Boulton, G. Farmakis, BP and D.A. Smith, ArXiv preprint:* 2403.01117

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Periodic Hilbert transform $\mathcal{H}$ on $[0,1]$:

$$\mathcal{H}f(x) = pv\int_0^1 \cot\left[\pi(x-y)\right] f(y)dy \quad \text{and} \quad \widehat{\mathcal{H}f}(k) = -i\,\mathrm{sgn}(k)\hat{f}(k)$$

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$$\mathcal{H}f(x) = i\sum_{n=1}^{\infty}\left[\widehat{f}(-n)e^{-2\pi inx} - \widehat{f}(n)e^{2\pi inx}\right], \qquad f \in L^2[0,1]$$

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Crucial if elementary identity:

$$\sum_{n=1}^{\infty} \widehat{f}(n)e^{2\pi i n x} = \frac{(Id + i\mathcal{H})f - \hat{f}(0)}{2}$$

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Periodic Hilbert transform $\mathcal{H}$ of a step function

$\mathcal{H}$ the periodic Hilbert transform:

$$\mathcal{H}g(x) = \frac{1}{2\pi}\ \mathrm{p.v.}\int_{-\pi}^{\pi} \cot\frac{x-y}{2} g(y)\,\mathrm{d}y \Longrightarrow \widehat{\mathcal{H}g}(k) = -i\,sgn(k)\hat{g}(k).$$

$$\widehat{iu_{xx}} = -ik^2\hat{u}(k) \quad vs \quad \widehat{\mathcal{H}u_x x}(k) = -ik|k|\hat{u}(k)$$

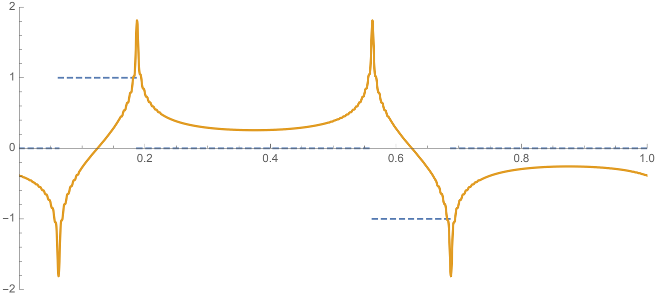
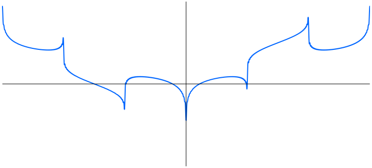
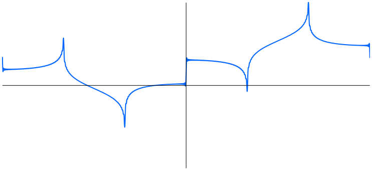


Figure: A step function (dashed) and its periodic Hilbert transform (solid). Where the given profile has a point of discontinuity, its periodic Hilbert transform displays an (infinite) logarithmic cusp

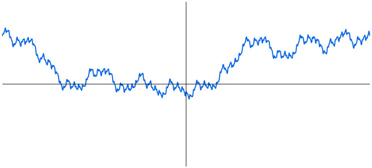
The linearised Benjamin-Ono equation $u_t = \mathcal{H} u_{xx}$ periodic, step initial condition



$t = \frac{\pi}{3}$



$t = \frac{\pi}{6}$



$t = 0.9$

The periodic linearised BO equation

$$BO: \; \widehat{iu_{xx}} = -ik^2\hat{u}(k) \quad vs \quad lS: \; \widehat{\mathcal{H}u_{xx}}(k) = -ik|k|\hat{u}(k)$$

Hilbert transform identity: for $g \in L^2(\mathbb{T})$

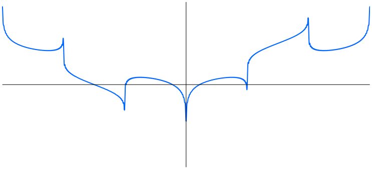
$$\sum_{n=1}^{\infty} \hat{g}(n)e_n(x) = \frac{(Id + i\mathcal{H})g - \langle g \rangle}{2},$$

Lemma *Assume* u_0 real-valued, *and WLOG* $\langle u_0 \rangle = 0$.
For u *solution of linear BO,* v *solution of free-space Schrödinger*

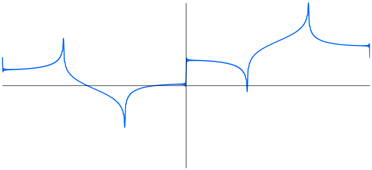
$$u(x,t) = Re\left[(Id + i\mathcal{H})' v(x,t)\right].$$

This implies the result on (cusp) revivals, both for the continuous/discontinuous dychotomy and the fractal dimension
The weak version appears to hold for the full nonlinear problem

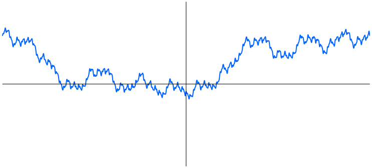
Linearised BO, 2π-periodic, step initial condition



$t = \frac{\pi}{3}$



$t = \frac{\pi}{6}$



$t = 0.9$

Numerical evaluation - fractal (box-counting) dimension

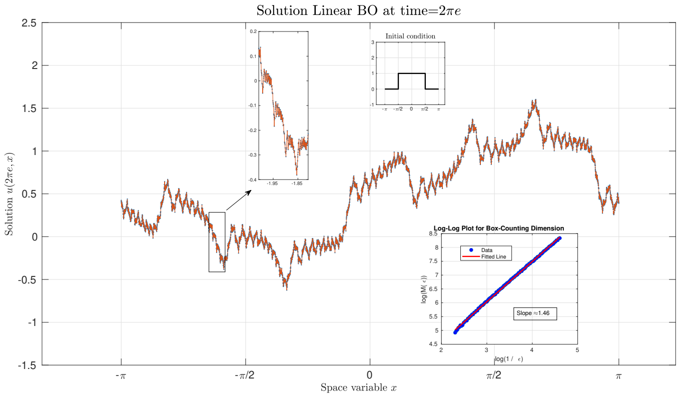
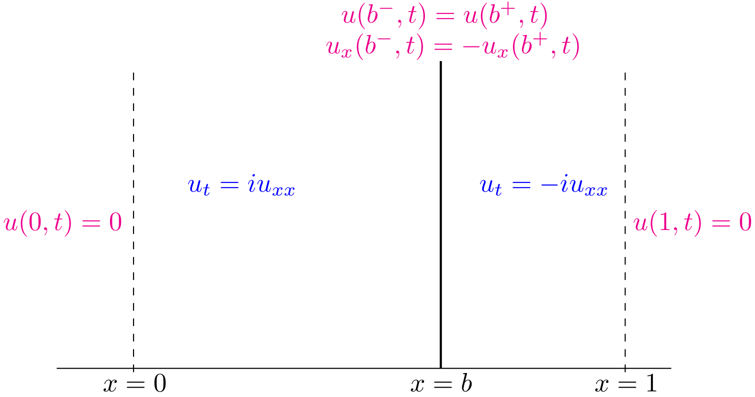


Figure: A step function (dashed) and its periodic Hilbert transform (solid). When a given profile has a point of discontinuity, its periodic Hilbert transform displays a logarithmic cusp [5]

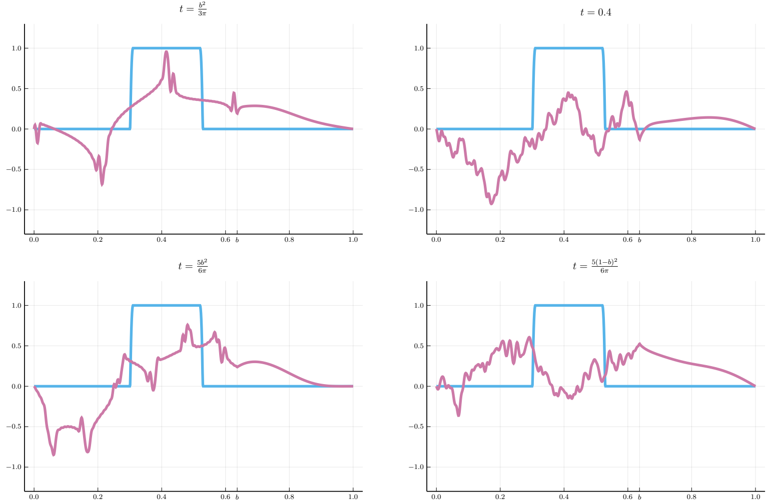
[5] *L. Boulton, B. Macpherson, BP, ArXiv preprint:* 2501.01322

More surprises: Schrödinger with a dislocation at $x = b$



Dislocation model - step initial condition, $b = 0.636619$

rational /irrational times, initial discontinuities to the left of b, 250 modes



blue: *initial condition* - magenta: *exact solution*

Summary: dispersive revivals

- for linear dispersive PDEs, **periodic**, initial discontinuities are propagated in the solution for a (measure zero) set of special values of the time
  but for almost all times the solution is continuous
- *polynomial dispersion*: jumps stay jumps
  non-polynomial dispersion of $d^0 \geq 2$: jumps may become cusps
- robust phenomenon that survives (in a weaker form) the perturbation by nonlinearity, quasi-periodicity, stochastic noise
- it can also survive, in weak cusp form, when the boundary conditions are not periodic
- Applications!?!

Thank yous ....

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