# On the ghost conjecture of Bergdall and Pollack, I

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# $p$-adic slopes of modular forms

- Fix a prime $p \geq 5$ and a positive integer $N$ that is prime to $p$.
- Let $v_p(\cdot)$ be the $p$-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$ and $|\cdot|_p = p^{-v_p(\cdot)}$ be the corresponding $p$-adic norm.
- For every integer $k \geq 2$, let $\mathrm{S}_k(\Gamma_1(N), \bar{\mathbb{Q}}_p)$ be the space of cuspforms of weight $k$ and level $\Gamma_1(N)$. For a normalized Hecke eigenform $f \in \mathrm{S}_k(\Gamma_1(N), \bar{\mathbb{Q}}_p)$ with $q$-expansion $f = \sum_{n \geq 1} a_n q^n$, its $p$-adic slope is defined to be the number $v_p(a_p)$.
- Motivations to study $p$-adic slopes of modular forms:
  1. computational perspective: Gouvêa conjecture, Gouvêa-Mazur conjecture;
  2. automorphic perspective: geometry of eigencurves (finiteness of irreducible components, spectral halo conjecture etc.);
  3. Galois perspective: study of the (crystalline) deformation of the local residual Galois representation $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$.

# Preliminaries I

- Let $\Delta = \mathbb{F}_p^\times$ and $\omega : \Delta \to \mathbb{Z}_p^\times$ be the Teichmüller character. We identify $\Delta$ with the torsion subgroup of $\mathbb{Z}_p^\times$ via $\omega$. Fix a topological generator $\gamma = \exp(p)$ of $1 + p\mathbb{Z}_p$.
- For every character $\varepsilon : \Delta \to \mathbb{Z}_p^\times$, let $\mathcal{W}_\varepsilon$ be the rigid analytic space parametrizing continuous $p$-adic valued characters $\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ such that $\kappa|_\Delta = \varepsilon$. For such a $\kappa$, we define its coordinate on $\mathcal{W}_\varepsilon$ to be

$$w_\kappa := \kappa(\gamma) - 1.$$

- Every integer $k \geq 2$ gives rise to a classical $p$-adic weight $\kappa_k : z \mapsto z^{k-2}$ with coordinate $w_k = \exp(p(k-2)) - 1$.
- Set $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$. For every $p$-adic weight $\kappa$, let $\mathrm{S}_\kappa^\dagger(\Gamma)$ be the sapce of overconvergent cuspforms of weight $\kappa$ and level $\Gamma$, equipped with a compact operator $U_p$. By a theorem of Coleman, we have

$$\mathrm{S}_k(\Gamma, \bar{\mathbb{Q}}_p) = \mathrm{S}_k^\dagger(\Gamma)^{U_p-\mathrm{slope} \lneq k-1}$$

So it suffices to determine the ‘overconvergent slopes’, i.e. the $p$-adic valuations of $U_p$-eigenvalues in the space $\mathrm{S}_k^\dagger(\Gamma)$, or equivalently, to determine the Newton polygon of the characteristic power series of the $U_p$-operator on $\mathrm{S}_k^\dagger(\Gamma)$.

# Preliminaries II

- Let $\mathrm{G}_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the Galois group of $\mathbb{Q}$, $\mathrm{G}_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ be its $p$-decomposition group and $\mathrm{I}_{\mathbb{Q}_p} \subset \mathrm{G}_{\mathbb{Q}_p}$ be the inertia subgroup. Let $\omega_1 : \mathrm{G}_{\mathbb{Q}} \to \mathbb{F}_p^\times$ be the mod $p$ cyclotomic character.
- Fix a continuous irreducible residual Galois representation $\bar{r} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ that is modular of level $N$. Assume that the restriction $\bar{r}|_{\mathrm{G}_{\mathbb{Q}_p}}$ is reducible and satisfies a certain generic assumption.
- Let $b$ be the unique integer in $\{2, \ldots, p\}$ such that $\det(\bar{r})|_{\mathrm{I}_{\mathbb{Q}_p}} = \omega_1^{b-1}$. Set $\varepsilon(b) = \omega^{b-2} : \Delta \to \mathbb{Z}_p^\times$.
- For every $p$-adic weight $\kappa \in \mathcal{W}_{\varepsilon(b)}$, let $\mathrm{S}_\kappa^\dagger(\bar{r})$ be the $\bar{r}$-isotypic component of the space $\mathrm{S}_\kappa^\dagger(\Gamma)$. It is stable under the $U_p$-operator and let

$$C_{\bar{r}}(w_\kappa, t) = \det(1 - tU_p|_{\mathrm{S}_\kappa^\dagger(\bar{r})})$$

be the characteristic power series of $U_p$ on $\mathrm{S}_\kappa^\dagger(\bar{r})$.

# Statement of the conjecture

- Bergdall and Pollack constructed a formal series $G_{\bar{r}}(w,t) \in \mathbb{Z}_p[\![w,t]\!]$ (called the ghost series), which only depends on the dimensions of certain subspaces of classical modular forms. This series will serve as a 'model' of the characteristic power series of the $U_p$-operator.

### Ghost conjecture of Bergdall-Pollack

For every $p$-adic weight $\kappa \in \mathcal{W}_{\varepsilon(b)}$, the Newton polygon of $G_{\bar{r}}(w_\kappa, t)$ coincides with the Newton polygon of $C_{\bar{r}}(w_\kappa, t)$.

- The assumption that $\bar{r}|_{\mathrm{G}_{\mathbb{Q}_p}}$ is reducible is crucial. The conjecture is false without this assumption.
- We will formulate a local version of ghost conjecture and prove it under mild assumptions.

# Set up

- Let $E/\mathbb{Q}_p$ be a finite extension with integer ring $\mathcal{O}$, a uniformizer $\varpi$ and residue field $\mathbb{F}$.
- Define the following subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\mathrm{K}_p := \mathrm{GL}_2(\mathbb{Z}_p) \supset \mathrm{Iw}_p := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \supset \mathrm{Iw}_{p,1} := \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}.$$

- Fix a reducible nonsplit residual Galois representation $\bar{\rho} : \mathrm{I}_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$:

$$\bar{\rho} \simeq \begin{pmatrix} \omega_1^{a+b+1} & * \neq 0 \\ 0 & \omega_1^b \end{pmatrix} \qquad \text{for } 1 \leq a \leq p-4 \text{ and } 0 \leq b \leq p-2.$$

Let $\sigma_{a,b} = \mathrm{Sym}^a \mathbb{F}^{\oplus 2} \otimes \det^b$ be the Serre weight of $\bar{\rho}$, considered as a right $\mathbb{F}$-representation of $\mathrm{GL}_2(\mathbb{F}_p)$ (the transpose of the usual left representation). Denote by $\mathrm{Proj}_{a,b}$ the projective envelope of $\sigma_{a,b}$ in the category of (right) $\mathbb{F}[\mathrm{GL}_2(\mathbb{F}_p)]$-module.

# Augmented modules

- An $\mathcal{O}[\![\mathrm{K}_p]\!]$-projective augmented module $\widetilde{\mathrm{H}}$ is a finitely generated *right* projective $\mathcal{O}[\![\mathrm{K}_p]\!]$-module equipped with an right $\mathcal{O}[\mathrm{GL}_2(\mathbb{Q}_p)]$-module structure such that the two induced $\mathcal{O}[\mathrm{K}_p]$-structures on $\widetilde{\mathrm{H}}$ coincide. We say that $\widetilde{\mathrm{H}}$ is of type $\bar{\rho}$ with multiplicity $m(\widetilde{\mathrm{H}})$ if
  - (Serre weight) $\overline{\mathrm{H}} := \widetilde{\mathrm{H}}/(\varpi, \mathrm{I}_{1+p\mathrm{M}_2(\mathbb{Z}_p)})$ is isomorphic to a direct sum of $m(\widetilde{\mathrm{H}})$ copies of $\mathrm{Proj}_{a,b}$ as a right $\mathbb{F}[\mathrm{GL}_2(\mathbb{F}_p)]$-module, where $\mathrm{I}_{1+p\mathrm{M}_2(\mathbb{Z}_p)}$ is the augmentation ideal of $\mathcal{O}[\![1+p\mathrm{M}_2(\mathbb{Z}_p)]\!]$.

  We say $\widetilde{\mathrm{H}}$ is primitive if $m(\widetilde{\mathrm{H}}) = 1$.
- $\widetilde{\mathrm{H}}$ naturally appears in the complete homology groups of certain Shimura varieties. Fix an absolutely irreducible residual Galois representation $\bar{r} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ such that $\bar{r}|_{\mathrm{I}_{\mathbb{Q}_p}} \cong \bar{\rho}$. Fix a neat tame level $K^p \subseteq \mathrm{GL}_2(\mathbb{A}_f^p)$. For an open compact subgroup $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$, write $Y(K)$ for the corresponding open modular curve over $\mathbb{Q}$. The *$\bar{r}$-localized completed homology* with tame level $K^p$ defined by

$$\widetilde{\mathrm{H}}(K^p)_{\mathfrak{m}_{\bar{r}}} := \varprojlim_m \mathrm{H}_1^{\mathrm{Betti}}\Big(Y\big(K^p(1+p^m\mathrm{M}_2(\mathbb{Z}_p))\big)_{\mathbb{C}}, \mathcal{O}\Big)^{\mathrm{cplx}=1}_{\mathfrak{m}_{\bar{r}}}.$$

  is an $\mathcal{O}[\![\mathrm{K}_p]\!]$-projective augmented module.

# Abstract classical forms I

- We will define various spaces of automorphic forms in our abstract setting that are analogues of notions in modular forms.

| modular forms | abstract automorphic forms |
|---|---|
| $\mathrm{S}_k(\Gamma_1(N), \bar{\mathbb{Q}}_p)_{\bar{r}}$ | $\mathrm{S}_k^{\mathrm{ur}}(\varepsilon_1)$ |
| $\mathrm{S}_k(\Gamma_1(N) \cap \Gamma_0(p), \bar{\mathbb{Q}}_p)_{\bar{r}}$ | $\mathrm{S}_k^{\mathrm{Iw}}(\psi)$ |
| $\mathrm{S}_k^{\dagger}(\Gamma_1(N) \cap \Gamma_0(p), \bar{\mathbb{Q}}_p)_{\bar{r}}$ | $\mathrm{S}_k^{\dagger}(\psi)$ |
| family of overconvergent modular forms | $\mathrm{S}^{\dagger,(\varepsilon)}$ |
| family of $p$-adic modular forms | $\mathrm{S}_{p\text{-adic}}^{(\varepsilon)}$ |

- Fix a character $\varepsilon = \varepsilon_1 \times \varepsilon_2 = \omega^{-s+b} \times \omega^{a+s+b} : \Delta^2 \to \mathbb{Z}_p^\times$ for some $s \in \{0, \ldots, p-2\}$.
- For $\alpha \in \mathbb{Z}_p$, let $\bar{\alpha}$ be its mod $p$ reduction. We view $\varepsilon$ as a character of $\mathrm{Iw}_p$ via the formula

$$\varepsilon\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \varepsilon(\bar{\alpha}, \bar{\delta}), \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Iw}_p.$$

# Abstract classical forms II

- For every integer $k \geq 2$, let $\mathcal{O}[z]^{\leq k-2}$ denote the space of polynomials in $\mathcal{O}$ of degree $\leq k-2$. It carries an action of the monoid $\mathrm{M}_2(\mathbb{Z}_p)^{\det\neq 0}$:

$$h|_{\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)}(z) = (\gamma z+\delta)^{k-2} h\Big(\frac{\alpha z+\beta}{\gamma z+\delta}\Big), \text{ for } \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}_p).$$

- We define the space of abstract classical forms of weight $k$ and character $\psi = \varepsilon \cdot (1 \times \omega^{2-k})$ to be the space

$$\mathrm{S}_k^{\mathrm{Iw}}(\psi) := \mathrm{Hom}_{\mathcal{O}[\mathrm{Iw}_p]}\big(\widetilde{\mathrm{H}}, \ \mathcal{O}[z]^{\leq k-2} \otimes \psi\big).$$

- Fix a decomposition of the double coset $\mathrm{Iw}_p \left(\begin{smallmatrix} p^{-1} & 0\\ 0 & 1\end{smallmatrix}\right)\mathrm{Iw}_p = \coprod_{j=0}^{p-1} v_j \mathrm{Iw}_p$ with $v_j = \left(\begin{smallmatrix} p^{-1} & 0\\ j & 1\end{smallmatrix}\right)$. Define the $U_p$-operator on the spaces $\mathrm{S}_k^{\mathrm{Iw}}(\psi)$ by the formula

$$U_p(\varphi)(x) = \sum_{j=0}^{p-1} \varphi(xv_j)|_{v_j^{-1}} \quad \text{for all } x \in \widetilde{\mathrm{H}} \text{ and } \varphi \in \mathrm{S}_k^{\mathrm{Iw}}(\psi).$$

# Abstract classical forms III

- For $k \geq 2$ satisfying $k \equiv 2 + a + 2s \bmod (p-1)$, the character $\psi = \varepsilon \cdot (1 \times \omega^{2-k})$ is of the form $\psi = \varepsilon_1 \times \varepsilon_1$ with $\varepsilon_1 = \omega^{-s+b}$. Define the space of abstract classical forms with $\mathrm{K}_p$-level of weight $k$ and central character $\varepsilon_1$ to be

$$\mathrm{S}_k^{\mathrm{ur}}(\varepsilon_1) := \mathrm{Hom}_{\mathcal{O}[\mathrm{K}_p]}\big(\widetilde{\mathrm{H}},\ \mathcal{O}[z]^{\leq k-2} \otimes \varepsilon_1 \circ \det\big).$$

| classical modular forms | abstract automorphic forms |
|---|---|
| $\mathrm{S}_k(\Gamma_1(N), \bar{\mathbb{Q}}_p)_{\bar{r}}$ | $\mathrm{S}_k^{\mathrm{ur}}(\varepsilon_1) = \mathrm{Hom}_{\mathcal{O}[\mathrm{K}_p]}\big(\widetilde{\mathrm{H}},\ \mathcal{O}[z]^{\leq k-2} \otimes \varepsilon_1 \circ \det\big)$ |
| $\mathrm{S}_k(\Gamma_1(N) \cap \Gamma_0(p), \bar{\mathbb{Q}}_p)_{\bar{r}}$ | $\mathrm{S}_k^{\mathrm{Iw}}(\psi) = \mathrm{Hom}_{\mathcal{O}[\mathrm{Iw}_p]}\big(\widetilde{\mathrm{H}},\ \mathcal{O}[z]^{\leq k-2} \otimes \psi\big)$ |
| $\mathrm{S}_k^{\dagger}(\Gamma_1(N) \cap \Gamma_0(p), \bar{\mathbb{Q}}_p)_{\bar{r}}$ | ? |

# Abstract automorphic forms I

- Let $\mathcal{C}^0(\mathbb{Z}_p;\mathcal{O}[\![w]\!])$ denotes the space of continuous functions on $\mathbb{Z}_p$ with values in $\mathcal{O}[\![w]\!]$.
- We can define right actions of the monoid

$$\mathbf{M}_1=\left\{\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in \mathrm{M}_2(\mathbb{Z}_p);\ p|\gamma,\ p\nmid\delta,\ \alpha\delta-\beta\gamma\neq 0\right\}.$$

  on $\mathcal{C}^0(\mathbb{Z}_p;\mathcal{O}[\![w]\!])$ and the Tate algebra $\mathcal{O}\langle w/p\rangle\langle z\rangle$. When endowed with such an action, these spaces will be denoted by $\mathcal{C}^0(\mathbb{Z}_p;\mathcal{O}[\![w]\!]^{(\varepsilon)})$ and $\mathcal{O}\langle w/p\rangle^{(\varepsilon)}\langle z\rangle$.
- We define the space of abstract $p$-adic forms and the space of family of abstract overconvergent forms to be

$$\begin{aligned}\mathrm{S}^{(\varepsilon)}_{p\text{-adic}} &:= \mathrm{Hom}_{\mathcal{O}[\mathrm{Iw}_p]}\big(\widetilde{\mathrm{H}},\ \mathcal{C}^0(\mathbb{Z}_p;\mathcal{O}[\![w]\!]^{(\varepsilon)})\big),\\ \mathrm{S}^{\dagger,(\varepsilon)} &:= \mathrm{Hom}_{\mathcal{O}[\mathrm{Iw}_p]}\big(\widetilde{\mathrm{H}},\ \mathcal{O}\langle w/p\rangle^{(\varepsilon)}\langle z\rangle\big).\end{aligned}$$

  We can define $U_p$-operator on these spaces by the same formula as on $\mathrm{S}_k^{\mathrm{Iw}}(\psi)$.
- For any $k\geq 2$, we have an $\mathbf{M}_1$-equivariant inclusion

$$\mathcal{O}[z]^{\leq k-2}\otimes\psi\hookrightarrow \mathcal{O}\langle w/p\rangle^{(\varepsilon)}\langle z\rangle\otimes_{\mathcal{O}\langle w/p\rangle,w\mapsto w_k}\mathcal{O}$$

  and a similar inclusion for $\mathcal{C}^0(\mathbb{Z}_p;\mathcal{O}[\![w]\!]^{(\varepsilon)})$.

# Abstract automorphic forms II

- For every integer $k$, we define the space of abstract overconvergent forms of weight $k$ and character $\psi = \varepsilon \cdot (1 \times \omega^{2-k})$ by evaluating $\mathrm{S}^{\dagger,(\varepsilon)}$ at $w = w_k := \exp((k-2)p) - 1$:

$$\mathrm{S}_k^{\dagger}(\psi) := \mathrm{S}^{\dagger,(\varepsilon)} \otimes_{\mathcal{O}\langle w/p\rangle, w \mapsto w_k} \mathcal{O}.$$

- When $k \geq 2$, we have a $U_p$-equivariant inclusion $\mathrm{S}_k^{\mathrm{Iw}}(\psi) \hookrightarrow \mathrm{S}_k^{\dagger}(\psi)$.
- The characteristic power series of the $U_p$-action on $\mathrm{S}^{\dagger,(\varepsilon)}$ and $\mathrm{S}_{p\text{-adic}}^{(\varepsilon)}$ are well-defined and are equal; we denote it by

$$C^{(\varepsilon)}(w,t) = \sum_{n \geq 0} c_n^{(\varepsilon)}(w) t^n \in \mathcal{O}[\![w,t]\!].$$

- For any integer $k$, the evaluation $C^{(\varepsilon)}(w_k, t) \in \mathcal{O}[\![t]\!]$ is the characteristic power series of $U_p$-operator on $\mathrm{S}_k^{\dagger}(\psi)$.

# A digestion of notations

| modular forms | abstract automorphic forms |
|---|---|
| $\mathrm{S}_k(\Gamma_1(N), \bar{\mathbb{Q}}_p)_{\bar{r}}$ | $\mathrm{S}_k^{\mathrm{ur}}(\varepsilon_1) = \mathrm{Hom}_{\mathcal{O}[\mathrm{K}_p]}\big(\widetilde{\mathrm{H}}, \mathcal{O}[z]^{\leq k-2} \otimes \varepsilon_1 \circ \det\big)$ |
| $\mathrm{S}_k(\Gamma_1(N) \cap \Gamma_0(p), \bar{\mathbb{Q}}_p)_{\bar{r}}$ | $\mathrm{S}_k^{\mathrm{Iw}}(\psi) = \mathrm{Hom}_{\mathcal{O}[\mathrm{Iw}_p]}\big(\widetilde{\mathrm{H}}, \mathcal{O}[z]^{\leq k-2} \otimes \psi\big)$ |
| $\mathrm{S}_k^{\dagger}(\Gamma_1(N) \cap \Gamma_0(p), \bar{\mathbb{Q}}_p)_{\bar{r}}$ | $\mathrm{S}_k^{\dagger}(\psi) = \mathrm{S}^{\dagger,(\varepsilon)} \otimes_{\mathcal{O}\langle w/p\rangle, w \mapsto w_k} \mathcal{O}$ |
| family of overconvergent modular forms | $\mathrm{S}^{\dagger,(\varepsilon)} = \mathrm{Hom}_{\mathcal{O}[\mathrm{Iw}_p]}\big(\widetilde{\mathrm{H}}, \mathcal{O}\langle w/p\rangle^{(\varepsilon)}\langle z\rangle\big)$ |
| family of $p$-adic modular forms | $\mathrm{S}_{p\text{-adic}}^{(\varepsilon)} = \mathrm{Hom}_{\mathcal{O}[\mathrm{Iw}_p]}\big(\widetilde{\mathrm{H}}, \mathcal{C}^0(\mathbb{Z}_p; \mathcal{O}[\![w]\!]^{(\varepsilon)})\big)$ |

# Local ghost series I

- From now on, we assume that $\tilde{\mathrm{H}}$ is primitive.
- For every integer $k \geq 2$, we set

$$d_k^{\mathrm{Iw}}(\psi) := \mathrm{rank}_{\mathcal{O}} \mathrm{S}_k^{\mathrm{Iw}}(\psi).$$

For $k \equiv 2 + a + 2s \bmod (p-1)$, we set

$$d_k^{\mathrm{ur}}(\varepsilon_1) := \mathrm{rank}_{\mathcal{O}} \mathrm{S}_k^{\mathrm{ur}}(\varepsilon_1) \quad \text{and} \quad d_k^{\mathrm{new}}(\varepsilon_1) := d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) - 2d_k^{\mathrm{ur}}(\varepsilon_1)$$

We know that $d_k^{\mathrm{new}}(\varepsilon_1)$ is always an even integer.

- For $k \equiv 2 + a + 2s \bmod (p-1)$, we define a sequence $(m_n^{(\varepsilon)}(k))_{n \geq 1}$ of integers as

$$\underbrace{0, \ldots, 0}_{d_k^{\mathrm{ur}}(\varepsilon_1)}, 1, 2, 3, \ldots, \tfrac{1}{2}d_k^{\mathrm{new}}(\varepsilon_1)-1, \tfrac{1}{2}d_k^{\mathrm{new}}(\varepsilon_1), \tfrac{1}{2}d_k^{\mathrm{new}}(\varepsilon_1)-1, \ldots, 3, 2, 1, 0, 0, \ldots,$$

# Local ghost series II

- Let $\bar{\rho} = \begin{pmatrix} \omega_1^{a+b+1} & * \neq 0 \\ 0 & \omega_1^b \end{pmatrix} : \mathrm{I}_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be a reducible, nonsplit, and generic residual representation with $a \in \{1, \dots, p-4\}$ and $b \in \{0, \dots, p-2\}$. Let $\widetilde{\mathrm{H}}$ be a primitive $\mathcal{O}[\![\mathrm{K}_p]\!]$-projective augmented module of type $\bar{\rho}$.
- Fix a character $\varepsilon = \omega^{-s+b} \times \omega^{a+s+b} : \Delta^2 \to \mathbb{Z}_p^\times$ as before. Define the *ghost series* of type $\bar{\rho}$ to be the formal power series

$$G^{(\varepsilon)}(w,t) = G_{\bar{\rho}}^{(\varepsilon)}(w,t) = 1 + \sum_{n=1}^{\infty} g_n^{(\varepsilon)}(w) t^n \in \mathbb{Z}_p[w][\![t]\!] \subset \mathcal{O}[\![w,t]\!],$$

where each coefficient $g_n^{(\varepsilon)}(w)$ is a product

$$g_n^{(\varepsilon)}(w) = \prod_{\substack{k \geq 2 \\ k \equiv 2+a+2s \bmod p-1}} (w - w_k)^{m_n^{(\varepsilon)}(k)} \in \mathbb{Z}_p[w]$$

- We call $w_k = \exp(p(k-2)) - 1$ a ghost zero of $g_n^{(\varepsilon)}(w)$ and the integer $m_n^{(\varepsilon)}(k)$ its ghost multiplicity.

# Local ghost conjecture

## Local ghost conjecture

Let $\bar{\rho} = \begin{pmatrix} \omega_1^{a+b+1} & * \neq 0 \\ 0 & \omega_1^b \end{pmatrix} : \mathrm{I}_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be a reducible, nonsplit, and generic residual representation with $a \in \{1, \ldots, p-4\}$ and $b \in \{0, \ldots, p-2\}$. Let $\widetilde{\mathrm{H}}$ be a primitive $\mathcal{O}[\![\mathrm{K}_p]\!]$-projective augmented module of type $\bar{\rho}$. Let $C^{(\varepsilon)}(w,t)$ be the characteristic power series of $U_p$-operator and $G^{(\varepsilon)}(w,t)$ be the ghost series for $\widetilde{\mathrm{H}}$ defined before. Then for every $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, we have $\mathrm{NP}(G^{(\varepsilon)}(w_\star, -)) = \mathrm{NP}(C^{(\varepsilon)}(w_\star, -))$.

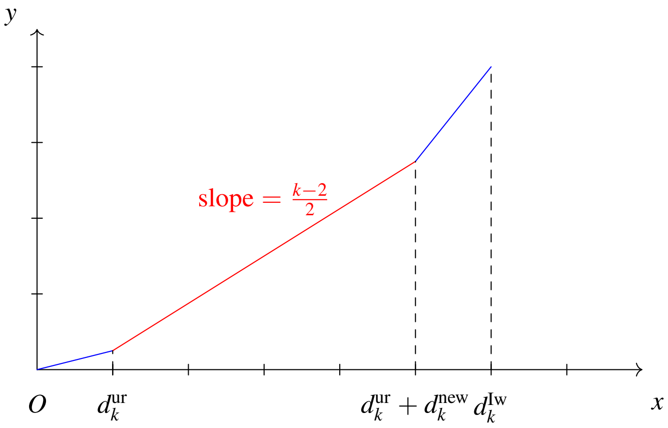
## Theorem (Liu-Truong-Xiao-Z.)

*The local ghost conjecture holds when* $p \geq 11$ *and* $2 \leq a \leq p-5$.

From now on, we assume that $b = 0$ and the matrix $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ acts trivially on $\tilde{\mathrm{H}}$. We will fix a character $\varepsilon : \Delta^2 \to \mathbb{Z}_p^\times$ and suppress it from notations.

# Intuition on ghost zeroes

- Let $f \in \mathrm{S}_k(\Gamma_0(Np), \bar{\mathbb{Q}}_p)$ be a normalized Hecke eigenform. We have the following facts about the slopes of classical modular forms:
  1. When $f$ is new at $p$, we have $a_p^2 = p^{k-2}$ and hence $f$ has slope $\frac{k-2}{2}$;
  2. The other $U_p$-eigenvalues in $\mathrm{S}_k(\Gamma_0(Np), \bar{\mathbb{Q}}_p)$ come in pairs: for a normalized eigenform $g \in \mathrm{S}_k(\Gamma_0(N), \bar{\mathbb{Q}}_p)$ with $T_p$-eigenvalue $a_p$, it has two $p$-stabilizations $f_\alpha(z) = f(z) - \beta f(pz), f_\beta(z) = f(z) - \alpha f(pz)$ in $\mathrm{S}_k(\Gamma_0(Np), \bar{\mathbb{Q}}_p)$ with $U_p$-eigenvalues $\alpha$ and $\beta$, where $\alpha, \beta$ are the roots of $X^2 - a_pX + p^{k-1}$. So the slopes of these two $p$-old forms sum to $k-1$. In particular, $v_p(a_p)$ can usually be read off from $v_p(\alpha)$ and $v_p(\beta)$.
- The slopes of $p$-oldforms behave very different from those of $p$-newforms. Let $f \in \mathrm{S}_k(\Gamma_0(N), \bar{\mathbb{Q}}_p)_{\bar{r}}$ with $\bar{r}|_{\mathrm{I}_{\mathbb{Q}_p}} \cong \bar{\rho}$ be a eigenform with $T_p$-eigenvalue $a_p$. Berger-Li-Zhu proved that $v_p(a_p) \leq \lfloor \frac{k-2}{p-1} \rfloor$ (conjecturally this can be strengthened to $\lfloor \frac{k-2}{p+1} \rfloor$).
- The Newton polygon of $U_p$-operator on $\mathrm{S}_k(\Gamma_0(Np), \bar{\mathbb{Q}}_p)_{\bar{r}}$ should have a line segment of length $d_k^{\mathrm{new}}$ and slope $\frac{k-2}{2}$. In particular, the point $(i, v_p(c_i(w_k)))$ is not a vertex of $\mathrm{NP}(C(w_k, -))$, for $i = d_k^{\mathrm{ur}} + 1, \dots, d_k^{\mathrm{ur}} + d_k^{\mathrm{new}} - 1$. These integers $i$'s are exactly those integers with the property $g_i(w_k) = 0$.




Newton polygon of $C(w_k, -)$

# Power basis of abstract automorphic forms

- $\widetilde{\mathrm{H}}$ is free of rank 2 as an $\mathcal{O}[\![\mathrm{Iw}_{p,1}]\!]$-module. There exists a basis $\{e_1, e_2\}$ of $\widetilde{\mathrm{H}}$ such that $\left(\begin{smallmatrix}\Delta & 0\\ 0 & \Delta\end{smallmatrix}\right) \subset \mathrm{Iw}_p$ acts on them via the characters $1 \times \omega^a$ and $\omega^a \times 1$ and $e_i\left(\begin{smallmatrix}0 & 1\\ p & 0\end{smallmatrix}\right) = e_{3-i}$ for $i = 1, 2$.
- For $i = 1, 2$ and $h(z) \in \mathcal{O}\langle w/p\rangle\langle z\rangle$, we use $e_i^* h(z)$ to denote the element in $\mathrm{S}^{\dagger,(\varepsilon)} = \mathrm{Hom}_{\mathcal{O}[\mathrm{Iw}_p]}\big(\widetilde{\mathrm{H}}, \mathcal{O}\langle w/p\rangle\langle z\rangle\big)$ which sends $e_i$ to $h(z)$ and $e_{3-i}$ to 0.
- We have a basis of the space $\mathrm{S}^{\dagger,(\varepsilon)}$ as well as the space $\mathrm{S}_k^{\dagger}(\psi)$ for every $k \in \mathbb{Z}$:

$$\mathbf{B} = \mathbf{B}^{(\varepsilon)} := \big\{e_1^* z^s, e_1^* z^{p-1+s}, \dots; e_2^* z^{\{a+s\}}, e_2^* z^{p-1+\{a+s\}}, \dots\big\}.$$

  It is called the *power basis* of the above spaces. When $k \geq 2$, the subsequence $\mathbf{B}_k$ consisting of terms whose power in $z$ is less than or equal to $k-2$ forms a basis of $\mathrm{S}_k^{\mathrm{Iw}}\big(\varepsilon \cdot (1 \times \omega^{2-k})\big)$. We order the elements in $\mathbf{B}$ with increasing degrees on $z$.
- The rank $d_k^{\mathrm{Iw}}(\psi) := \mathrm{rank}_{\mathcal{O}} \mathrm{S}_k^{\mathrm{Iw}}(\psi)$ can be computed from the above expression for all $k \geq 2$. For $k \equiv 2 + a + 2s \bmod (p-1)$, the rank $d_k^{\mathrm{ur}}(\varepsilon_1) = \mathrm{rank}_{\mathcal{O}} \mathrm{S}_k^{\mathrm{ur}}(\varepsilon_1)$ can be computed using representation theory of $\mathrm{GL}_2(\mathbb{F}_p)$.

# Atkin-Lehner involution

- For $k \equiv 2 + a + 2s \bmod (p-1)$, write $\tilde{\varepsilon}_1 = \varepsilon_1 \times \varepsilon_1 = \varepsilon \cdot (1 \times \omega^{2-k})$. We have a well-defined Atkin-Lehner involution on the space of classical automorphic forms $\mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) = \mathrm{Hom}_{\mathcal{O}[\mathrm{Iw}_p]}\big(\widetilde{\mathrm{H}},\ \mathcal{O}[z]^{\leq k-2} \otimes \tilde{\varepsilon}_1\big)$:

$$\begin{aligned} \mathrm{AL}_{(k,\tilde{\varepsilon}_1)} : \mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) &\longrightarrow \mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) \\ \varphi &\longmapsto \Big( \mathrm{AL}_{(k,\tilde{\varepsilon}_1)}(\varphi) : x \mapsto \varphi\Big(x\big(\begin{smallmatrix} 0 & p^{-1} \\ 1 & 0 \end{smallmatrix}\big)\Big)\Big|_{\big(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\big)} \Big). \end{aligned}$$

- Let $\mathbf{B}_k = \{\mathbf{e}_1, \dots, \mathbf{e}_{d_k^{\mathrm{Iw}}}\}$ be the power basis of $\mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$. The degree of an element $\mathbf{e}_i$ is the exponent of $z$ in $\mathbf{e}_i$.
- Let $\mathrm{L}_k \in \mathrm{M}_{d_k^{\mathrm{Iw}}}(\mathcal{O})$ be the matrix of the map $\mathrm{AL}_{(k,\tilde{\varepsilon}_1)}$ under the basis $\mathbf{B}_k$. Then

$$\mathrm{L}_k = \begin{pmatrix} 0 & \cdots & 0 & p^{\deg \mathbf{e}_1} \\ 0 & \cdots & p^{\deg \mathbf{e}_2} & 0 \\ 0 & \cdots & 0 & 0 \\ p^{\deg \mathbf{e}_{d_k^{\mathrm{Iw}}}} & \cdots & 0 & 0 \end{pmatrix}$$

# $p$-stabilization of abstract classical forms I

We define the following four maps

$$\mathrm{S}_k^{\mathrm{ur}}(\varepsilon_1) = \mathrm{Hom}_{\mathcal{O}[\![\mathrm{K}_p]\!]}\big(\widetilde{\mathrm{H}},\ \mathcal{O}[z]^{\leq k-2} \otimes \tilde{\varepsilon}_1\big)$$

$$\iota_1 \downarrow \quad \iota_2 \downarrow \qquad \uparrow \mathrm{proj}_2 \quad \uparrow \mathrm{proj}_1$$

$$\mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) = \mathrm{Hom}_{\mathcal{O}[\![\mathrm{Iw}_p]\!]}\big(\widetilde{\mathrm{H}},\ \mathcal{O}[z]^{\leq k-2} \otimes \tilde{\varepsilon}_1\big)$$

given by, for $\phi \in \mathrm{S}_k^{\mathrm{ur}}(\varepsilon_1)$, $\varphi \in \mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$, and $x \in \widetilde{\mathrm{H}}$,

$$\iota_1(\phi) = \phi, \quad \mathrm{proj}_1(\varphi)(x) = \sum_{j=0,\dots,p-1,\star} \varphi(xu_j)\big|_{u_j^{-1}},$$

$$\iota_2(\phi)(x) = \mathrm{AL}_{(k,\tilde{\varepsilon}_1)}(\iota_1(\phi))(x), \mathrm{proj}_2(\varphi)(x) = \mathrm{proj}_1(\mathrm{AL}_{(k,\tilde{\varepsilon}_1)}(\varphi))(x).$$

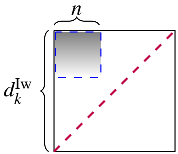
Here $u_j = \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix}$ for $j = 0, \dots, p-1$ and $u_\star = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ form a set of coset representatives of $\mathrm{Iw}_p \backslash \mathrm{K}_p$.

# $p$-stabilization of abstract classical forms II

- For classical modular forms, we have the stabilization maps $\mathrm{S}_k(\Gamma_0(N)) \to \mathrm{S}_k(\Gamma_0(Np)), f(z) \mapsto f(z)$ and $f(z) \mapsto f(pz)$. The two embeddings $\iota_1, \iota_2 : \mathrm{S}_k^{\mathrm{ur}}(\varepsilon_1) \to \mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$ are their analogues in our abstract context.
- The KEY equality is:

$$U_p = \underbrace{\iota_2 \circ \mathrm{proj}_1}_{\mathrm{rank} \leq d_k^{\mathrm{ur}}} - \underbrace{\mathrm{AL}_{(k,\tilde{\varepsilon}_1)}}_{\text{antidiagonal}} : \mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) \to \mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1).$$

The matrix $\mathrm{U}_k^{\mathrm{Iw}}$ of the $U_p$-operator on $\mathrm{S}_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1)$ with respect to the power basis $\mathbf{B}_k$ is the sum of the anti-diagonal matrix $-\mathrm{L}_k$ and a matrix with rank $\leq d_k^{\mathrm{ur}}$.




$\Longrightarrow$ Rank of upper left $n \times n$ submatrix of $\mathrm{U}_k^{\mathrm{Iw}}$ is $\leq n - m_n(k)$

# $p$-stabilization of abstract classical forms III

- Let $\mathrm{U}^\dagger \in \mathrm{M}_\infty(\mathcal{O}\langle w/p\rangle)$ be the matrix of $U_p$-operator on $\mathrm{S}^\dagger$ with respect to the power basis $\mathbf{B}$. The upper left $d_k^{\mathrm{Iw}} \times d_k^{\mathrm{Iw}}$ submatrix of $\mathrm{U}^\dagger|_{w=w_k}$ is $\mathrm{U}_k^{\mathrm{Iw}}$.
- The determinant of upper left $n \times n$ submatrix of $\mathrm{U}^\dagger$ has zero of multiplicity $m_n(k)$ at $w = w_k$.
- The matrix $\mathrm{U}^\dagger \in \mathrm{M}_\infty(\mathcal{O}\langle w/p\rangle)$ has a (row) Hodge bound so that one should expect
$$c_n(w) \approx \det\big(\text{ upper-left } n \times n \text{ minor of } \mathrm{U}^\dagger\big).$$
- In summary it is natural to expect $c_n(w) \approx g_n(w)$.

# Properties of ghost series

- When studying the local conjecture, it is natural and necessary to consider the following equation:
  given $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$ and a positive integer $n$, how can we determine whether the point $(n, v_p(g_n(w_\star)))$ is a vertex of the Newton polygon $\mathrm{NP}(G(w_\star, -))$?
- When $k \equiv 2 + a + 2s \bmod (p-1)$ and $m_n(k) \neq 0$, it is clear that $(n, v_p(g_n(w_k)))$ is not a vertex. The intuition is that when $w_\star$ is sufficiently $p$-adically close to such a $w_k$, then $(n, v_p(g_n(w_\star)))$ cannot be a vertex either.
- Fix $k \equiv 2 + a + 2s \bmod (p-1)$. Write $g_{n,\hat{k}}(w) := g_n(w)/(w - w_k)^{m_n(k)}$. Set

$$\Delta'_{k,\ell} = \Delta^{(\varepsilon)\prime}_{k,\ell} := v_p\big(g_{\frac{1}{2}d_k^{\mathrm{Iw}}+\ell,\hat{k}}(w_k)\big) - \tfrac{k-2}{2}\ell, \quad \text{for } \ell = -\tfrac{1}{2}d_k^{\mathrm{new}}, \dots, \tfrac{1}{2}d_k^{\mathrm{new}}.$$

  Let $\underline{\Delta}_k = \underline{\Delta}_k^{(\varepsilon)}$ denote the convex hull of the points $(\ell, \Delta'_{k,\ell})$ for $\ell = -\frac{1}{2}d_k^{\mathrm{new}}, \dots, \frac{1}{2}d_k^{\mathrm{new}}$, and let $(\ell, \Delta_{k,\ell})$ denote the corresponding points on $\underline{\Delta}_k$.
- We have the ghost duality equalities

$$\Delta'_{k,\ell} = \Delta'_{k,-\ell} \quad \text{and} \quad \Delta_{k,\ell} = \Delta_{k,-\ell} \quad \text{for all } \ell = -\tfrac{1}{2}d_k^{\mathrm{new}}, \dots, \tfrac{1}{2}d_k^{\mathrm{new}}.$$

# near Steinberg range I

- For $k \equiv 2 + a + 2s \bmod (p-1)$ and $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, let $L_{w_\star,k}$ denote the largest number (if it exists) in $\{1, \ldots, \frac{1}{2}d_k^{\mathrm{new}}\}$ such that

$$v_p(w_\star - w_k) \geq \Delta_{k,L_{w_\star,k}} - \Delta_{k,L_{w_\star,k}-1}.$$

  When such $L_{w_\star,k}$ exists, we call the interval

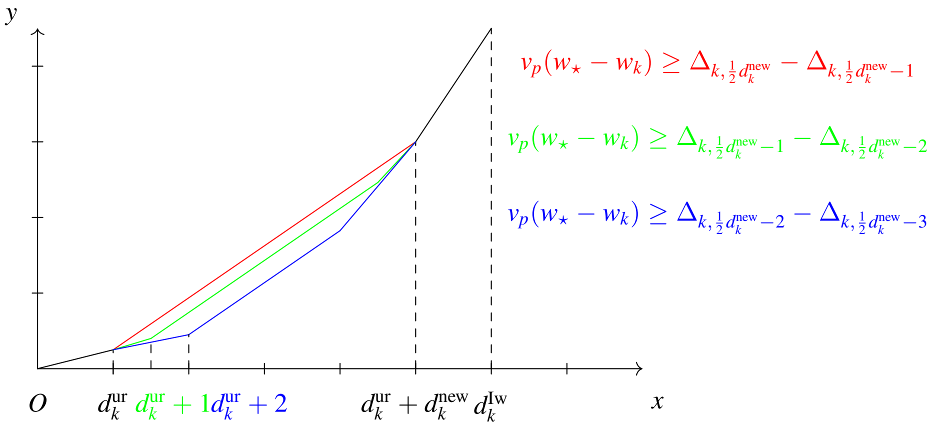
$$\mathrm{nS}_{w_\star,k} := \left(\tfrac{1}{2}d_k^{\mathrm{Iw}} - L_{w_\star,k},\ \tfrac{1}{2}d_k^{\mathrm{Iw}} + L_{w_\star,k}\right)$$

  the near Steinberg range for $(w_\star, k)$.
- For a positive integer $n$, we say that $(w_\star, n)$ is *near-Steinberg* if $n$ belongs to the near-Steinberg range $\mathrm{nS}_{w_\star,k}$ for some $k$.

### Proposition

*For every $n \in \mathbb{N}$, the point $\big(n, v_p(g_n(w_\star))\big)$ is a vertex of $\mathrm{NP}(G(w_\star, -))$ if and only if $(w_\star, n)$ is not near-Steinberg.*

# near Steinberg range II



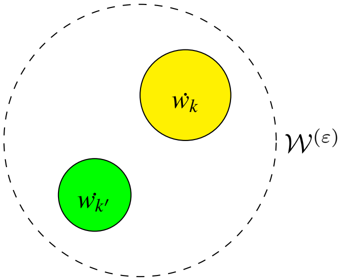

Newton polygon of $G(w_\star, -)$ when $w_\star$ leaves away from $w_k$

# near Steinberg range III

For a fixed $n \in \mathbb{N}$, the set of elements $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$ for which $\big(n, v_p(g_n(w_\star))\big)$ is a vertex of $\mathrm{NP}\big(G(w_\star, -)\big)$ form a quasi-Stein open subset of the weight disk $\mathcal{W}^{(\varepsilon)}$

$$\mathrm{Vtx}_n^{(\varepsilon)} := \mathcal{W}^{(\varepsilon)} \setminus \bigcup_k \Big\{ w_\star \in \mathfrak{m}_{\mathbb{C}_p} \;\Big|\; v_p(w_\star - w_k) \geq \Delta_{k, |\frac{1}{2} d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) - n| + 1} - \Delta_{k, |\frac{1}{2} d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) - n|} \Big\},$$

where the union is taken over all $k = k_\varepsilon + (p-1)k_\bullet$ with $k_\bullet \in \mathbb{Z}$ such that $n \in \big(d_k^{\mathrm{ur}}(\varepsilon_1), d_k^{\mathrm{Iw}}(\tilde{\varepsilon}_1) - d_k^{\mathrm{ur}}(\varepsilon_1)\big)$, i.e. $w_k$ is a ghost zero of $g_n(w)$.




Shape of the space $\mathrm{Vtx}_n^{(\varepsilon)}$

# Lagrange interpolation I

- The key tool to compare the Newton polygon of the characteristic power series

$$C(w,t)=\sum_{n\geq 0} c_n(w)t^n \in \mathcal{O}[\![w,t]\!]$$

and the ghost series

$$G(w,t)=\sum_{n\geq 0} g_n(w)t^n \in \mathbb{Z}[w][\![t]\!]$$

is Lagrange interpolation formula.

- Fix a positive integer $n$ and a positive integer $k\equiv 2+a+2s \bmod (p-1)$ with $m_n(k)>0$, let $g_{n,\hat{k}}(w)=g_n(w)/(w-w_k)^{m_n(k)}$ as before. We consider the formal expansion in $E[\![w-w_k]\!]$:

$$\frac{c_n(w)}{g_{n,\hat{k}}(w)}=\sum_{i\geq 0} A_{k,i}^{(n)}(w-w_k)^i,$$

and let $A_k^{(n)}(w)=\sum_{i=0}^{m_n(k)-1} A_{k,i}^{(n)}(w-w_k)^i$ be its truncation up to the term $(w-w_k)^{m_n(k)-1}$.

# Lagrange interpolation II

## Langrange interpolation formula

There exists $h_n(w) \in E\langle w/p\rangle$, such that we have an equality in $E\langle w/p\rangle$:

$$c_n(w) = \sum_{\substack{k \equiv 2+a+2s \bmod (p-1) \\ m_n(k)\neq 0}} \left(A_k^{(n)}(w) \cdot g_{n,\hat{k}}(w)\right) + h_n(w) \cdot g_n(w).$$

It is called the Lagrange interpolation of $c_n(w)$ along $g_n(w)$.

The main technical result of our proof is the following:

## Proposition

*To prove the local ghost conjecture, it suffices to prove that for any positive integer $n$ and every ghost zero $w_k$ of $g_n(w)$, we have*

$$v_p(A_{k,i}^{(n)}) \geq \Delta_{k,\frac{1}{2}d_k^{\mathrm{new}}-i} - \Delta'_{k,\frac{1}{2}d_k^{\mathrm{new}}-m_n(k)} \quad \textit{for} \quad i = 0, 1, \ldots, m_n(k) - 1.$$

# Lagrange interpolation III

$$c_n(w) = \sum_{\substack{k \equiv 2+a+2s \bmod (p-1) \\ m_n(k) \neq 0}} \underbrace{\left(A_k^{(n)}(w) \cdot g_{n,\hat{k}}(w)\right)}_{\textit{error term}} + \underbrace{h_n(w) \cdot g_n(w)}_{\textit{main term}}.$$

- The estimate $v_p(A_{k,i}^{(n)}) \geq \Delta_{k,\frac{1}{2}d_k^{\mathrm{new}}-i} - \Delta'_{k,\frac{1}{2}d_k^{\mathrm{new}}-m_n(k)}$ implies $v_p(A_{k,i}^{(n)}) > 0$. Since $c_n(w) \in \mathcal{O}[\![w]\!]$ and $g_n(w) \in \mathcal{O}[w]$, we have $h_n(w) \in \mathcal{O}[\![w]\!]$. We can actually show $h_n(w)$ is a unit in $\mathcal{O}[\![w]\!]$. Hence $v_p(h_n(w_\star)) = 1$ for all $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$.
- Assuming the estimate in the proposition, we can prove the following facts for all $w_\star \in \mathfrak{m}_{\mathbb{C}_p}$, every positive integer $n$ and every ghost zero $w_k$ of $g_n(w)$:
  1. The point $\left(n, v_p\left(A_k^{(n)}(w_\star) g_{n,\hat{k}}(w_\star)\right)\right)$ lies on or above the Newton polygon $\mathrm{NP}(G(w_\star, -))$; and
  2. moreover if $\left(n, v_p(g_n(w_\star))\right)$ is a vertex of $\mathrm{NP}(G(w_\star, -))$, then $v_p\left(A_k^{(n)}(w_\star) g_{n,\hat{k}}(w_\star)\right) > v_p\left(g_n(w_\star)\right)$.

  The first statement implies that $\mathrm{NP}(C(w_\star, -))$ always lies on or above $\mathrm{NP}(G(w_\star, -))$, while the second statement says that these two polygons meet at all vertices of $\mathrm{NP}(G(w_\star, -))$. So the local ghost conjecture follows!

# Ingredients to prove the technical proposition

- Let $\underline{\zeta} = \{\zeta_1, \ldots, \zeta_n\}$ and $\underline{\xi} = \{\xi_1, \ldots, \xi_n\}$ be two sets of $n$ positive integers. We apply the Lagrange interpolation to $\det(\mathrm{U}^\dagger(\underline{\zeta} \times \underline{\xi}))$ along $g_n(w)$ and we have

$$\det\left(\mathrm{U}^\dagger(\underline{\zeta} \times \underline{\xi})\right) = \sum_{\substack{k \equiv k_\varepsilon \mod (p-1) \\ m_n(k) \neq 0}} \left(A_k^{(\underline{\zeta} \times \underline{\xi})}(w) \cdot g_{n,\hat{k}}(w)\right) + h_{\underline{\zeta} \times \underline{\xi}}(w) \cdot g_n(w).$$

  We prove a similar estimate:

$$v_p(A_{k,i}^{(\underline{\zeta} \times \underline{\xi})}) \geq \Delta_{k, \frac{1}{2} d_k^{\mathrm{new}} - i} - \Delta'_{k, \frac{1}{2} d_k^{\mathrm{new}} - m_n(k)} + \tfrac{1}{2}\left(\deg(\underline{\zeta}) - \deg(\underline{\xi})\right).$$

- A decomposition $\mathrm{U}^\dagger = \mathrm{T}_k - \mathrm{L}_k$ which allows us to express $\det\left(\mathrm{U}^\dagger(\underline{\zeta} \times \underline{\xi})\right)$ in term of determinants of smaller minors;
- A refined halo bound on the infinite matrix $\mathrm{U}^\dagger \in \mathrm{M}_\infty(\mathcal{O}\langle w/p \rangle)$.

Thank you!