Classical Function Theory in Modern Mathematics

ICMS, Bayes Centre, Edinburgh

1–5 July 2024

The programme is subject to change. All times are British Summer Time (BST).

MONDAY 1 July 2024: Differential Equations	
09:15–09:30	Registration and refreshments
09:30–09:45	Welcome and housekeeping
09:45–10:45	**Alan Sokal** (University College London) *Some conjectures concerning the zeros of the deformed exponential function*
10:45–11:15	Coffee break
11:15–12:15	**Eveliina Peltola** (Aalto University and University of Bonn) *On geometric properties of conformally invariant curves*
12:15–14:30	Lunch
14:30–15:00	**Lakshmi Priya** (Tel Aviv University) *Almost sharp lower bound for the nodal volume of harmonic functions*
15:00–16:00	**Masaki Tsukamoto** (Kyoto University) *Rate distortion dimension of random Brody curves*
16:00–16:30	Coffee break
16:30–17:30	**Patrick Ng** (University of Hong Kong) *Brody holomorphic curves on the degree six Fermat surface*
17:30–19:00	Drinks reception and poster session

TUESDAY 2 July 2024: Complex Dynamics	
09:30–10:30	**Walter Bergweiler** (University of Kiel) *The escaping set*
10:30–11:00	Coffee break
11:00–12:00	**Gwyneth Stallard** (Open University) *Shrinking targets and recurrent behaviour for forward compositions of inner functions*
12:00–14:30	Lunch
14:30–15:00	**James Waterman** (Stony Brook University) *A counterexample to Eremenko's conjecture*
15:00–16:00	**Melkana Brakalova-Trevithick** (Fordham University) *When the circular dilatation at a point equals one*
16:00–16:30	Coffee break
16:30–17:30	**Mitsuhiro Shishikura** (Kyoto University) *Reconstructing the dynamics via dynamical charts from a sequence of near-parabolic or cylinder renormalization*
18:00–19:00	Public lecture and recital (Room G.03, ground floor)

WEDNESDAY 3 July 2024: Geometry	
09:30–10:30	**Mario Bonk** (UCLA) *Lamé functions and special elliptic integrals*
10:30–11:00	Coffee break
11:00–11:30	**Ilmari Kangasniemi** (University of Cincinnati) *On the theory of quasiregular values*
11:30–12:30	**Yilin Wang** (IHES) *Jordan curves with the piecewise geodesic property*
12:30 onwards	Lunch and free afternoon

THURSDAY 4 July 2024: Meromorphic Functions	
09:30–10:30	**Mikhail Sodin** (Tel Aviv University) *Equivariant Weierstrass theorem*
10:30–11:00	Coffee break
11:00–12:00	**Aimo Hinkkanen** (University of Illinois Urbana-Champaign) *Moments and positivity*
12:00–14:30	Lunch
14:30–15:00	**Adi Glucksam** (Northwestern University) *Multi-fractal spectrum of planar harmonic measure*
15:00–16:00	**Oleg Ivrii** (Tel Aviv University) *Critical values of inner functions*
16:00–16:30	Coffee break
16:30–17:30	**Problem session** *Chair: Phil Rippon*
19:00 onwards	Workshop dinner (St. Trinnean's and Nelson Rooms)

FRIDAY 5 July 2024: Geometry	
09:30–10:30	**Dmitri Panov** (King's College London) *Spherical surfaces and their moduli spaces*
10:30–11:00	Coffee break
11:00–12:00	**Andrei Gabrielov** (Purdue University) *Classification of generic spherical quadrilaterals*
12:00 onwards	Lunch and end of the workshop

Abstracts

Walter Bergweiler (University of Kiel)

The escaping set

Abstract: The escaping set of an entire function consists of the points that tend to infinity under iteration. Much of the research on the escaping set was related to a question of Eremenko who asked whether every connected component of the escaping set is unbounded. This question was recently resolved by Marti-Pete, Rempe and Waterman. However, there are many open questions about the escaping set that remain. In this talk, we discuss some of these problems.

Melkana Brakalova-Trevithick *(Fordham University)*

When the circular dilatation at a point equals one

Abstract: We discuss geometric and analytic conditions implying certain local behavior of quasiconformal mappings at a point in the plane where the circular dilatation equals one, e.g. conformality, $C^{1+\alpha}$ conformality, asymptotic homogeneity, weak conformality, or maximal stretch for the q.c. map at that point. Some results include extensions of the Teichmüller-Wittich-Belinskii theorem. Besides being of interest by themselves, they enjoy applications in Nevanlinna theory, modulus of continuity studies, complex dynamics, the theory of p-integrable Teichmüller spaces, some of which are highlighted.

Mario Bonk (UCLA)

Lamé functions and special elliptic integrals

Abstract: In a recent paper Eremenko et al. investigated metrics on tori that have constant positive curvature and one conic singularity. They showed that this is related to a class of elliptic differentials characterized by the property that on the underlying torus they have only one zero and only double poles with vanishing residua. These differentials turn out to be connected to Lamé functions. In my talk I will provide an analytic perspective for these results and show that this also leads to the consideration of periods of differentials with only simple poles.

Andrei Gabrielov (Purdue University)

Classification of generic spherical quadrilaterals

Abstract: A spherical polygon is a surface homeomorphic to a closed disk equipped with a Riemannian metric of constant positive curvature, with conic singularities on the boundary. Classification of generic spherical triangles goes back to Klein (1890). Non-generic spherical triangles were classified by Eremenko (2004). Spherical quadrilaterals with the sides on either two or three circles were classified by Eremenko et al (2014-16). Classification of generic spherical quadrilaterals (with the sides on 4 distinct great circles) was completed in (AG 2023). The space of generic quadrilaterals with prescribed angles at the corners consists of finitely many open curves. At the endpoints of these curves, quadrilaterals may degenerate, or converge to a spherical quadrilateral with the sides on a four-circle configuration with a triple intersection. In the latter case, the family extends beyond the quadrilateral with the triple intersection to another one-parametric family of generic spherical quadrilaterals.

Adi Glucksam (Northwestern University)

Multi-fractal spectrum of planar harmonic measure

Abstract: In this talk, I will define various notions of the multi-fractal spectrum of harmonic measures and discuss finer features of the relationship between them and properties of the corresponding conformal maps. Furthermore, I will describe the role of multifractal formalism and dynamics in the universal counterparts. This talk is based on a joint work with I. Binder.

Aimo Hinkkanen (University of Illinois Urbana-Champaign)

Moments and positivity

Abstract: In classical moment theory one studies the moments of positive measures. We start with signed measures on the real axis and obtain conditions in terms of their moments and absolute moments for the measure to be non-negative.

As applications, we obtain conditions in terms of moments for a real-valued function on the real axis to be non-negative, and necessary and sufficient conditions for polynomials in one and several real variables to be non-negative for all real values of their variables. We further obtain conditions in terms of moments for polynomials and some entire functions to have only real zeros.

Oleg Ivrii (Tel Aviv University)

Critical values of inner functions

Abstract: Let $\mathscr J$ be the space of inner functions of finite entropy endowed with the topology of stable convergence. We prove that an inner function $F \in \mathscr J$ possesses a radial limit (and in fact, a minimal fine limit) in the unit disk at $\sigma(F')$ a.e.~point on the unit circle. We use this to show that the singular value measure $\nu(F) = \sum_{c \in \crit F} (1-|c|)\cdot \delta_{F(c)} + F_*(\sigma(F'))$ varies continuously in F. Our analysis involves a surprising connection between Beurling-Carleson sets and angular derivatives. (This is joint work with Uri Kreitner.)

Ilmari Kangasniemi (University of Cincinnati)

On the theory of quasiregular values

Abstract: A quasiregular (QR) map is a Sobolev map $f: \mathbb{R}^n \to \mathbb{R}^n$ satisfying the distortion inequality $|Df(x)|^n \leq K \det(Df(x))$ at almost every x, where $K \geq 1$ is a constant. QR maps form a higher-dimensional class of maps with many similar geometric properties as single-variable holomorphic maps. In this talk, we consider a generalization of the distortion inequality of the form $|Df(x)|^n \leq K \det(Df(x)) + \Sigma(x)|f(x) - y|^n$, where Σ is a real-valued weight function and $y \in \mathbb{R}^n$ is a fixed point. Our recent results show that under various L^p-integrability assumptions on Σ, this condition can be used to prove single-value counterparts to many fundamental results of QR-maps at the point y. The list of generalized results includes the QR-versions of the open mapping theorem, Liouville theorem, and Picard theorem. Joint work with Jani Onninen.

Patrick Ng (University of Hong Kong)

Brody holomorphic curves on the degree six Fermat surface

Abstract: A holomorphic map from the complex line to the n-dimensional complex projective space is called a Brody curve if its spherical derivative is bounded. In 2010, Eremenko applied potential theory to study Brody curves omitting n hyperplanes in general position and showed that these

curves have growth order at most one, normal type. In this talk, we will characterize Brody curves on the degree six Fermat surface in the three dimensional complex projective space based on Eremenko's potential theoretical method. This is a joint work with Sai Kee Yeung.

Dmitri Panov (King's College London)

Spherical surfaces and their moduli spaces

Abstract: A spherical surface is a Riemann surface with a curvature 1 metric and a finite number of conical singularities. It can always be glued from a collection of spherical triangles by isometric identification of their sides. Contrary to the hyperbolic case, when the theory is almost identical to the theory of Riemann surfaces, the case of spherical surfaces is wide open. I will speak about recent results in the area, such as a full description of the moduli space of spherical metrics with one conical singularity on a torus (joint work with Gabrielle Mondello and Alex Eremenko), the description of possible conical angles of a spherical metric on a 2-sphere, disconnectedness of the moduli spaces and their unboundedness (joint work with Gabirelle Mondello).

Eveliina Peltola (Aalto University and University of Bonn)

On geometric properties of conformally invariant curves

Abstract: How to construct a canonical random conformally invariant path in two dimensions? Motivated by Loewner's classical theory of dynamics of slit domains, Schramm introduced random Loewner evolutions to model canonical random curves via evolutions of conformal maps. While the initial usage of such Schramm-Loewner evolutions (SLEs) was to describe critical interfaces in statistical physics models and their relation to conformal field theory (CFT), SLE type curves quickly turned out to be ubiquitous in various problems in probability theory and mathematical physics, and to have intricate connections to complex geometry and beyond.
This talk highlights the geometric aspects of SLE type curves. As examples, we shall mention versions of the Loewner energy (the anticipated action functional of these canonical curve models, or more rigorously, the rate function in large deviations principles for the random curves), the classification of real rational functions with prescribed critical points (a special case of the Shapiro-Shapiro conjecture in real enumerative geometry, first proven by Eremenko and Gabrielov), and the emergence of the Virasoro algebra (the symmetry algebra of CFTs) from complex deformations of boundary components of bordered Riemann surfaces.
Based mainly on joint works with Sid Maibach (Bonn) and Yilin Wang (IHES).

Lakshmi Priya (Tel Aviv University)

Almost sharp lower bound for the nodal volume of harmonic functions

Abstract:

In this talk, I will discuss the relation between the growth of harmonic functions and their nodal volume. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function, where $n \geq 2$. One way to quantify the growth of u in the ball $B(0,1) \subset \mathbb{R}^n$ is via the *doubling index* N, defined by

$$\sup_{B(0,1)} |u| = 2^N \sup_{B(0,\frac{1}{2})} |u|.$$

I will present a result, obtained jointly with A. Logunov and A. Sartori, where we prove an almost sharp result, namely:

$$\mathcal{H}^{n-1}(\{u = 0\} \cap B(0,2)) \gtrsim_{n,\varepsilon} N^{1-\varepsilon},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$ dimensional Hausdorff measure.

Mitsuhiro Shishikura (Kyoto University)
Reconstructing the dynamics via dynamical charts from a sequence of near-parabolic or cylinder renormalization

Abstract: For germs of irrationally indifferent fixed points of analytic functions, one can define a cylinder renormalization by taking first return map to a certain fundamental domain. We discuss how to reconstruct the original dynamics from a sequence of cylinder renormalization. When there is a priori bounds for the sequence such as near-parabolic renormalization for high type rotation numbers, this reconstruction can be carried out with some bounds. We also discuss how to extend this for a more general type of a priori bounds.

Mikhail Sodin (Tel Aviv University)
Equivariant Weierstrass theorem

Abstract: Consider a map Z from the space E of entire functions to the space D of discrete subsets of the complex plane that maps entire functions to their zero sets. The Weierstrass theorem asserts that this map is surjective. In the natural topologies, the spaces E and D are Polish, the complex plane acts continuously on both of them by translations, and the map Z is equivariant with respect to these actions. Does it admit an equivariant (right) inverse?

In a joint work in progress with Konstantin Slutsky and Aron Wennman, we prove the existence of an equivariant Borel inverse if the action of the complex plane on D is free. Such an inverse cannot be made continuous and no such inverse exists if the action is not free.

Alan Sokal (University College London)
Some conjectures concerning the zeros of the deformed exponential function

Abstract: I discuss some interesting conjectures concerning the zeros of the deformed exponential function $F(x,y) = \sum\limits_{n=0}^\infty {x^n \over n!} y^{n(n1)/2}$. Some of these are related to more general conjectures concerning the coefficientwise nonnegativity of the Taylor expansion for the leading root of certain series $f(x,y) = \sum\limits_{n=0}^\infty \alpha_n x^n y^{n(n-1)/2}$.

Gwyneth Stallard (Open University)
Shrinking targets and recurrent behaviour for forward compositions of inner functions

Abstract: We give sharp results about the behaviour of the boundary extensions of forward compositions of inner functions, in relation to shrinking targets and recurrence, inspired by fundamental results about iteration of inner functions. We also discuss examples which show the sharpness of our results and illustrate behaviours that cannot occur in the simpler case of iteration. (This is joint work with Anna Miriam Benini, Vasso Evdoridou, Núria Fagella and Phil Rippon.)

Masaki Tsukamoto (Kyoto University)
Rate distortion dimension of random Brody curves

Abstract: Our main purpose is to propose an ergodic theoretic approach to the study of entire holomorphic curves. In particular, we develop a "holomorphic curve analogue" of the theory of Axiom A diffeomorphisms. Brody curves are one-Lipschitz holomorphic maps from the complex plane to the complex projective space. They naturally form a dynamical system, and “random Brody curves” refers to invariant probability measures on it. We study their geometric and dynamical properties. Our first main theorem claims that the rate distortion dimension of random Brody curves is bounded by the integral of a "potential function". This is analogous to the Ruelle inequality of

smooth ergodic theory. Our second main theorem claims that there exists a rich variety of invariant probability measures attaining equality in this "Ruelle inequality for Brody curves". The proof is based on the "variational principle" for mean dimension with potential. This approach is motivated by the thermodynamic formalism for Axiom A diffeomorphisms.

Yilin Wang (IHES)

Jordan curves with the piecewise geodesic property

Abstract: We consider Jordan curves passing through a given set of n points on the Riemann sphere with the property that each curve segment is the hyperbolic geodesic in the complement of the rest of the curve. This family of "piecewise geodesic Jordan curves" is reminiscent of the canonical embeddings of pairs of arcs studied by Bonk and Eremenko. We show that for a given set of points and a homotopy class of Jordan curves relative to these points, there is a unique piecewise geodesic curve that is continuously differentiable. These curves are minimizers of the Loewner energy and give a specific complex projective structure on the n-punctured sphere with real and parabolic holonomy around the punctures. Similar to a result of Takhtajan-Zograf, we also obtain a new type of accessory parameters and show that they can be expressed as differentials of the Loewner energy.

James Waterman (Stony Brook University)

A counterexample to Eremenko's conjecture

Abstract: In 1989, Eremenko investigated the set of points that escape to infinity under iteration of a transcendental entire function, the so-called escaping set. He proved that every component of the closure of the escaping set is unbounded and conjectured that all the components of the escaping set are unbounded. Much of the recent work on the iteration of transcendental entire functions is involved in investigating properties of the escaping set, motivated by Eremenko's conjecture. We will discuss constructing a transcendental entire function with a bounded connected component of the escaping set, providing a counterexample to Eremenko's conjecture. This is joint work with David Martí-Pete and Lasse Rempe.

Posters

Andrew Brown (University of Liverpool)
Slow-growing counterexamples to the strong Eremenko conjecture, or, "How slow can you grow?"

(William) Assheton Don (Open University)
Wandering Cauliflower

Josef Greilhuber (Stanford University)
Cones on which few harmonic functions can vanish

Anna Jové (Universitat de Barcelona)
Dynamics on the boundary of transcendental Fatou Components

Xing-Yu Li (Shantou University)
Completely regular growth solutions to linear differential equations with exponential polynomials coefficients

Techheang Meng (UCL)
Functions with unbounded slow growth Nevanlinna characteristic in the upper halfplane

Julia Münch (University of Liverpool)
Extending expanding Thurston maps

Aapo Pajala (Aalto University)
Real loci (or nets) of real rational functions

Nikolai Prochorov (Institut de Mathématiques de Marseille)
Dynamical line complexes and Thurston theory

Ruicen Qiu (Peking University)
Discrete harmonic measure for distance expanding dynamical systems

Alex Rodriguez (Stony Brook University)
Large disks in quadrilaterals

Lingrui Wang (SCMS, Fudan University)
Hausdorff dimension of non-escaping set of exponential family