Removal lemmas, dense and sparse

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Caltech

Triangle removal

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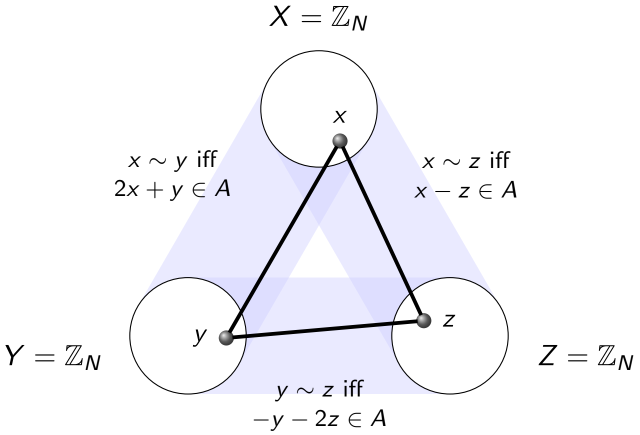
Application: Roth's theorem

A subset A of $[n] := \{1, 2, \ldots, n\}$ with no 3-term arithmetic progression has size $o(n)$.

Triangle removal $\implies$ Roth



More applications of triangle removal

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Theorem

A 3-uniform hypergraph on n vertices with girth greater than 3 has $o(n^2)$ edges.

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'A tripartite graph between vertex sets X, Y and Z, each pair of which induces a sufficiently random-like graph, contains asymptotically the same number of triangles xyz as if the graphs were truly random.'

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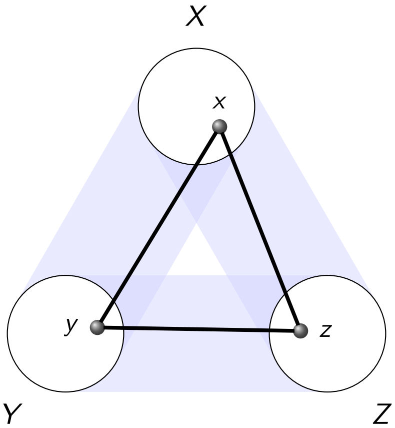
The combination of these two tools is usually known as the regularity method.

Proving the removal lemma



Bounds for the removal lemma

Triangle removal lemma - quantitative version

For every $\epsilon > 0$, there exists $\delta > 0$ such that any graph on n vertices with at most δn^3 triangles can be made triangle-free by deleting at most ϵn^2 edges.

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Open problem

Improve these bounds.

Other removal lemmas

Graph removal lemma

For any fixed graph H, any graph on n vertices with $o(n^{v(H)})$ copies of H can be made H-free by deleting $o(n^2)$ edges.

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The bounds in the hypergraph removal lemma are of Ackermann type. For instance, in the 3-uniform case we have the following.

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3-uniform simplex removal

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Improve the bounds for the hypergraph removal lemma.

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Induced removal lemma (Alon–Fischer–Krivelevich–Szegedy, 2000)

For any fixed graph H, any graph on n vertices with $o(n^{v(H)})$ induced copies of H can be made induced-H-free by modifying $o(n^2)$ edges.

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The Brown–Erdős–Sós problem

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Problem (Special case of Brown–Erdős–Sós)

For which v and e is $f(n, v, e) = o(n^2)$?

The Brown–Erdős–Sós problem

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Theorem (C.–Gishboliner–Levanzov–Shapira, 2023)

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The proof of this last result uses hypergraph removal at every uniformity to prove a result about 3-uniform hypergraphs.

Conjecture (Gowers–Long, 2021)

For any fixed $e \geq 3$, there exists $\epsilon > 0$ such that

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In particular, the $e = 5$ case would solve the following question of Ruzsa in the positive.

Question (Ruzsa, 1993)

Is there $\epsilon > 0$ such that the largest subset A of $[n]$ with no non-trivial solution to $2x + 2y = 3z + w$ has size at most $n^{1-\epsilon}$?

Induced matching lemma

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The following little known problem asks for a generalisation to hypergraphs.

Conjecture (Frankl–Rödl)

An n-vertex linear 3-graph which is the union of at most n induced matchings has $o(n^2)$ edges.

Sparse removal lemmas

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Can we develop a regularity method for sparse graphs?

Finding a sparse regularity lemma is comparatively straightforward, going back to work of Kohayakawa and Rödl in the 1990s. The difficulty lies with finding appropriate counting lemmas.

Counting lemmas in random graphs

Theorem (Work of many, including C.–Gowers–Samotij–Schacht)

A counting lemma holds with high probability for subgraphs of a binomial random graph. E.g., for triangles, a counting lemma holds with high probability for subgraphs of $G_{n,p}$ where $p \gg n^{-1/2}$.

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Theorem (Kohayakawa–Łuczak–Rödl)

For any $\epsilon > 0$, there exist positive constants δ and C such that if $p \geq Cn^{-1/2}$, then the following holds a.a.s. in $G_{n,p}$. Every subgraph of $G_{n,p}$ which contains at most $\delta p^3 n^3$ triangles may be made triangle-free by removing at most ϵpn^2 edges.

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Extended to all (balanced) hypergraphs by C.–Gowers.

Szemerédi's theorem in random sets

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For any $k \geq 3$ and $\delta > 0$, there exists a positive constant C such that if $p \geq Cn^{-1/(k-1)}$, then the following holds a.a.s. in $[n]_p$. Every subset of $[n]_p$ of size at least δpn contains a k-AP.

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Unlike in the dense case, this does not follow automatically from the sparse removal lemma.

Theorem (C.–Fox–Zhao)

A counting lemma (and hence a removal lemma) holds for subgraphs of sufficiently pseudorandom (hyper)graphs.

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This does carry across to the arithmetic setting.

Corollary (Green–Tao, C.–Fox–Zhao)

Let A be a sufficiently pseudorandom subset of the integers. Then A is Szemerédi, that is, any subset A' of A with positive relative density contains arbitrarily long arithmetic progressions.

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Such a 'relative Szemerédi theorem' is the main component in the proof of the Green–Tao theorem, that the primes contain arbitrarily long arithmetic progressions.

Definition

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Theorem (Kohayakawa–Rödl–Schacht–Skokan)

For any $\epsilon > 0$, there exist positive constants δ and c such that if $\lambda \leq cp^3 n$, where $p = d/n$, then any (n, d, λ)-graph G has the property that every subgraph of G which contains at most $\delta p^3 n^3$ triangles may be made triangle-free by removing at most ϵpn^2 edges.

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This would potentially allow removal lemmas in pseudorandom sets down to density $n^{-1/3}$, which might (maybe) make the following conjecture approachable.

Conjecture

There are 3-APs of Friedlander–Iwaniec primes, that is, primes of the form $x^2 + y^4$.

Regularity in C_4-free graphs

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'Theorem' (C.–Fox–Sudakov–Zhao)

'In graphs with few 4-cycles, a counting lemma holds for 5-cycles.'

Applications: removal lemmas

Theorem (C.–Fox–Sudakov–Zhao)

Any C_4-free graph on n vertices with $o(n^{5/2})$ 5-cycles can be made 5-cycle-free by deleting $o(n^{3/2})$ edges.

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Writing $p = n^{-1/2}$, this can be restated as saying that for any $\epsilon > 0$, there exists $\delta > 0$ such that any C_4-free graph on n vertices with at most $\delta p^5 n^5$ 5-cycles may be made 5-cycle-free by removing at most $\epsilon p n^2$ edges.

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Applications: number theory

Theorem (C.–Fox–Sudakov–Zhao)

Fix positive integers a and b. Every subset of $[n]$ without a non-trivial solution to the equation

$$ax_1 + ax_2 + bx_3 = ax_4 + (a+b)x_5$$

has size $o(n^{1/2})$. Here a trivial solution is one of the form $(x_1, \ldots, x_5) = (x, y, y, x, y)$ or (y, x, y, x, y) (or (y, y, x, x, y) if $a = b$) for some $x, y \in \mathbb{Z}$.

For example, this holds for

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Open problem

Can the bound be improved to $n^{1/2-\epsilon}$ for some $\epsilon > 0$?

Applications: number theory

Theorem (C.–Fox–Sudakov–Zhao)

Fix positive integers $a_1, \ldots, a_4$. The maximum size of a Sidon subset of $[n]$ without a solution in distinct variables to the equation

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is $o(n^{1/2})$ and at least $n^{1/2-o(1)}$.

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An alternative proof of this result was given by Sean Prendiville using Fourier methods.

Applications: compactness

On the previous slide, we are simultaneously avoiding

(a) solutions to the Sidon equation $x_1 + x_2 = x_3 + x_4$ and

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There exist Sidon sets of size $(1 + o(1))n^{1/2}$, as well as sets of size $n^{1-o(1)}$ avoiding (b). However, our theorem shows that by simultaneously avoiding non-trivial solutions to both equations, the maximum size is substantially reduced.

Applications: compactness

On the previous slide, we are simultaneously avoiding

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This is the first example showing that 'compactness' does not hold for linear equations. A similar example for graphs remains an important open problem.

Which graphs can be counted in C_4-free graphs?

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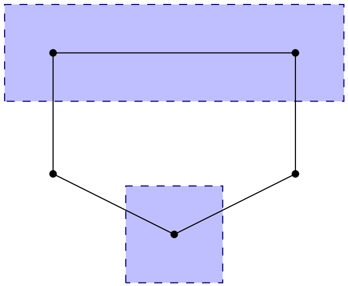
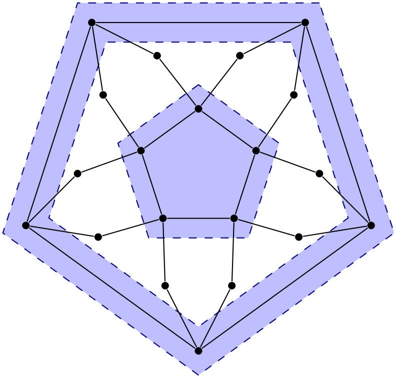
Sufficient that H has an 'islands and bridges' decomposition.

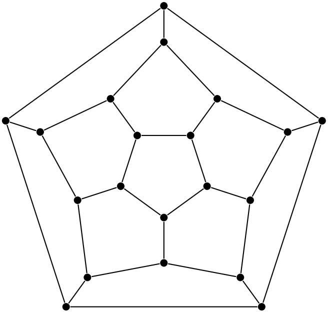
Islands and bridges

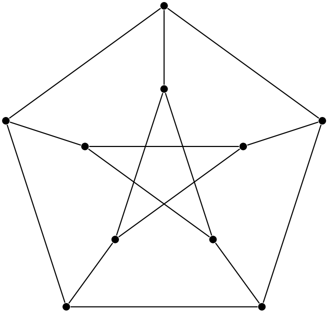


Islands and bridges



Can these graphs be counted?





A final problem

Open problem

Prove a removal lemma for 7-cycles in graphs with few 6-cycles.

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Potential application

A 3-uniform hypergraph on n vertices with girth greater than 7 has $o(n^{4/3})$ edges.

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Potential application

The maximum size of a B_3-set in $[n]$ without a solution in distinct variables to the equation

$$x_1 + \cdots + x_6 = 6x_7$$

is $o(n^{1/3})$.

Thank you for listening!