AIMS | African Institute for Mathematical Sciences SENEGAL

On the nonlinear thermomechanical analysis of a stayed-beam having fractional viscoelastic properties in complex environment

Presented by:
Eliane Raïssa FANKEM

The African Institute for Mathematical Sciences SENEGAL,
In collaboration with the Laboratory of Mechanics, Materials and Structures,
University of Yaoundé I

"Retreat for Women in Applied Mathematics 2025"
Monday 13 - Friday 17 January 2025
Edinburgh, UK

Remark

- Haiti in 2010, Japan in 2011, China in 2008, 2014 and in 2016
- Nepal in 2015, Ecuador in 2016, Italy in 2016 and 2019.
- Cameroon in December 2019.

Consequences

- a large number of injuries and deaths in humans and animals
- left behind extensive material damage and a weak economy or heavy financial losses

Illustration

Context

— Vibrations in man-made structures are a central problem in mechanical engineering ; this results from external or internal excitations that they face during their live.

— Considerable efforts have been devoted to the study of nonlinear vibrating structures. This is generally achieved with many techniques and methods of which some will be presented in this part of the work.

Idea to solve that problems

We are interested by the ways of building structures with high capacities and good resistance abilities.

1 General Introduction

2 From Structural control systems to Self-vibration control of Structures
- Dynamic of structures
- Structural control systems
- Self-vibration Control
- An inverted pendulum with multi-branching view as self-controlled system : Modelling and vibration absorber capacity
- Problematic of this presentation

3 On the nonlinear thermomechanical analysis of a cables stayed-beam having fractional viscoelastic properties in complex environment
- Modelling
- Modal equation
- Some results

Models of elastic beams

They are four main theories about elastic beams modelling see the Table 1.1 :

Table – 1.1 Four beam theories

Beam models	**Bending moment**	**Lateral displacement**	**Shear deformation**	**Rotary inertia**
Euler - Bernoulli	$\surd$	$\surd$	$\times$	$\times$
Rayleigh	$\surd$	$\surd$	$\times$	$\surd$
Shear	$\surd$	$\surd$	$\surd$	$\times$
Timoshenko	$\surd$	$\surd$	$\surd$	$\surd$

Models of elastic beams

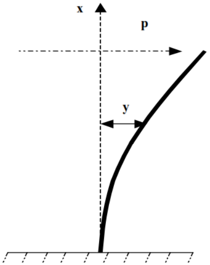


Fig 1 : Elastic beam

- Euler-Bernoulli theory

$$m\frac{\partial^2 y(x,t)}{\partial t^2} + EI\frac{\partial^4 y(x,t)}{\partial x^4} = 0 \quad (1)$$

- Shear theory

$$m\frac{\partial^2 y(x,t)}{\partial t^2} - k_s GA\left(\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\partial \alpha(x,t)}{\partial x}\right) = 0 \quad (2a)$$

$$EI\frac{\partial^2 \alpha(x,t)}{\partial x^2} + k_s GA\left(\frac{\partial y(x,t)}{\partial x} - \alpha(x,t)\right) = 0 \quad (2b)$$

Models of elastic beams

- Rayleigh theory

$$m\frac{\partial^2 y(x,t)}{\partial t^2} + EI\frac{\partial^4 y(x,t)}{\partial x^4} - \rho I\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2} = 0 \quad (3)$$

- Timoshenko theory

$$m\frac{\partial^2 y(x,t)}{\partial t^2} = k_s GA\left(\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{\partial \alpha(x,t)}{\partial x}\right) \quad (4a)$$

$$\rho I\frac{\partial^2 \alpha(x,t)}{\partial t^2} = k_s GA\left(\frac{\partial y(x,t)}{\partial x} - \alpha(x,t)\right) + EI\frac{\partial^2 \alpha(x,t)}{\partial x^2} \quad (4b)$$

Eliminating α, we obtain the uncoupled equations of motion given by

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + m\frac{\partial^2 y(x,t)}{\partial t^2} - \rho I\left(1 + \frac{E}{k_s G}\right)\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{k_s G}\frac{\partial^4 y(x,t)}{\partial t^4} = 0 \quad (5)$$

Model of rigid beams

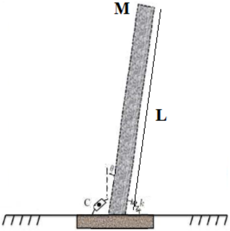


Fig 2 : An inverted pendulum

Under the action of the external excitation, the motion of the inverted pendulum is given by

$$J\frac{d^2\theta}{dt^2} + C\frac{d\theta}{dt} + k\theta - \frac{1}{2}MgL\sin\theta = M'(t) \tag{6}$$

Before the Structural control systems, we had the classic control systems : passive, active, semi-active and hybrid.

What is Structural Control ?

It is the control of selected response variable of a structure subjected to dynamics loading

- Such variables may be displacements or their time derivatives (velocities, accelerations) and/or forces

- Full controllability can be achieved in mode control and the control of rigid body mechanism

- For rigid body control, a structural system must consist of an assemblage of rigid bodies

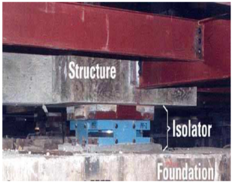
According to seismic control these structural control techniques have been proposed :

– Base Isolation

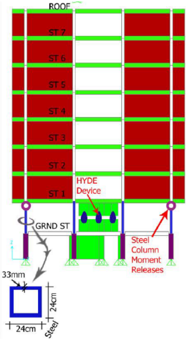
– Hyde (Hysteretic device) system

– Tendon system

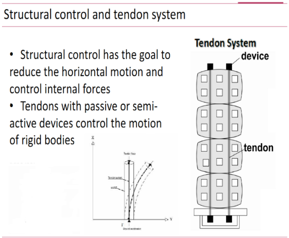
– Tuned Mass Damped, etc.

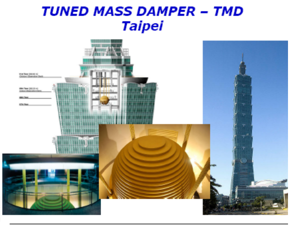


Base isolation system.



Hyde (Hysteretic device) system.





Tuned Mass Damper

What is Self-control of vibration ?

•A self-control system is a system which has the ability to maintain or turn back itself in a suitable stage whatever what disturb it and put it away from that stage.

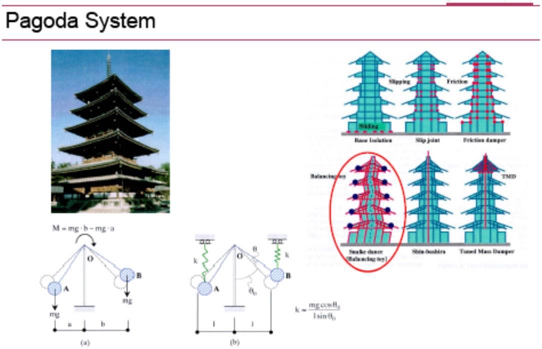
•Self-control is also known as maintained self-oscillation, self-excited, self-induced, spontaneous, autonomous.

•These structures do not need any external help (added after the building of structure) or internal system (structural control system, etc) to be controlled.

•This new system is suitable for high-rise buildings because there generally have flexible and low damping characteristics.

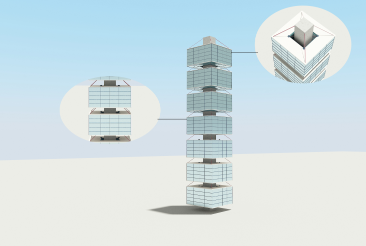
This fact is already scientifically explained, but no modern structure has been built with this robust structural system, which belongs to a class of seismic control concepts. These concepts rely on the control of rigid body motions allowing for a drastic reduction in kinetic and potential energy in the structure, thus leading to a very robust behaviour ? ?

This fact is already scientifically explained, but no modern structure has been built with this robust structural system, which belongs to a class of seismic control concepts. These concepts rely on the control of rigid body motions allowing for a drastic reduction in kinetic and potential energy in the structure, thus leading to a very robust behaviour.



Modern pagoda system

Here the idea is to consider a structure as the one of figure, made by a central column (a rigid body : a Cantilever beam) and at different level we joint pendulums which represent each floor and study the behavior of the structure obtained.



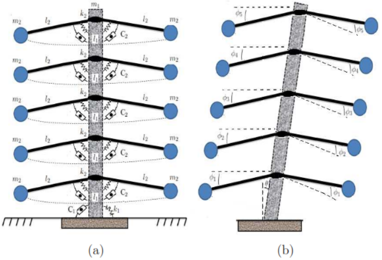


Fig 7 : Studied Model (a) At rest, (b)Disturbed.

The system of motion is given by (with *k* the floor index) :

$$\left\{\begin{array}{l}
\left[\frac{25}{4}Ml_1^2 + 110m_1l_1^2 - 2m_1l_1^2\cos^2\phi_0\left(\sin^2\phi_1 + \sin^2\phi_2 + \sin^2\phi_3 + \sin^2\phi_4 + \sin^2\phi_5\right)\right]\ddot{\theta} \\
+\left[C_1 - 4m_1l_1l_2\cos\phi_0\left(\dot{\phi}_1\cos\phi_1 + 2\dot{\phi}_2\cos\phi_2 + 3\dot{\phi}_3\cos\phi_3 + 4\dot{\phi}_4\cos\phi_4 + 5\dot{\phi}_5\cos\phi_5\right)\right]\dot{\theta} \\
+\left[\begin{array}{l} 2m_1l_1^2\cos^2\phi_0\left(\begin{array}{l}\cos\phi_1\sin\phi_1 + 4\cos\phi_2\sin\phi_2 + 9\cos\phi_3\sin\phi_3 \\ +16\cos\phi_4\sin\phi_4 + 25\cos\phi_5\sin\phi_5\end{array}\right) \\ -2m_1l_1l_2\sin\phi_0\left(\sin\phi_1 + 2\sin\phi_2 + 3\sin\phi_3 + 4\sin\phi_4 + 5\sin\phi_5\right)\end{array}\right]\dot{\theta}^2 \\
+K_1\theta + \left(30m_1 - \frac{5}{2}M\right)gl_1\sin\theta + \left(\frac{l_1}{l_2}\cos\phi_0\sin\phi_1 - 1\right)C_2\dot{\phi}_1 \\
+\left(2\frac{l_1}{l_2}\cos\phi_0\sin\phi_2 - 1\right)C_2\dot{\phi}_2 + \left(3\frac{l_1}{l_2}\cos\phi_0\sin\phi_3 - 1\right)C_2\dot{\phi}_3 \\
+\left(4\frac{l_1}{l_2}\cos\phi_0\sin\phi_4 - 1\right)C_2\dot{\phi}_4 + \left(5\frac{l_1}{l_2}\cos\phi_0\sin\phi_5 - 1\right)C_2\dot{\phi}_5 \\
+\left(\frac{l_1}{l_2}\cos\phi_0\sin\phi_1 - 1\right)2K_2\phi_1 + \left(2\frac{l_1}{l_2}\cos\phi_0\sin\phi_2 - 1\right)2K_2\phi_2 \\
+\left(3\frac{l_1}{l_2}\cos\phi_0\sin\phi_3 - 1\right)2K_2\phi_3 + \left(4\frac{l_1}{l_2}\cos\phi_0\sin\phi_4 - 1\right)2K_2\phi_4 \\
+\left(5\frac{l_1}{l_2}\cos\phi_0\sin\phi_5 - 1\right)2K_2\phi_5 - 2m_1l_1l_2\cos\phi_0\left(\begin{array}{l}\dot{\phi}_1^2\cos\phi_1 + 2\dot{\phi}_2^2\cos\phi_2 \\ +3\dot{\phi}_3^2\cos\phi_3 + 4\dot{\phi}_4^2\cos\phi_4 \\ +5\dot{\phi}_5^2\cos\phi_5\end{array}\right) \\
-m_1gl_1\sin(2\phi_0)\left[\begin{array}{l}\sin\phi_1\sin(\theta+\phi_1) + 2\sin\phi_2\sin(\theta+\phi_2) + \\ 3\sin\phi_3\sin(\theta+\phi_3) + 4\sin\phi_4\sin(\theta+\phi_4) \\ +5\sin\phi_5\sin(\theta+\phi_5)\end{array}\right] = 0 \\
2m_1l_2^2\ddot{\phi}_k + C_2\dot{\phi}_k + 2K_2\phi_k - \left(\begin{array}{l}2m_1gl_2 \\ \sin\phi_0\end{array}\right)\sin(\theta+\phi_k) + \left(\begin{array}{l}2(k)m_1l_1l_2 \\ \dot{\theta}^2\cos\phi_0\end{array}\right)\cos\phi_k = \\
\qquad 2\left(\begin{array}{l}(k)m_1l_1l_2\cos\phi_0\sin\phi_k \\ -m_1l_2^2\end{array}\right)\ddot{\theta}
\end{array}\right.$$

For numerical purpose, we started by studying the autonomous system (the system without any external force)

The energy study is :

$$
\begin{aligned}
E_m &= \left[1 - 2\Gamma\left(J_{\phi_1} + J_{\phi_2} + J_{\phi_3} + J_{\phi_4} + J_{\phi_5}\right)\right]\dot{\theta}^2 + \Omega_1^2\theta^2 - 2\beta_1^2\cos\theta \\
&+2K_{\phi_1}\cos\left(\theta + \phi_1\right) + 2K_{\phi_2}\cos\left(\theta + \phi_2\right) + 2K_{\phi_3}\cos\left(\theta + \phi_3\right) \\
&+2K_{\phi_4}\cos\left(\theta + \phi_4\right) + 2K_{\phi_5}\cos\left(\theta + \phi_5\right) \\
&+\Gamma\left[\begin{array}{l} \dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2 + \dot{\phi}_4^2 + \dot{\phi}_5^2 + \Omega_2^2\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 + \phi_5^2\right) \\ +2\beta_2^2\left[\begin{array}{l} \cos\left(\theta + \phi_1\right) + \cos\left(\theta + \phi_2\right) + \cos\left(\theta + \phi_3\right) \\ + \cos\left(\theta + \phi_4\right) + \cos\left(\theta + \phi_5\right) \end{array}\right] \end{array}\right]
\end{aligned}
\qquad (7)
$$

For numerical purpose, we started by studying the system without any external force but after moving it from his initial position :
Angular Displacement

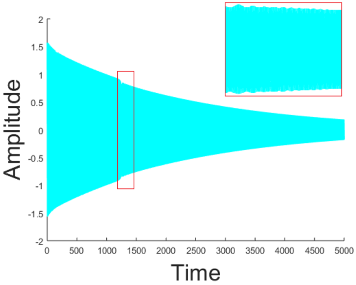


Fig 8 a- Central column.

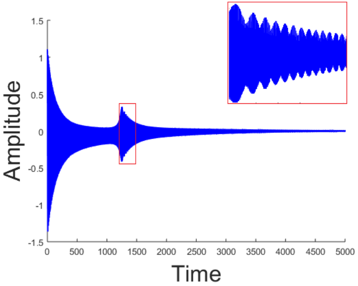


Fig 8 b- And level.

Energy

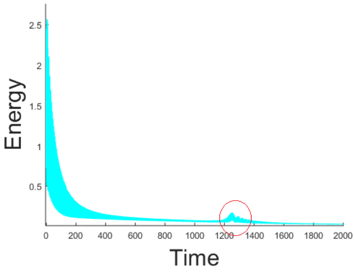


Fig 9 a- With one floor.

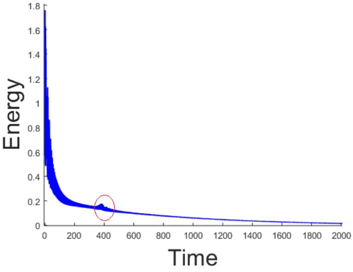


Fig 9 b- With two floors.

Autonomous case

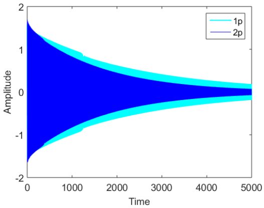


Fig 10 a- 1 and 2 floors

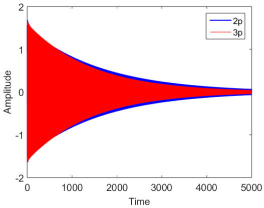


Fig 10 b- 2 and 3 floors

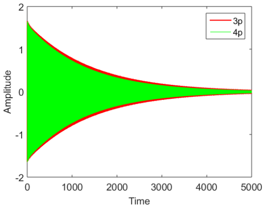


Fig 10 c- 3 and 4 floors

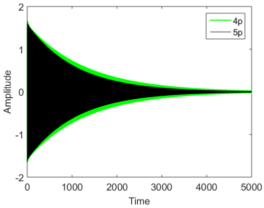


Fig 10 d- 4 and 5 floors

After a brief review of the evolution observed in the control of mechanical structures, let's talk about our most recent work which concerns the nonlinear thermomechanical analysis of a cables stayed-beam having fractional viscoelastic properties in complex environment.

The Euler Bernoulli beam is modelled here by a fractional material involving Kelvin-Voigt with real order fractional derivatives. The homogeneous beam is simply supported and placed in an environment subjected to temperature change.

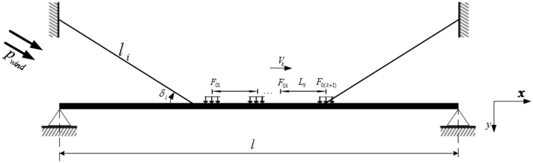


Fig 11 : Structural configuration

Taking into account the external loads on the structure and the action of the cables, the application of Newton's second law on the motion leads to the following equation of dynamics

$$\rho S\frac{\partial^2 y(x,t)}{\partial t^2}+EI\left(\frac{\partial^4 y(x,t)}{\partial x^4}-\frac{3}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 y(x,t)}{\partial x^2}\left(\frac{\partial y(x,t)}{\partial x}\right)^2\right)\right)+\xi\frac{\partial y(x,t)}{\partial t}+(\theta_{NL}+H^c)\frac{\partial^2 y}{\partial x^2}$$
$$-\frac{ES}{2l}\frac{\partial^2 y}{\partial x^2}\int_0^l\left(\frac{\partial y(x,t)}{\partial x}\right)^2 dx+\sum_{i=1}^{N_c}K_i^c\delta(x-\frac{il}{N_c})y(x,t)=p_{wind}+p_{mov} \tag{8}$$

$y(x,t)$ denotes the transversal displacement of the beam, l the length of the beam, S the cross-section area, I the moment of inertia, ρ is the density of the material, and ξ dissipation coefficient of the beam.

The wind load is taking into account by the term p_{wind} which is the aerodynamic force due to the wind, determined according to the quasi-steady theory and given by

$$p_{wind} = \frac{1}{2}\rho_a U^2 b \left[A_1 \left(\frac{\dot{y}}{U} \right) + A_3 \left(\frac{\dot{y}}{U} \right)^3 \right], \tag{9}$$

Where, $U = \bar{u} + u(t)$ is the wind speed devised into the turbulent part $u(t)$ and stationary part $\bar{u}$ (average wind speed in the region); b is the thickness of the beam, A_1 and A_3 are the aerodynamic coefficients given in (Anague *et al.*, 2019). Considering very large than turbulence, a Taylor expansion allows us to obtain

$$p_{wind} = \frac{1}{2}\rho_a \bar{u}^2 b \left[A_1 (1 + \frac{u(t)}{\bar{u}}) \frac{\dot{y}}{\bar{u}} + A_3 \left(1 - \frac{u(t)}{\bar{u}} \right) \left(\frac{\dot{y}}{\bar{u}} \right)^3 \right] \tag{10}$$

The moving load problem modelled here by p_{mov} , is the force due to a platoon moving loads made up of K moving loads. The mathematical expression of the loads due to this type of traffic, which takes into account the roughness of the deck is given by

$$p_{mov} = \sum_{k=1}^{K} F_{0k} \cos(\Omega_k t) \delta \left(x - Vt - \sum_{k'=1}^{k} L_{k'} \right) \tag{11}$$

$L_k > 0$ is the distance between loads (for example cars) $(k = 1, 2, 3....K; L_1 = 0)$, F_{0k} represents the amplitude, and Ω_k the frequency due to the roughness of the road and V_k the travel speed of the k^{th} load

This modal equation of the system is

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) + (\xi - \chi_1(\bar{u} + u(t)))\dot{q}_n(t) + \eta_n^\beta D_t^\beta q_n(t) - D_t^\beta \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \gamma_{1nijk}^\beta q_i(t) q_j(t) q_k(t)$$
$$- \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \gamma_{2nijk} q_i(t) q_j(t) q_k(t) - \left(\frac{1}{\bar{u}} - \frac{u(t)}{\bar{u}^2}\right) \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \chi_{3nijk} \dot{q}_i(t) \dot{q}_j(t) \dot{q}_k(t)$$
$$= \sum_{k=1}^{K} f_{0k} \left[\sin(\Omega_{2k} t + \varphi_k) + \sin(\Omega_{1k} t + \varphi_k)\right] \tag{12}$$

where,

$$\omega_n^2(\theta_{nl}) = \frac{2E_0 I}{\rho S l \omega_0^2} \int_0^l \phi_n(x) \phi''''_n(x) dx + 2\left(l\theta_{nl} + \frac{2H^c}{\rho S l \omega_0^2}\right) \int_0^l \phi''_n(x) \phi_n(x) dx + \sum_{i=1}^{N_c} \frac{2K_i^c}{\rho S l \omega_0^2} \phi_n^2(x_i)$$

$$\gamma_{1nijk}^\beta = \frac{\mu \omega_0^{\beta-2}}{\rho} \left(\frac{3Il}{S} \int_0^l \phi_n(x) \left(\phi''_i(x) \phi'_j(x) \phi'_k(x)\right)'' dx + \int_0^l \phi_n(x) \phi''_i(x) \int_0^l \phi'_j(x) \phi'_k(x) dx dx \right)$$

$$\gamma_{2nijk} = \frac{E_0}{\rho \omega_0^2} \left(\frac{3Il}{S} \int_0^l \phi_n(x) \left(\phi''_i(x) \phi'_j(x) \phi'_k(x)\right)'' dx + \int_0^l \phi_n(x) \phi''_i(x) \int_0^l \phi'_j(x) \phi'_k(x) dx dx \right)$$

$$\eta_n^\beta = \frac{2\mu I \omega_0^{\beta-2}}{\rho S l} \int_0^l \phi_n(x) \phi''''_n(x) dx; \chi_{3nijk} = \frac{\rho_a b A_3 l \omega_0}{\rho S u_0} \int_0^l \phi_n(x) \phi_i(x) \phi_j(x) \phi_k(x) dx$$

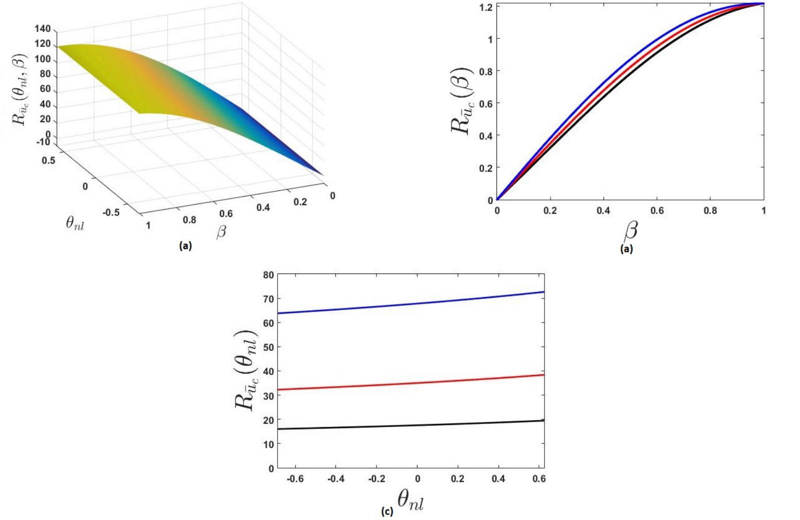
$$\xi = \frac{c}{\rho S \omega_0}; \chi_1 = \frac{\rho_a b A_1 u_0}{2\rho S \omega_0}; \omega_0^2 = \frac{E_0 I \pi^4}{\rho S l^4}; f_{0k} = \frac{F_{0k}}{\rho S l^2 \omega_0^2}; v_k = \frac{V_k}{l \omega_0}; \omega_k = \frac{\Omega_k}{\omega_0}$$

$$l_k = \frac{L_k}{l}; u_0 = 1.0 m.s^{-1}; \varphi_k = \pi \sum_{k'=1}^{k} l_{k'}; \Omega_{2k} = \pi v_k + \omega_k; \Omega_{1k} = \pi v_k - \omega_k.$$

The non-dimensional non-linear thermal stress is written as $\theta_{nl} = \frac{\theta_{NL}}{\rho S l^2 \omega_0^2}$. For the sake of simplicity, let us consider negligible the roughness of the road and the mechanical vibrations of loads i.e. $\omega_k = 0$, thus $\Omega_{2k} = \Omega_{1k}$. For this model, a set of loads of the same speed is considered, thus $v_k = v$ and then $\Omega_{2k} = \Omega_{1k} = \Omega_1$. Following this, the equation of the dynamics at the first order of vibrations is written

$$\begin{aligned} \ddot{q} + \omega_n^2(\theta_{nl})q + (\xi - \chi_1(\bar{u} + u(t)))\dot{q} + \eta^\beta D_t^\beta q - \gamma_1^\beta D_t^\beta q^3 - \gamma_2 q^3 \\ -\chi_3 \left(\frac{1}{\bar{u}} - \frac{u(t)}{\bar{u}^2}\right) \dot{q}^3 = \sum_{k=1}^{K} 2f_{0k} \sin(\Omega_1 t + \varphi_k) \end{aligned} \tag{14}$$

The multi-scale method is an alternative to solve this last equation containing the fractional real order.



Critical speed variation (a) 3D plot ; (b) versus β for θ nl = -0.68 (black) ; θ nl = 0.0 (red) ; θ nl = +0.61(blue)and (c) versus θ nl for β = 0.1 (black) ; β = 0.2 (red) ; β = 0.4 (blue)

Some references

[1] Fujita K, Hanazato T and Sakamoto I (2004) Earthquake response monitoring and seismic performance of five-storied timber pagoda. In : 13th world conference on earthquake engineering, Vancouver, Canada, 1-6 August 2004

[2] Dorka UE (2004) Erdbebensicherung durch structural control. Stahlbau 73 : 661-667

[3] Avossa AM, Giacinto DD, Malangone P, et al. (2018) Seismic retrofit of a multispan prestressed concrete girder bridge with friction pendulum devices. Shock and Vibration 2018 : 1-22

[4] Dorka UE (2014) Seismic control for elevated roads. Building Materials and Structures 57 : 9-20

[5] E.R. Fankem, B.R. Nana Nbendjo, P. Woafo, U. Dorka. "An inverted pendulum with multi-branching view as self-controlled system : Modelling and vibration absorber capacity", Journal of Vibration and Control (2020) DOI :10.1177/1077546320908141

[6] T.B. Djuitchou Yaleu, E.R. Fankem, B.R. Nana Nbendjo. "On the Nonlinear Thermomechanical Analysis of a Stayed-Beam Having Fractional Viscoelastic Properties in Complex Environment", Journal of Applied Nonlinear Dynamics (2024) DOI :10.5890/JAND.2024.06.012

Thank you for your kind attention