Additive Combinatorics Without Addition

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Based on joint work with
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ICMS workshop on Additive Combinatorics

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Ramsey theory

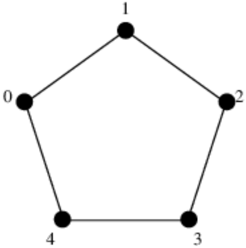
Ramsey theory contains many deep results which show that every very large structure contains a large well-organized substructure.

Ramsey's theorem guarantees that every very large graph contains a large clique or independent set.

Ramsey numbers

Definition (Ramsey number)

The *Ramsey number* $r(n)$ is the minimum N such that every graph on N vertices contains a clique or independent set of size n.



The 5-cycle has no clique or independent set of size 3.

Every 6-vertex graph has a clique or independent set of size 3.

Hence, $r(3) = 6$.

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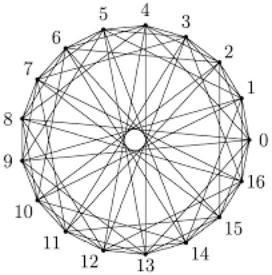

The Paley graph P_{17} has no clique or independent set of size 4. Furthermore, $r(4) = 18$.

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Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

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$$43 \leq r(5) \leq 48.$$
$$102 \leq r(6) \leq 147.$$

Ramsey graphs

Definition (Ramsey graphs)

A graph on N vertices is *C-Ramsey* if it has no clique or independent set of size $C \log_2 N$.

Theorem (Erdős-Szekeres 1935)

There is no $\frac{1}{2}$-Ramsey graph.

Theorem (Campos-Griffiths-Morris-Sahasrabudhe 2023+)

There is $\varepsilon > 0$ such that there is no $(\frac{1}{2} + \varepsilon)$-Ramsey graph.

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$$\mathbb{P}[G \text{ is not 2-Ramsey}] \le \mathbb{E}[\# \text{ cliques or ind. sets of order } n]$$

$$= 2^{1-\binom{n}{2}}\binom{N}{n} = o(1).$$

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Problem (Erdős)

Explicitly construct C-Ramsey graphs for some constant C.

Searching for hay in a haystack

Sven Sachsalber hunts for a needle in a haystack in a 2014 performance art piece. Photo: Palais de Tokyo, Paris.

Paley graphs

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For a prime $N \equiv 1 \pmod{4}$, the *Paley graph* P_N has vertex set $\mathbb{Z}_N$ and vertices $x \neq y$ are adjacent if $x - y$ is a quadratic residue.

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Theorem (Montgomery 1972)

Assuming GRH, $\omega(P_N) \geq c \log N \log\log N$ for infinitely many N.

Theorem (Graham-Ringrose 1990)

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Theorem (Hanson-Petrides, Di Benedetto-Solymosi-White 2021)

$\omega(P_N) \leq (\sqrt{2N-1}+1)/2$ for all N.

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Question

Are uniform random Cayley graphs Ramsey?

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Theorem (Green 2005, Mrazović 2017)

Asymptotically almost surely, the clique number of a uniform random Cayley graph on $\mathbb{F}_2^d$ with $N = 2^d$ is $\Theta(\log N \log \log N)$.

Clique number of random Cayley graphs

Theorem

The clique number of a uniform random Cayley graph on any group G of order N is asymptotically almost surely $O(\log N \log\log N)$.

On Alon's conjecture

Theorem

For almost all N, all abelian groups G of order N have a Cayley graph which is C-Ramsey.

Counting sets with small product set

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Theorem

In any group G of order N, the number of subsets $A \subset G$ with $|A| = n$ and $|AA^{-1}| \leq Kn$ is at most $N^{C(K+\log n)}(CK)^n$.

From additive combinatorics to edge-colored graphs

Consider a group G of order N. Color the edges of the complete graph on G by assigning each edge (x, y) the color $\{xy^{-1}, yx^{-1}\}$. This edge-coloring of K_N is such that each color is 1 or 2-regular. A Cayley graph on G is the edge-union of some color classes.

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What can we say about $\omega(G_c(p))$ if c is Δ-bounded?

Counting sets with few colors

Theorem

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By applying Vizing's theorem, we may assume $\Delta = 1$ (that is, the edge-coloring is proper).

Theorem

If an edge-coloring c of K_N is Δ-bounded, then a.a.s.

$$\omega(G_c(p)) = O_{p,\Delta}(\log N \log \log N).$$

Improved communication through repetition

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The n^{th} power G^n of a graph $G = (V, E)$ has vertex set V^n and $(u, v) \in E(G^n)$ if $u \neq v$ and for each i, $u_i = v_i$ or $(u_i, v_i) \in E(G)$.

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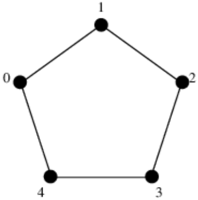

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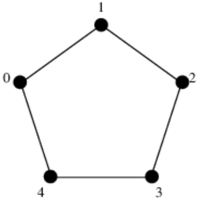

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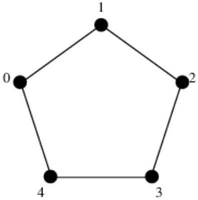

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$\alpha(C_5^2) = 5$ given by the ind. set $\{(0,0), (1,2), (2,4), (3,1), (4,3)\}$.
More generally, if G is self-complementary, then $\alpha(G^2) \geq |G|$.
Indeed, $\{(x, \pi(x)) : x \in V(G)\}$ for π an isomorphism from G to its complement is an independent set in G^2.

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They made the following stronger conjecture, as it would give an analogous result for the Witsenhausen rate for dual source coding.

Conjecture (Alon and Orlitsky '95)

There exists self-complementary Ramsey Cayley graphs.

Going beyond uniform random

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Let $N = 5^d$. A uniform random symmetric $S \subset \mathbb{F}_5^d$ a.a.s. contains the nonzero elements of a subspace of order $\Theta(\log N \log\log N)$ and hence G_S a.a.s. contains a clique of that order.

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It suffices to pick a random self-complementary Cayley graph on $\mathbb{F}_5^d$ in which each possible clique A has $|A-A| \geq |A|\log|A|$ and A has probability of being a clique at most $2^{-\Omega(|A-A|)}$.

Going beyond uniform random

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Theorem

There is a C-Ramsey self-complementary Cayley graph on $\mathbb{F}_5^d$.

For each nonzero $x \in \mathbb{F}_5^d$, randomly pick exactly one of $\{x, 4x\}$ or $\{2x, 3x\}$ to be a subset of the generating set S. This guarantees:

1. S is symmetric.
2. G_S is self-complementary with isomorphism $\phi(x) = 2x$.
3. If $x \in S$, then $2x \notin S$.

(3) implies if A is a clique, then $|A + 2 \cdot A| = |A|^2$, so the Plünnecke-Ruzsa inequality implies

$$|A|^2 = |A + 2 \cdot A| \leq |A + A + A| \leq |A - A|^3 |A|^{-2},$$

yielding $|A - A| \geq |A|^{4/3}$.

Theorem

Every finite vector space of characteristic at least five has a $(2 + o(1))$-Ramsey Cayley graph.

Theorem

Every finite vector space of characteristic $\equiv 1 \pmod 4$ has a self-complementary $(2 + o(1))$-Ramsey Cayley graph.

General random graph models

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Definition: Δ-independent random graphs

Suppose a random graph G is such that each pair e of vertices appears as an edge of G with probability p_e, and appears independently of all edges apart from those in a graph G_e.
We say G is Δ-*independent* if $\Delta(G_e) \leq \Delta$ for each pair e.

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Theorem

Suppose $0 < p < 1$ is fixed. Let G be a Δ-independent random graph on N vertices with $p_e = p$ for all pairs e.

1. If $\Delta = N^{o(1)}$, then $\omega(G) \geq (2 - o(1)) \log_{1/p} N$ a.a.s.
2. If $\Delta = O(1)$, then $\omega(G) \leq O(\log N \log\log N)$ a.a.s.

Conjectures

Conjecture (Alon 1989)

There is a constant C such that every finite group has a Cayley graph which is C-Ramsey.

An important step in this direction is the following:

Toy Conjecture

There is a two-coloring of $\mathbb{F}_2^d \setminus \{0\}$ such that there is no subspace of size Cd whose nonzero elements are monochromatic.

Thank you