Higher-Order Fourier Analysis

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I will be talking mostly about discrete questions and techniques though techniques (and problems) in ergodic theory form an important part of the subject.

Gowers norms on G

Let G be a finite abelian group. Mostly we'll talk about the cases $G = \mathbb{Z}/N\mathbb{Z}$ and $G = \mathbb{F}_p^n$ for $\mathbb{F}_p$ some fixed finite field.

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Let $f : G \to \mathbb{C}$ be a function. Define

$$\|f\|_{U^k} := \Big(\mathbb{E}_{n,h_1,\dots,h_k \in G} \prod_{\vec{\omega} \in \{0,1\}^k} \mathcal{C}^{|\omega|} f(n + \vec{\omega} \cdot \vec{h})\Big)^{1/2^k}.$$

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$$\|f\|_{U^2(G)} := \Big(\mathbb{E}_{n,h_1,h_2} f(n)\overline{f(n+h_1)f(n+h_2)}f(n+h_1+h_2)\Big)^{1/4}.$$

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If $f(x) = e^{2\pi i\phi(x)}$,

$$\|f\|_{U^k}^{2^k} = \mathbb{E}_{h_1,\dots,h_k}\mathbb{E}_x e^{2\pi i \Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_k}\phi(x)},$$

where $\Delta_h\psi(x) := \psi(x) - \psi(x+h)$.

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Exercise: If $f : [N] \to \mathbb{C}$ is 1-bounded and $\|f\|_{U^k} = 1$, then $f(x) = e^{2\pi i \phi(x)}$ for some degree $(k-1)$ polynomial phase $\phi : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

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Theorem (Generalised von Neumann Theorem)

Suppose that $A \subseteq \mathbb{Z}/N\mathbb{Z}$ is a set of size αN such that the number of pairs (x, d) such that $x, x+d, \dots, x+(k-1)d$ is not within εN^2 of $\alpha^k N^2$. Then $\|f_A\|_{U^{k-1}} \gg_k \varepsilon$, where $f_A = 1_A - \alpha$.

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Gowers norms control any 'finite-complexity' linear pattern.

Inverse theorems

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Here $\hat{G}$ is the group of characters $\gamma : G \to \mathbb{C}$, and

$$\hat{f}(\gamma) = \mathbb{E}_{x \in G} f(x)\overline{\gamma(x)} = \frac{1}{|G|}\sum_{x \in G} f(x)\overline{\gamma(x)}.$$

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Therefore

$$\delta^4 \leqslant \sum_\gamma |\hat{f}(\gamma)|^4 \leqslant \sup_\gamma |\hat{f}(\gamma)|^2 \sum_\gamma |\hat{f}(\gamma)|^2 \leqslant \sup_\gamma |\hat{f}(\gamma)|^2.$$

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$$\sum_j \Big| \sum_{x \in P_j} f(x) e^{-2\pi i q_j(x)} \Big| \gg_\delta N. \tag{1}$$

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Inequality (1) is not a necessary and sufficient condition for $\|f\|_{U^3} \approx 1$. It is impossible to check for arithmetic functions such as $f = \mu$.

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Suppose that p is odd and that $f : \mathbb{F}_p^n \to \mathbb{C}$ is 1-bounded. Suppose that $\|f\|_{U^3(\mathbb{F}_p^n)} \geqslant \delta$. Then there is some quadratic phase $q : \mathbb{F}_p^n \to \mathbb{F}_p$ such that

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For the $U^3[N]$-norm, the most obvious generalisation fails: there exist 1-bounded functions $f : [N] \to \mathbb{C}$ with $\|f\|_{U^3[N]} \approx 1$ which do not correlate with any quadratic phase $e^{2\pi i q(x)}$.

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Conversely, if this holds then $\|f\|_{U^3[N]} \approx 1$.

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$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \{x\} & \{z - y\lfloor x \rfloor\} \\ 0 & 1 & \{y\} \\ 0 & 0 & 1 \end{pmatrix}$$

$a = -\lfloor x \rfloor$, $b = -\lfloor y \rfloor$, $c = -\lfloor z - y\lfloor x \rfloor \rfloor$.

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Complexity: $\dim(G)$, structure constants of G, Lipschitz/smoothness properties of F (but *not* anything to do with $p(n)$).

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The notion of polynomial nilsequence, which is important for further analysis, was not fully clarified until a little later.

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In this paper they point out a major oversight in their earlier work from 1984. They say that the need for nilpotent groups was pointed out to them by Furstenberg and Weiss.

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The notion of nilsequence $F(p(n))$ itself (with $p(n) = a^n$) was introduced by Bergelson, Host and Kra (2005).

Higher $U^k[N]$ norms

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Gowers (1997) had previously proven a 'local' version of this.

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Suppose $X = (X, \mu, T)$ *ergodic. They define a (Host-Kra) seminorm* $|\!|\!| f |\!|\!|_{s+1}$ *and a 'factor'* Z_s *of* X *such that* $|\!|\!| f |\!|\!|_{s+1} = 0$ *if and only if* $\mathbb{E}(f|Z_s) = 0$, *and show that* Z_s *is an inverse limit of nilsystems.*

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Szegedy (et al) has given a different proof of the inverse theorem.

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Recall the U^3 inverse theorem over finite fields.

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Suppose that $f : \mathbb{F}_p^n \to \mathbb{C}$ *is* 1*-bounded and that* $\|f\|_{U^3(\mathbb{F}_p^n)} \geqslant \delta$. *Then there is some quadratic phase* $q : \mathbb{F}_p^n \to \mathbb{F}_p$ *such that* $|\mathbb{E}_{x \in \mathbb{F}_p^n} f(x) e^{-2\pi i q(x)/p}| \gg_{\delta,p} 1$.

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Conjecture

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Instead of a 'classical' polynomial phase $\phi(x)/p$ where $\phi : \mathbb{F}_p^n \to \mathbb{F}_p$, one should work with 'non-classical' polynomial phases $\psi : \mathbb{F}_p^n \to \mathbb{R}/\mathbb{Z}$, which may not be the same thing for $p < k$.

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Conjecture (Updated $U^k(\mathbb{F}_p^n)$ inverse conjecture)

Suppose that $f : \mathbb{F}_p^n \to \mathbb{C}$ is 1-bounded and that $\|f\|_{U^{s+1}(\mathbb{F}_p^n)} \geqslant \delta$. Then there is some degree s (non-classical) polynomial phase $\psi : \mathbb{F}_p^n \to \mathbb{R}/\mathbb{Z}$ such that $|\mathbb{E}_{x\in\mathbb{F}_p^n} f(x) e^{-2\pi i \psi(x)}| \gg_{\delta,s,p} 1$.

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Alternative recent proof by Candela, Gonzalez-Sanchez and Szegedy.

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Subsequent advances in additive combinatorics, particularly Sanders' work, allows for the bounds to be improved to quasipolynomial. (Most importantly, $\dim G \ll \log^C(1/\delta)$, $\delta' \sim \exp(-\log^C(1/\delta))$).

Quantitative issues – the $U^3(\mathbb{F}_p^n)$-norm

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Proof gives extremely weak bounds for U^4-norm, and no bounds for U^5 and higher, due to use of ultraproduct arguments.

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Leng–Sah–Sawhney (2024) have given quasipolynomial bounds, essentially matching what is known in the $U^3[N]$ case.

Quantitative issues – finite field U^k for $k \geqslant 4$

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Theorem (TZ 2011)

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It is certainly natural to conjecture that we can take $\delta' = \delta^{C_{s,p}}$.

Further topics - nilsequences

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Recently Leng (2023) made significant progress on the quantitative aspects.

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- Work in the infinitary realm (Klurman, Moreira, Richter)

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- Inverse theorems for Higher-dimensional norms. The simplest one (Austin) is $\mathbb{E}_h \|\Delta_h f\|_{U^2}^4$.