Nonlinearity stability of active suspension model

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Macroscopic behaviour from microscopic laws

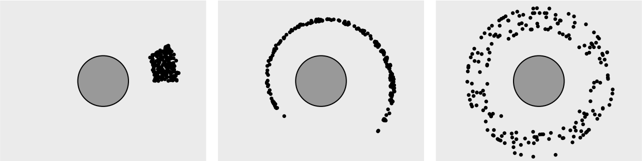
Statistical physics: many particle systems

- Microscopic laws: reversible
- Macroscopic laws: irreversible (thermodynamic)

Kinetic theory: density over phase space (x, v)

- with collisions by Boltzmann and Maxwell
  $\Rightarrow$ H theorem
- collisionless by Jeans (gravitational) and Vlasov (plasmas)
  $\Rightarrow$ reversible
  $\Rightarrow$ Landau (1946) damping

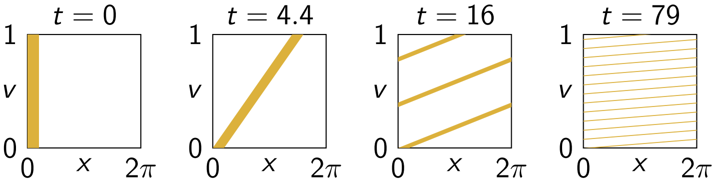
$t = 0$ $t = 10$ $t = 100$



Phase mixing for free transport

Density $f(t,x,v)$ evolves over phase space $(x,v) \in \mathbb{T} \times \mathbb{R}$ as

$$\partial_t f + v\partial_x f = 0.$$



Fourier transform $x \to k$:

$$\partial_t f_k + \mathrm{i}kvf_k = 0 \quad \Rightarrow \quad f_k(t,v) = \mathrm{e}^{-\mathrm{i}kvt} f_k^{\mathrm{in}}(v)$$

Spatial density

$$\rho_k(t) = \int_{v\in\mathbb{R}} f_k(t,v)\,\mathrm{d}v = \int_{v\in\mathbb{R}} \mathrm{e}^{-\mathrm{i}kvt} f_k^{\mathrm{in}}(v)\,\mathrm{d}v$$

decays if f^{in} has regularity.

Active suspension model

Model [Saintillan, Shelley '08]

Active particles (bacteria) in a Stokes fluid described by

- position $x \in \mathbb{T}^3$,
- orientation $v \in \mathbb{S}^2$.

Each particle moves forward $\Rightarrow$ Induced velocity field u.

Density $f(t,x,v)$ evolves as

$$\begin{cases} \partial_t f + (v+u)\cdot\nabla_x f + \nabla_v \cdot \Big(\mathbb{P}_{v^\perp}\left[(\gamma E(u) + W(u))v\right] f\Big) = \nu \Delta_v f, \\ -\Delta_x u + \nabla_x q = \alpha \nabla_x \cdot \displaystyle\int_{\mathbb{S}^2} f(t,x,v)\, v\otimes v \,\mathrm{d}v, \\ \nabla_x \cdot u = 0. \end{cases}$$

$E(u) = \frac{1}{2}\left[\nabla_x u + (\nabla_x u)^T\right]$ and $W(u) = \frac{1}{2}\left[\nabla_x u - (\nabla_x u)^T\right]$

Pusher: $\alpha < 0$ **Puller:** $\alpha > 0$

Oberservations for the active suspension model

Popular model. For pushers, simulations show a phase transition more or less observed in experiments:

- For $\gamma|\alpha|$ small enough, the incoherent state $f(v) = \frac{1}{4\pi}$ is stable.
- For $\gamma|\alpha|$ big enough, emergence of a new collective behaviour. Changes the rheology of the suspension.

Goal: Recover observations analytically

First step: Understand the linearised behaviour

In the incoherent regime, we exhibit a mixing phenomenon, both at $\nu = 0$ and $\nu \gg 1 > 0$ (much harder). The fact that $v \in \mathbb{S}^2$ changes deeply the behaviour with respect to usual settings.

Linearised model

Linearise around the incoherent state:

- Fourier modes $x \to k$ decouple
- Can rescale k to $|k| = 1$ in adimensional form

Fixed mode $k \in \mathbb{S}^2$, perturbation $f = f(t, v)$, $v \in \mathbb{S}^2$, evolves as

$$\begin{cases} \partial_t f + \mathrm{i} v \cdot k f - \dfrac{3\Gamma}{4\pi} v \otimes v : E(u) = \nu \Delta_v f, \\ u = \mathbb{P}_{k^\perp} \mathrm{i} k \Sigma, \\ \Sigma := \epsilon \displaystyle\int_{S^2} f(t, v) v \otimes v \, \mathrm{d}v \end{cases}$$

where $\epsilon = \pm 1$ (pullers $\epsilon = 1$, pushers $\epsilon = -1$) and strength number Γ.

Linear results

Theorem ($\nu = 0$: mixing)

If $\epsilon = 1$ (pullers), for any Γ, as $t \to \infty$

$$|u(t)| = O(t^{-2}), \qquad \|\psi\|_{H^{-1-}} = O(t^{-1}).$$

If $\epsilon = -1$ (pushers), there exists Γ_c such that

- *For $\Gamma < \Gamma_c$, the same stability result holds.*
- *For $\Gamma > \Gamma_c$, there exist unstable eigenmodes.*

Theorem ($\nu > 0$: mixing followed by enhanced dissipation)

If $\epsilon = 1$ or $\epsilon = -1$ and $\Gamma < \Gamma_c$, then for ν small enough

$$|u(t)| + \frac{1}{t}\|\psi\|_{H^{-1-}} \lesssim \min\left(\frac{|\ln t|^M}{t^2}, \mathrm{e}^{-\eta\nu^{\frac{1}{2}}t}\right).$$

Remarks

Remark: Decay due to mixing is at fixed polynomial rate, even for analytic f_{in}.
Strong difference with usual results due to $v \in \mathbb{S}^2$ instead of $v \in \mathbb{R}^3$.
Remark: Contemporary paper by [Albritton-Ohm] on the same model.

- $\nu = 0$: Under stability condition from dispersion relation, L^2 decay as
$$\int_{t=0}^{\infty} |u(t)|^2 (1+t)^{3-\epsilon} \, \mathrm{d}t < \infty$$
- $\nu > 0$: No analogue of our theorem. Only result of enhanced dissipation under stringent assumption $\Gamma \ll \nu^{1/2}$.

Decay by phase mixing and diffusion

Key step

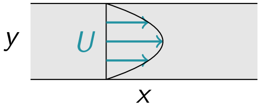
Understand phase mixing and diffusion (Fourier mode $x \to k$):

$$(\partial_t - L_1) f_k = (\partial_t + \mathrm{i}k\cdot v - \nu \Delta_v) f_k = 0, \quad f_k = f_k(t,v), t \in \mathbb{R}^+, v \in \mathbb{S}^2$$

where we can rescale $k \in \mathbb{S}^2$.

Challenge: Phase mixing is degenerate at poles $\pm k$

Similar to mixing through Poiseuille flow:



$$U = \begin{pmatrix} y(1-y) \\ 0 \end{pmatrix}$$

Common theme for phase mixing of trapped particles.

Decay by pure phase mixing

$$(\partial_t - L_1) f = (\partial_t + \mathrm{i} k \cdot v - \nu \Delta_v) f = 0, \qquad f = f(t,v), t \in \mathbb{R}^+, k, v \in \mathbb{S}^2.$$

Proposition (inviscid decay)

For $\nu = 0$ and $\delta > 0$ and weight $F : \mathbb{S}^2 \to \mathbb{R}$

$$\left| \int_{\mathbb{S}^2} f(t,v)\, F(v)\, \mathrm{d}v \right| \lesssim \frac{1}{(1+t)} \|F\|_{H^{1+\delta}} \|f\|_{H^{1+\delta}},$$

$$\left| \int_{\mathbb{S}^2} f(t,v)\, F(v)\, \nabla(k \cdot v)\, \mathrm{d}v \right| \lesssim \frac{1}{(1+t)^2} \|F\|_{H^{2+\delta}} \|f\|_{H^{2+\delta}}.$$

Idea: Solve explicitly and use stationary phase.

Decay by phase mixing and diffusion

$$(\partial_t - L_1)f = (\partial_t + \mathrm{i}k \cdot v - \nu\Delta_v)f = 0, \quad f = f(t, v), t \in \mathbb{R}^+, k, v \in \mathbb{S}^2.$$

Small degenerate diffusion $(0 < \nu \ll 1)$?

(collisions in kinetic theory (Boltzmann/Landau operator), viscosity in fluids)

Hypocoercivity: Decay by combination of transport and degenerate dissipation.

Decay rate for $\nu \ll 1$: Faster as simple diffusion as

- transport pushes perturbations to high Fourier frequencies,
- dissipation is faster for high Fourier frequencies.

Proposition (enhanced dissipation)

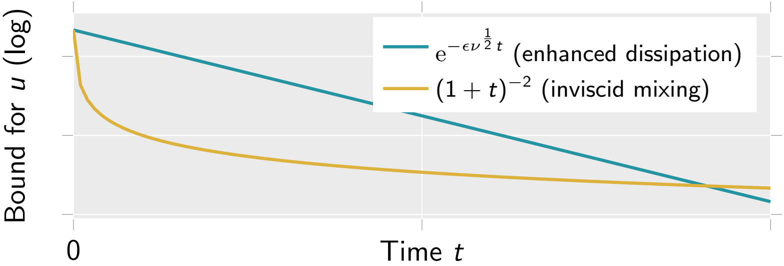
There exists $\nu_0, \lambda > 0$ *such that for* $0 < \nu < \nu_0$

$$\|f(t,\cdot)\|_{L^2(\mathbb{S}^2)} \lesssim \mathrm{e}^{-\lambda\nu^{\frac{1}{2}}t}\|f^{\mathrm{in}}\|_{L^2(\mathbb{S}^2)}.$$

Idea: Use functional with commutator brackets.

Decay of induced velocity field u (macroscopic quantity)

Velocity field $u(t) = \int_{\mathbb{S}^2} f(t,v)\,F(v)\,\nabla(k\cdot v)\,\mathrm{d}v$ (macroscopic)



Proposition (Persistence of phase mixing)

For $0 < \nu \ll 1$ *and* $t \lesssim \nu^{-1/2}|\log\nu|$

$$\left|\int_{\mathbb{S}^2} f(t,v)\,F(v)\,\mathrm{d}v\right| \lesssim_{\log} \frac{1}{(1+t)}\|F\|_{H^{1+\delta}}\|f\|_{H^{1+\delta}},$$

$$\left|\int_{\mathbb{S}^2} f(t,v)\,F(v)\,\nabla(k\cdot v)\,\mathrm{d}v\right| \lesssim_{\log} \frac{1}{(1+t)^2}\|F\|_{H^{2+\delta}}\|f\|_{H^{2+\delta}}.$$

Proof ideas

Consider the hypocoercive functional (a, b, c suitable constants):

$$E(t) = \frac{1}{2}\Big[\|f\|^2 + a\nu t\|\nabla f\|^2 + 2b\nu t^2 \Re\langle \mathrm{i}\nabla(k\cdot v)f, \nabla f\rangle + c\nu t^3\|\nabla(k\cdot v)f\|^2\Big]$$

In considered time-frame:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \frac{\nu}{2}\|\nabla f\|^2 + \frac{a\nu^2 t}{2}\|\nabla\nabla f\|^2 + \frac{b\nu t^2}{2}\|\nabla(k\cdot v)f\|^2 + \frac{c\nu^2 t^3}{2}\|\nabla(\nabla(k\cdot v)f)\|^2 \lesssim \text{OK}$$

Enhanced dissipation follows from interpolation

Lemma (interpolation)

For $\sigma \in (0, 1]$

$$\sigma^{1/2}\|g\|^2 \leq \frac{\sigma}{2}\|\nabla g\|^2 + 2\|\nabla(k\cdot v)\,g\|^2.$$

Proof ideas for mixing

Use vector-field method:

Inviscid case ($\nu = 0$)

Consider Jf for $J = \nabla + \mathrm{i}t\nabla(k \cdot v)$:

$$(\partial_t + \mathrm{i}k \cdot v)f = 0 \quad \Rightarrow \quad (\partial_t + \mathrm{i}k \cdot v)Jf = 0$$

Control on Jf $\Rightarrow$ time decay

With viscosity:

$$(\partial_t - L_1)Jf + \nu Jf = 2\mathrm{i}\nu t\big(\nabla(k \cdot v)f + (k \cdot v)\nabla f\big).$$

Expected bounds

$$\|f(t)\|_{L^2} \lesssim 1, \quad \|\nabla f(t)\|_{L^2} \lesssim t$$

would yield

$$\|Jf(t)\|_{L^2} \le C\left(1 + \nu \int_0^t s(1+s)\,\mathrm{d}s\right), \qquad \forall t \le \nu^{-1/2}.$$

Proof ideas for mixing

Idea: Use viscosity adapted vector fields

$$J_\nu f = \alpha(t)\nabla f + \mathrm{i}\beta(t)\nabla(k\cdot v)f$$

where $\beta' = \alpha$ and $\alpha' = -2i\nu\beta$ so that

$$\alpha(t) = \cosh(\sqrt{-2\mathrm{i}\nu}\,t) \text{ and } \beta(t) = \frac{1}{\sqrt{-2\mathrm{i}\nu}}\sinh(\sqrt{-2\mathrm{i}\nu}\,t).$$

New error

$$\Big(\partial_t + \mathrm{i}(k\cdot v) - \nu\Delta\Big)J_\nu f + \nu J_\nu f = 2\mathrm{i}\beta\nu\nabla([k\cdot v - 1]f)$$

localised **away** from pole $v = k$.

Volterra equation

To conclude for the linearised evolution, use Duhamel's formula to get Volterra equation for u:

$$u(t) + \int_0^t K_\nu(t-s)u(s)\,\mathrm{d}s = g(t)$$

where

$$K_\nu(t)w \cdot \bar{w} = \frac{3\epsilon\Gamma}{4\pi}\int_{\mathbb{S}^2} \mathrm{e}^{L_1 t}(\mathbb{P}_{k^\perp}k \cdot w)(\mathbb{P}_{k^\perp}k \cdot \bar{w})\,\mathrm{d}v$$

$$g(t) = \mathrm{i}\epsilon \int_{S^2} \mathrm{e}^{L_1 t} f_{\mathrm{in}}(k,v)\mathbb{P}_{k^\perp}k\,\mathrm{d}v$$

Key point: Obtain $O(t^{-2})$ decay for u!
Steps:

1. Prove $O(t^{-2})$ decay for K_ν and g
2. Identify condition on Γ to transfer decay to u

Transfer of decay in Volterra equation

Classical theory for Volterra equation
$u(t) + \int_0^t K(t-s)u(s)\,\mathrm{d}s = g(t)$:

Theorem (Paley-Wiener, see [Gripenberg et al])

If $g \in L^p(\mathbb{R}^+)$ and $K \in L^1(\mathbb{R}^+)$, and if its Laplace transform satisfies

$$\det(I + \mathcal{L}K(z)) \neq 0, \qquad \forall \Re z \geq 0 \tag{Lap}$$

then the Volterra equation has a unique solution in $u \in L^p(\mathbb{R}^+)$.

Not quantitative and for exponential decay. We show

Theorem (Quantitative version)

If $g, K \in O(t^{-\alpha})$, $\alpha > 1$ and (Lap), then $u \in O(t^{-\alpha})$.

Remark: Already known? Various quantitative statements in literature.

Solution of the Volterra equation

Our proof is to write

$$(\tilde{u}, \tilde{g}) := (1 + \epsilon t)^{\alpha}(u, g),$$

$$k(t, s) := \left(\frac{1 + \epsilon t}{1 + \epsilon s}\right)^{\alpha} K(t - s) 1_{s<t},$$

$$\tilde{u}(t) + \int_{s=0}^{t} k(t, s)\tilde{u}(s)\,\mathrm{d}s = \tilde{g}(t).$$

Aim: Show that $\tilde{u}$ is bounded knowing that $\tilde{g}$ is bounded.
Use that k satisfies

$$k(t, s) = 0 \quad \text{for } s \geq t, \qquad \|k\| := \sup_t \int_{\mathbb{R}^+} |k(t, s)|\,\mathrm{d}s < \infty.$$

This forms a *Banach algebra* for products

$$k_1 \star k_2(t, s) = \int_{\tau=0}^{\infty} k_1(t, \tau) k_2(\tau, s)\,\mathrm{d}\tau$$

Resolvent for Volterra equation

In this algebra, find the *resolvent* r satisfying

$$r + r \star k = r + k \star r = k.$$

If k has a resolvent the solution is $\tilde{u} = \tilde{g} - r \star \tilde{g}$.
Obtain the resolvent for small enough ϵ as perturbation from a von Neumann series of the kernel $K(t-s)1_{s<t}$ which has a resolvent $R(t-s)1_{s<t}$.

Last point: Spectral condition (Lap)
Use complex analysis for a Penrose style argument. Here one studies the winding number of $\det(I + \mathcal{L}K)$.

Nonlinear stability result

Theorem

Let $s > \frac{7}{2}$*. Assume linear stability.* $\exists C_0, \nu_0, \delta_0 > 0$ *such that* $\forall \nu \le \nu_0$ *and all initial data* ψ^{in}

$$\|\psi^{in}\|_{H^s_x L^2_p} \le \delta_0 \nu^{\frac{3}{2}}$$

there exists a global solution ψ *satisfying*

$$\sup_{t \ge 0} \|\psi(t)\|^2_{H^s_x L^2_p} + \nu \int_0^\infty \|\nabla_p \psi(t)\|^2_{H^s_x L^2_p} \,\mathrm{d}t \le C_0\, \nu^{-1} \|\psi^{in}\|^2_{H^s_x L^2_p}.$$

Difficulties

Recall full equation:

$$\begin{cases} \partial_t f + (v+u)\cdot\nabla_x f + \nabla_v \cdot \Big(\mathbb{P}_{v^\perp}\left[(\gamma E(u) + W(u))v\right] f\Big) = \nu\Delta_v f, \\ -\Delta_x u + \nabla_x q = \alpha \nabla_x \cdot \displaystyle\int_{\mathbb{S}^2} f(t,x,v)\, v\otimes v \,\mathrm{d}v, \\ \nabla_x \cdot u = 0. \end{cases}$$

Main difficulty: Cannot treat $u \cdot \nabla_x f$ as error term in linear theory (x regularity)

- Need to use $\nabla \cdot u = 0$!
- Need to take all modes together

Advection-diffusion equation

Given velocity field v with (bootstrap) assumption

$$\sup_{t\geq 0}\|v(t)\|_{H^s}+\left(\int_0^\infty \|v(t)\|_{H^s}^2\mathrm{d}t\right)^{\frac{1}{2}}\leq \epsilon\nu^{\frac{5}{4}} \tag{H}$$

Evolution

$$\partial_t g+(v+p)\cdot\nabla_x g=\nu\Delta_p g.$$

Theorem

Let $s>\frac{5}{2}$, $0<s'<s+\frac{1}{4}$. $\exists C_0,\epsilon,\nu_0,\eta_1>0$. For $\nu\leq\nu_0$:

$$\|g(t)\|_{H^s_xL^2_p}\leq C_0\mathrm{e}^{-\eta_1\nu^{\frac{1}{2}}t}\|g^{\mathrm{in}}\|_{H^s_xL^2_p},$$

$$\sum_{k\neq 0}|k|^{2s'}\left|V_k[g_k(t)]\right|^2\lesssim\left(\frac{\nu^{\frac{1}{2}}}{\min\{1,\nu^{\frac{1}{2}}t\}}\right)^3\|g^{\mathrm{in}},\nabla_p g^{\mathrm{in}},\nabla_p^2 g^{\mathrm{in}}\|_{H^s_xL^2_p}^2.$$

Ideas for the advection-diffusion equation

Study each Fourier mode k: Need long time results for enhanced dissipation and mixing (with localisation).
For enhanced dissipation, functional for mode k

$$E_{\chi,k}(Y_k) = \|Y_k\chi\|^2 + \left(\frac{\nu}{|k|}\right)^{\frac{1}{2}} a_k\|\nabla_p Y_k\chi\|^2 + 2b_k\Re\langle \mathrm{i}\nabla_p(p\cdot\hat{k})Y_k\chi, \nabla_p Y_k\chi\rangle + \left(\frac{\nu}{|k|}\right)^{-\frac{1}{2}} c_k\|\nabla_p(p\cdot\hat{k})Y_k\chi\|^2$$

where $(a_k, b_k, c_k) := (a, b, c)(h)$ with $h = \nu^{\frac{1}{2}}|k|^{\frac{1}{2}}t$ and

$$a(h) = A\min(h,1), \quad b(h) = B\min(h^2,1), \quad c = C\min(h^3,1).$$

Covering the advection

Use summed quantity:

$$E_{\chi,s}(Y) = \sum_k |k|^{2s} E_{\chi_k,k}(Y_k)$$

Simplified typical error term from velocity field v (no localisation):

$$\begin{aligned}
&\sum_{k,\ell} |k|^{2s} \Re\langle \mathrm{i} k v_{k-\ell} Y_\ell, Y_k\rangle \\
&= \frac{1}{2} \sum_{k,\ell} \Re \left\langle \mathrm{i} \left(|k|^{2s} k - |\ell|^{2s} \ell \right) \cdot v_{k-\ell} Y_\ell, Y_k \right\rangle \\
&= \frac{1}{2} \sum_{k,\ell} \Re \left\langle \mathrm{i} \left(|k|^{2s} - |\ell|^{2s} \right) v_{k-\ell} \cdot \ell Y_\ell, Y_k \right\rangle
\end{aligned}$$

Use the gain from the difference.