Synergies between analysis, geometry, mechanics, and topology in nonlinear elasticity theory

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Jack Carr Annual Lecture
22 January 2025

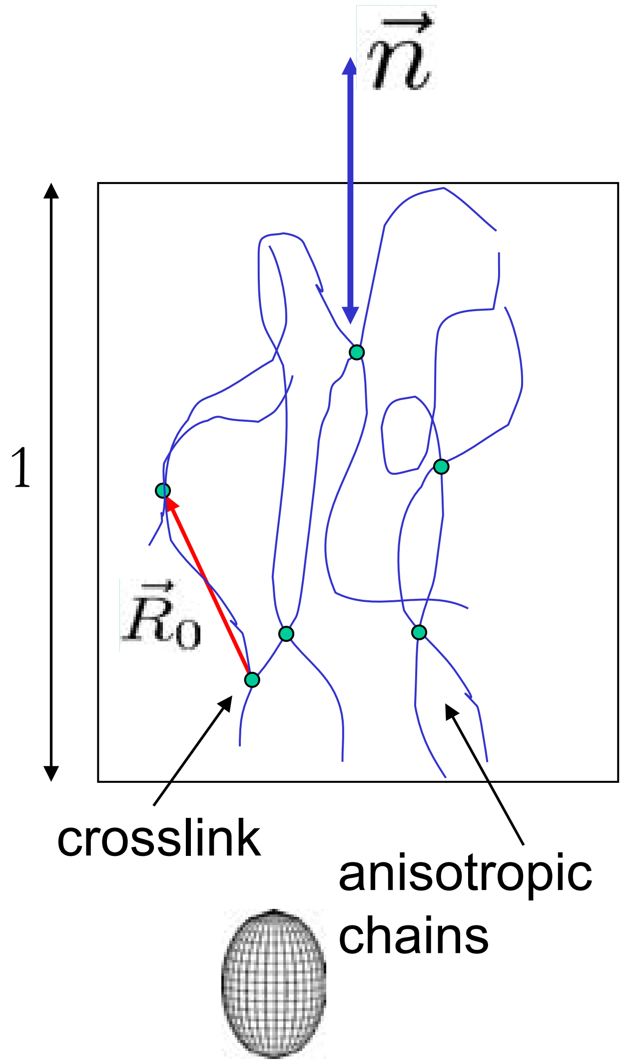
Agencia Nacional de Investigación y Desarrollo
FONDECYT 1231401 - Center for Mathematical Modeling Basal FB210005




Nematic elastomers

Rubbery networks composed of long, crosslinked polymer chains that are also liquid crystalline.

- M. Warner & E.M. Terentjev: Nematic elastomers – a new state of matter?, *Progress in Polymer Science* **21** (1996) 853–891.
- M. Warner & E.M. Terentjev: *Liquid Crystal Elastomers*, Clarendon Press, Oxford, 2003.
- A. DeSimone & G. Dolzmann: Macroscopic response of nematic elastomers via relaxation of a class of $SO(3)$-invariant energies, *Arch. Rational Mech. Anal.* **161** (2002) p. 181.
- S. Conti, A. DeSimone & G. Dolzmann: Soft elastic response of stretched sheets of nematic elastomers: a numerical study, *J. Mech. Phys. Solids* **50** (2002) p. 1431.



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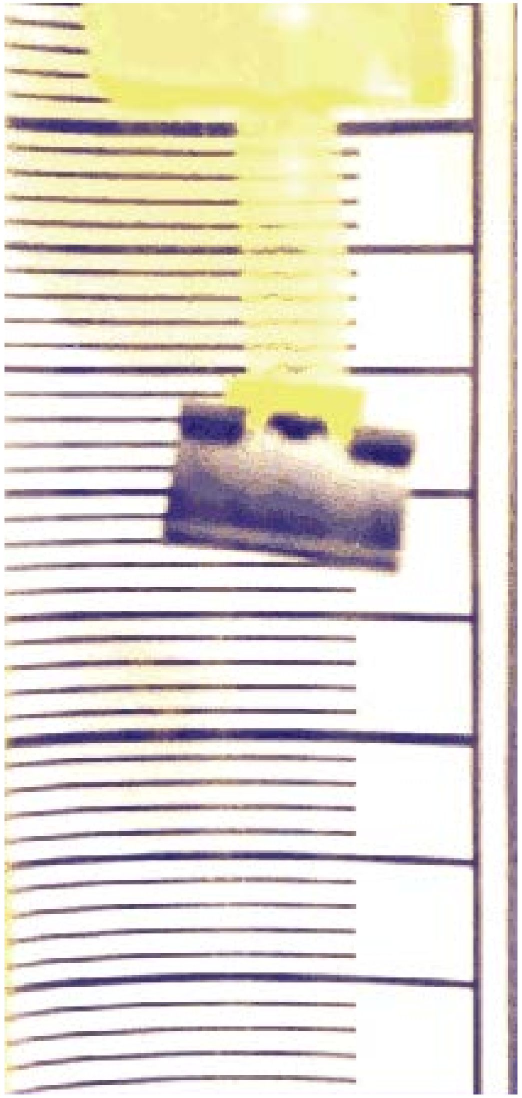
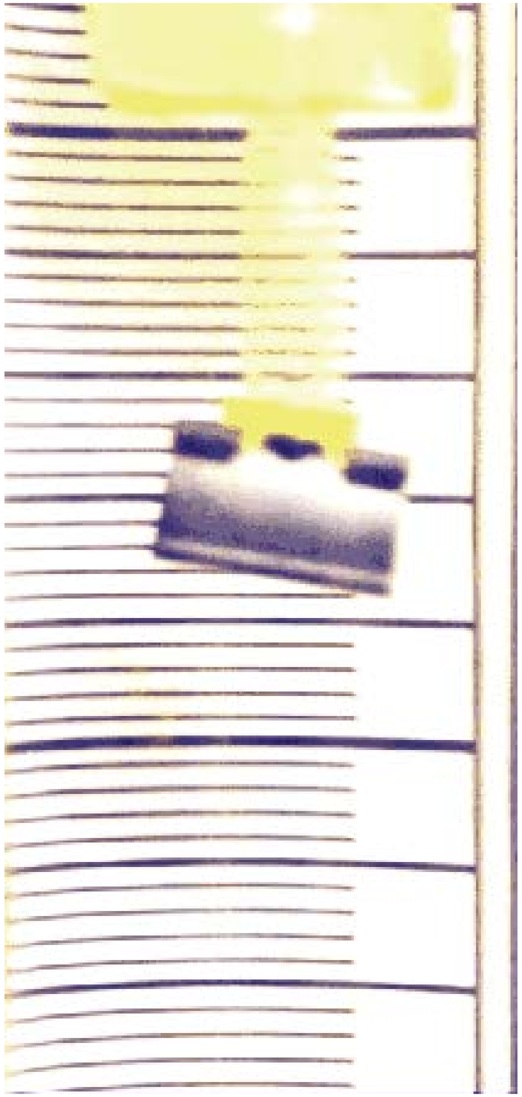
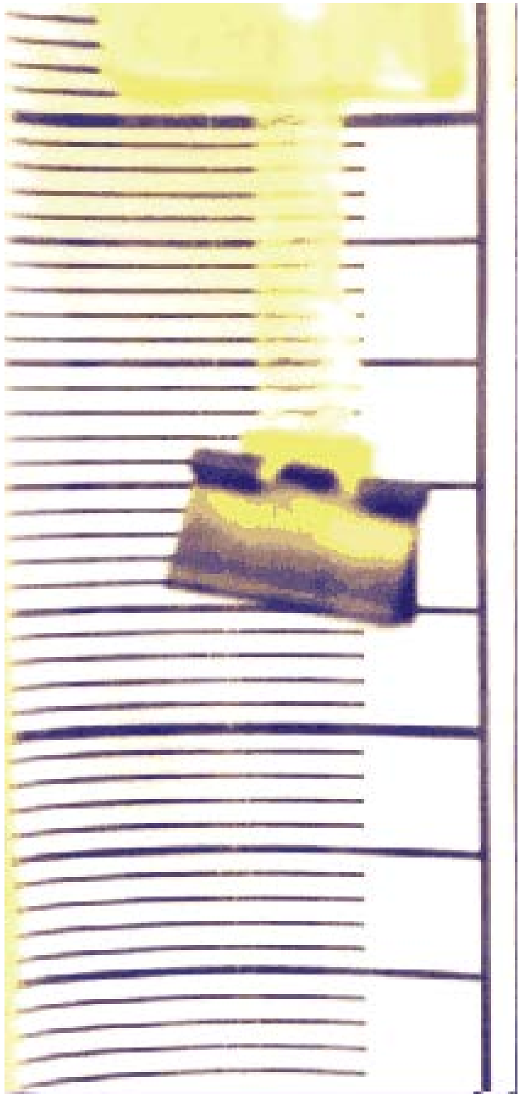
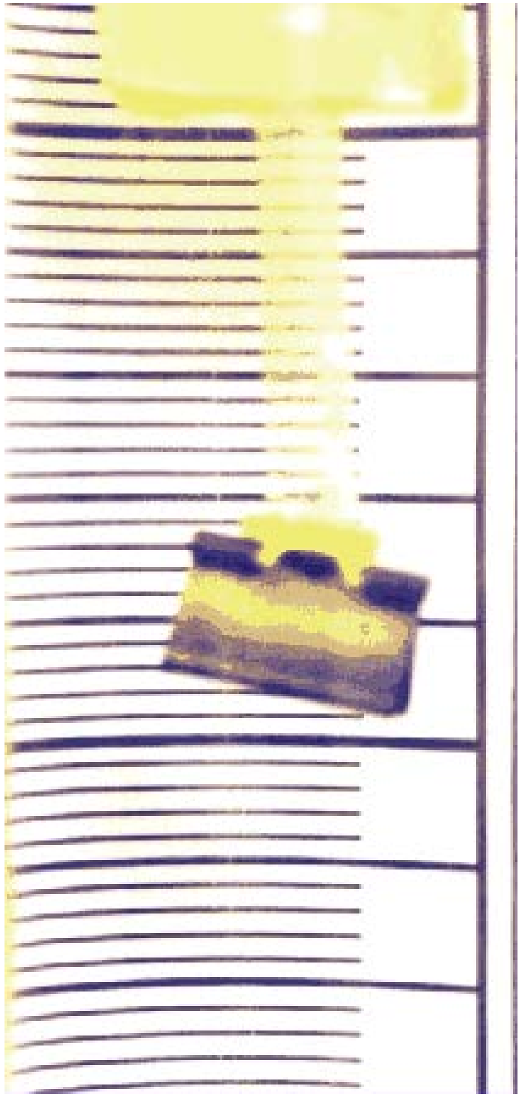
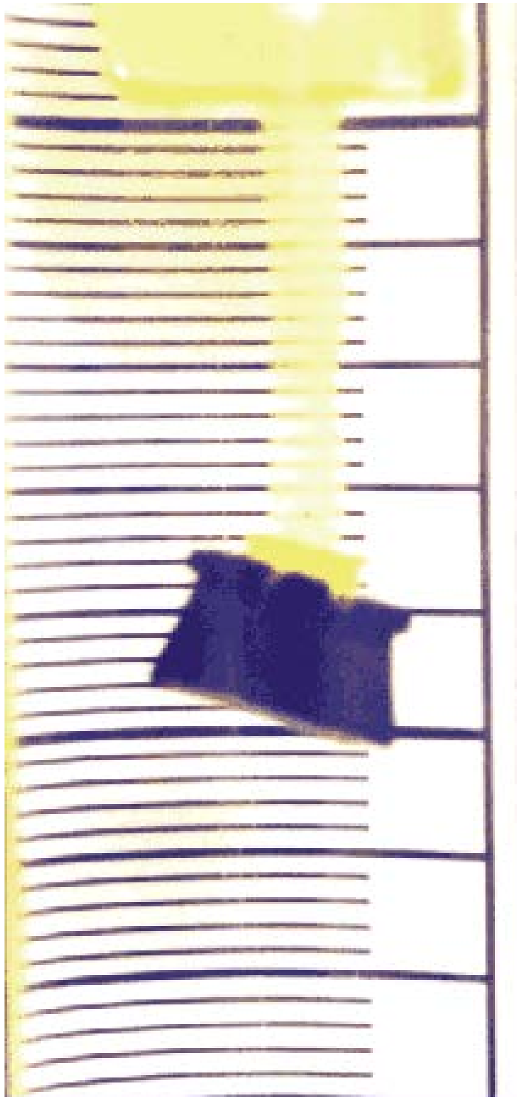
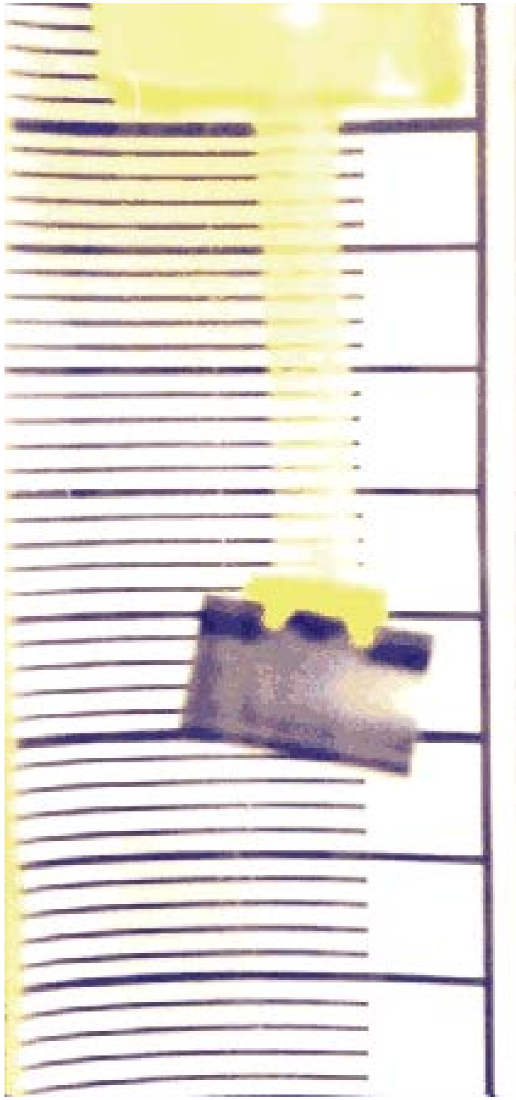


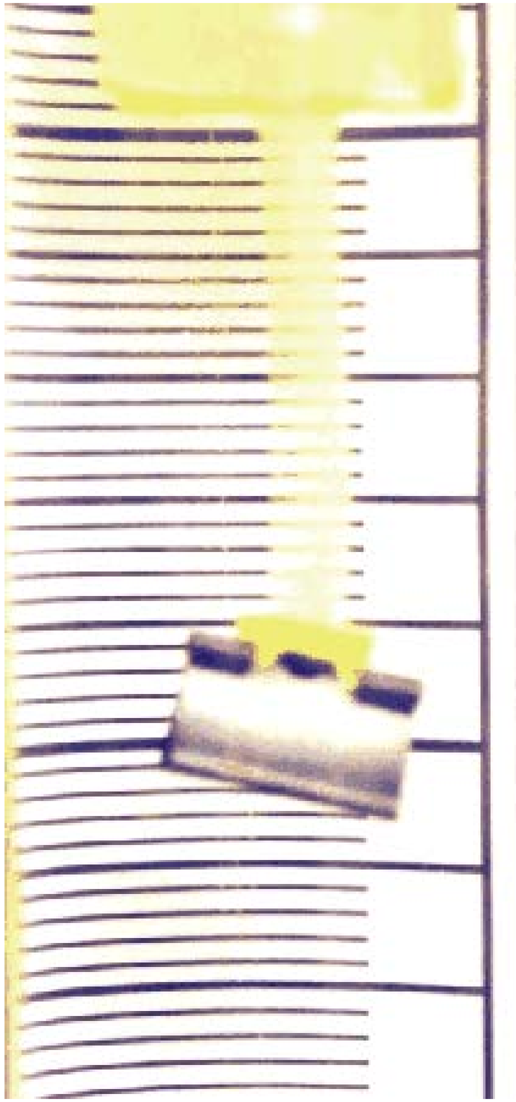


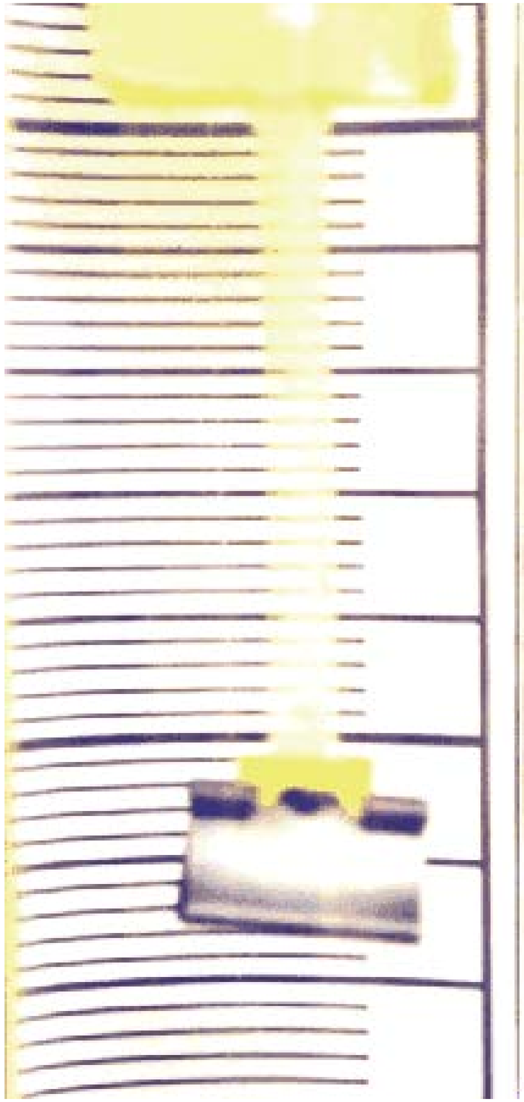


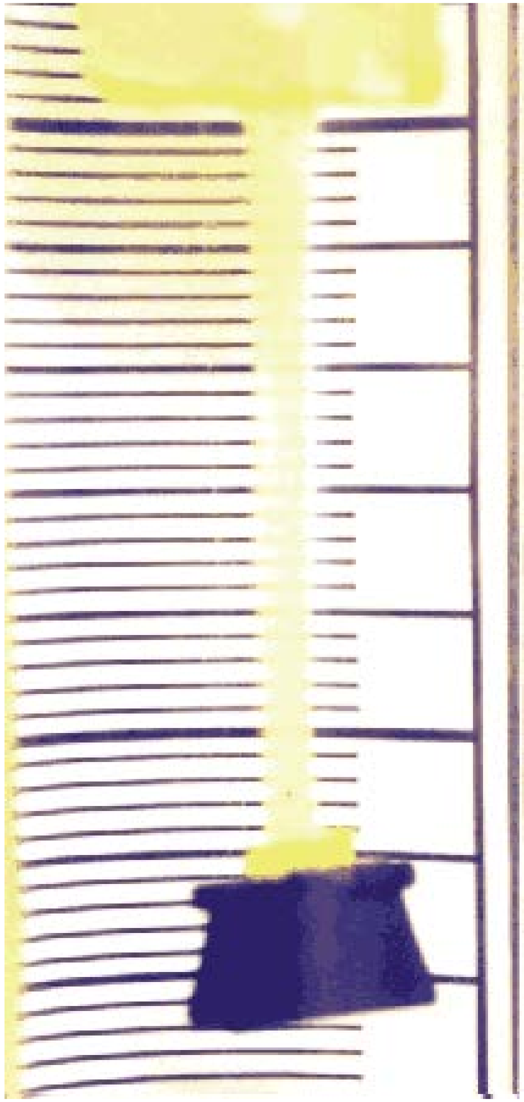


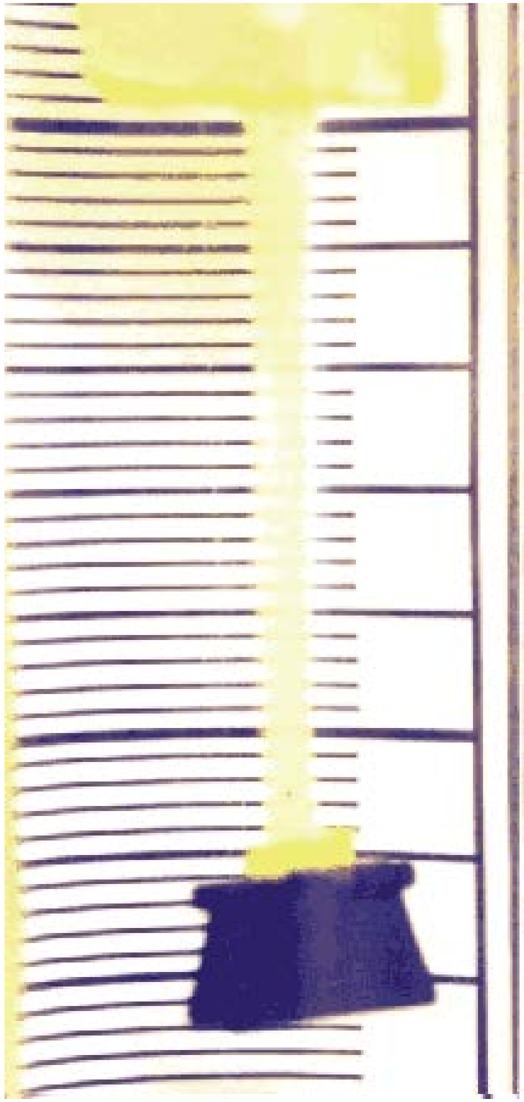


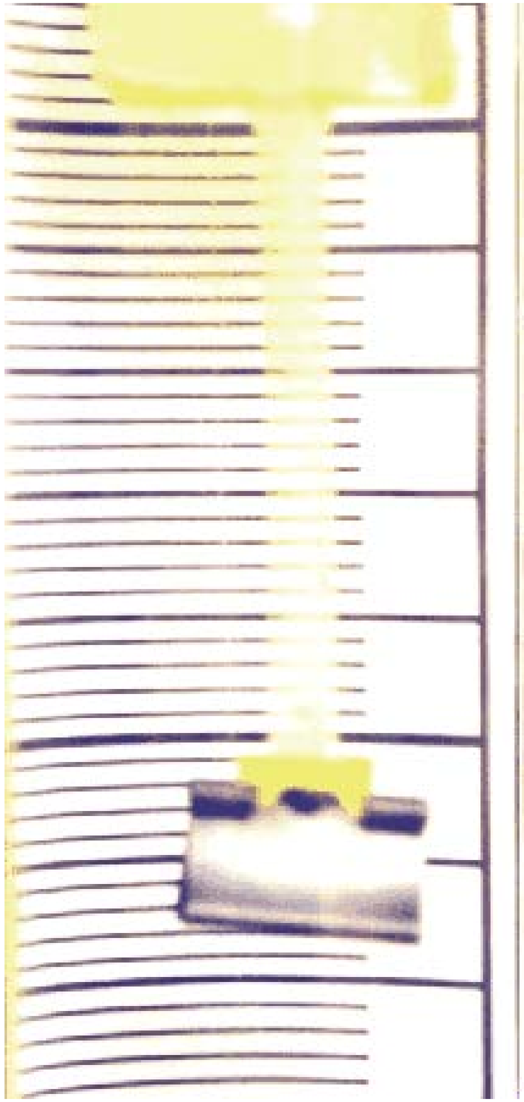


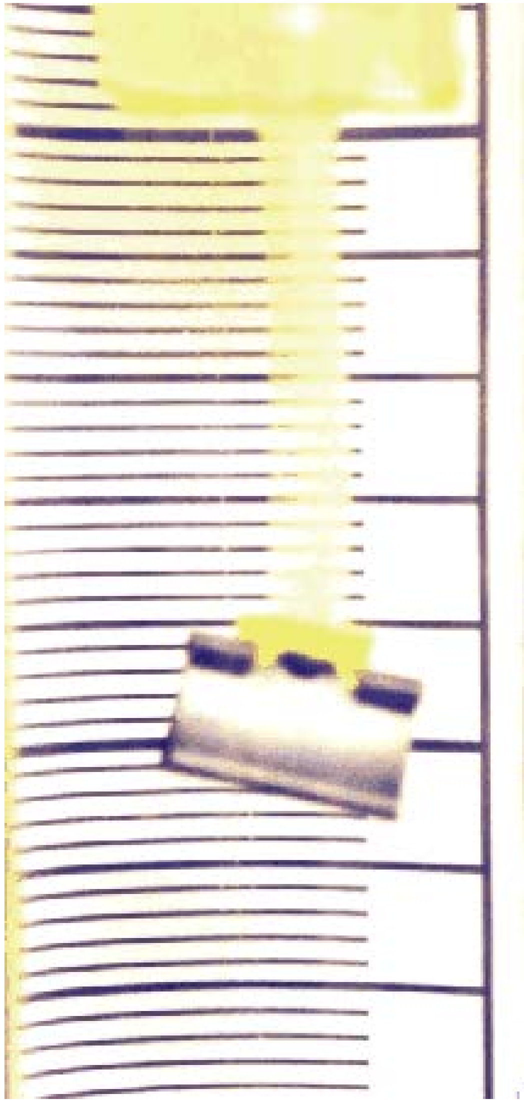


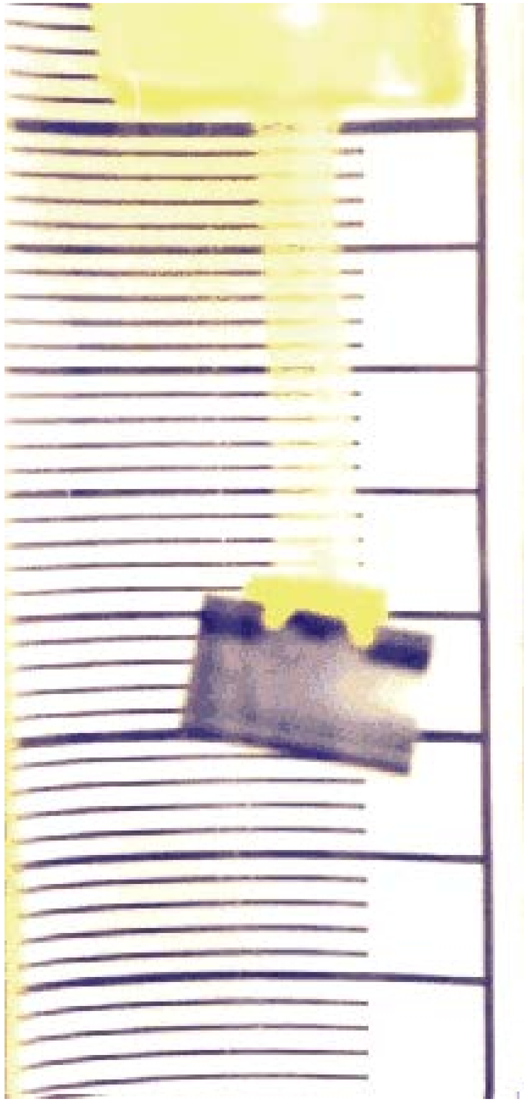


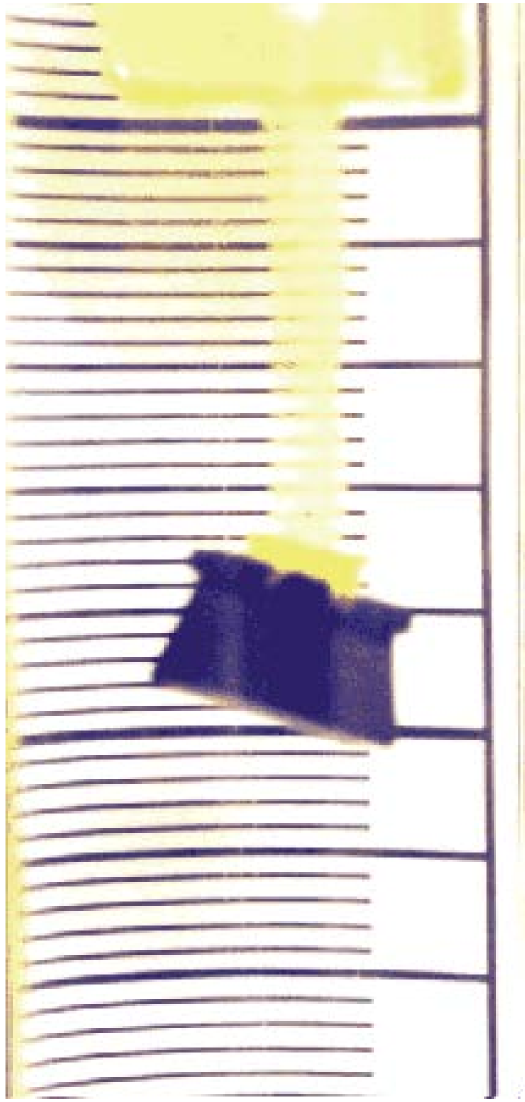


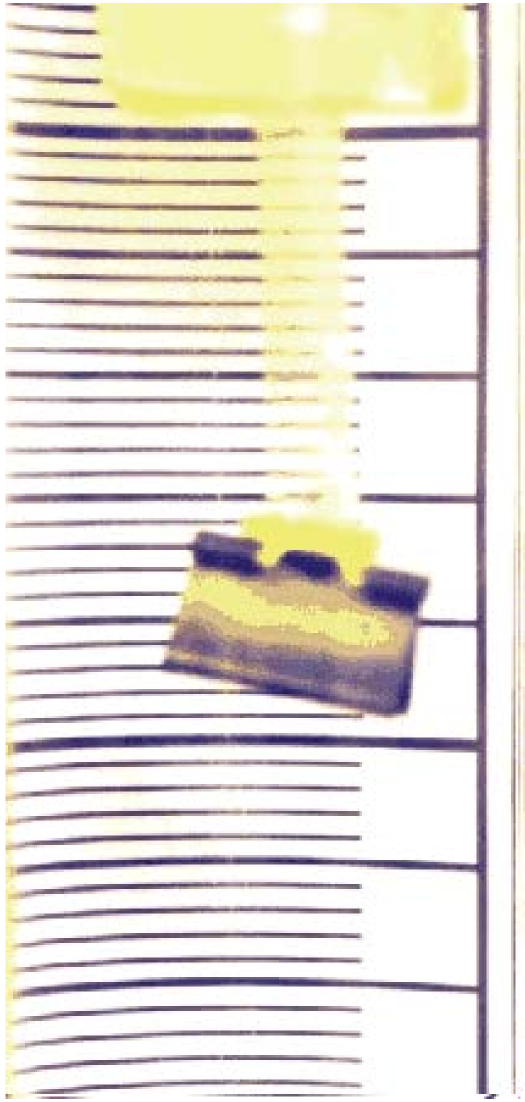


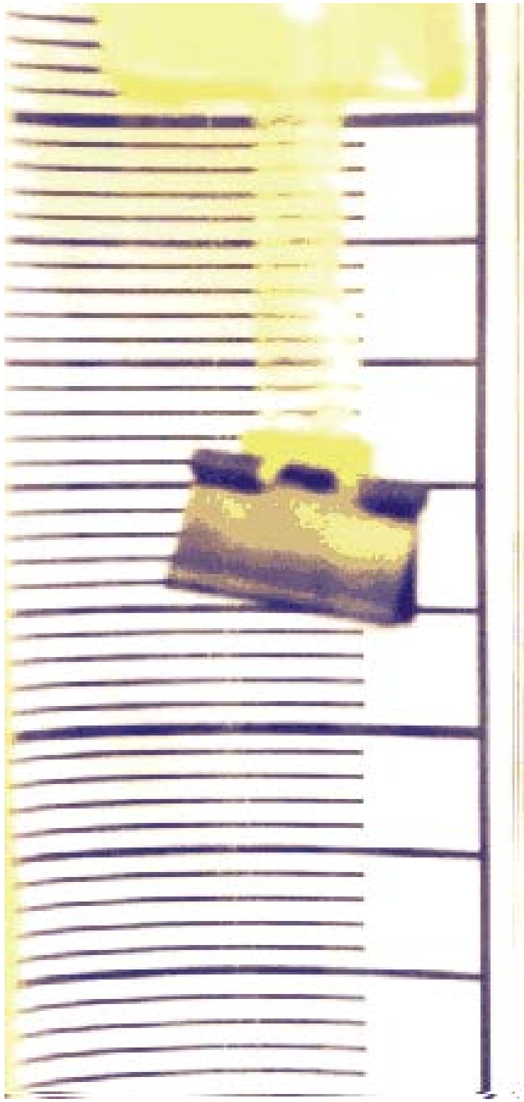


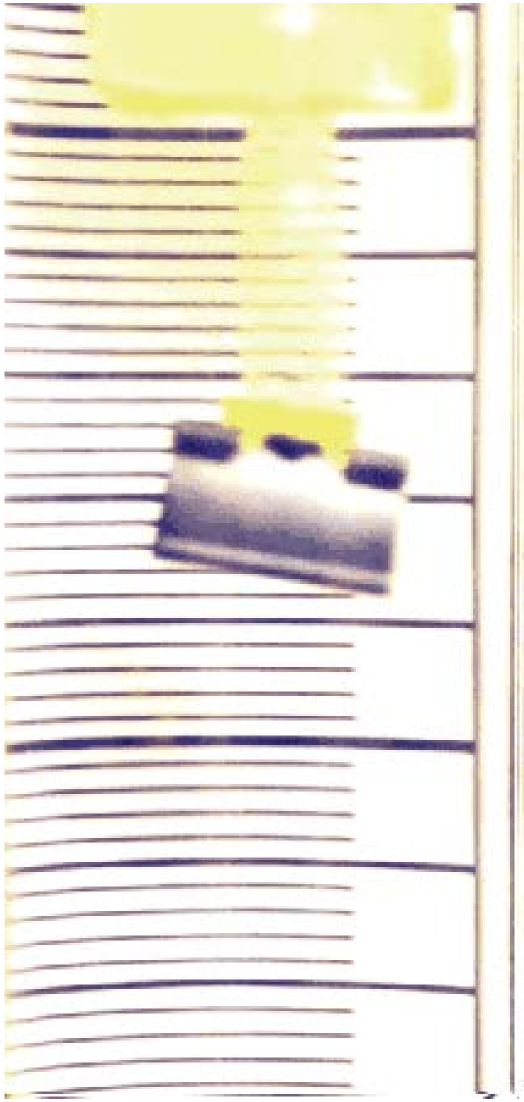


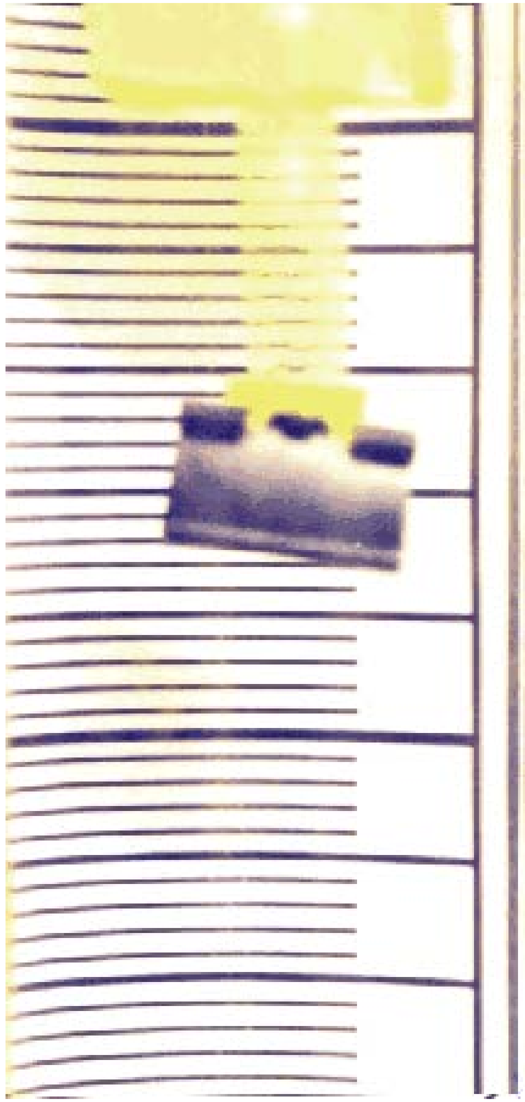












Applications

- Chemical, mechanical, and bio-medical sensors
- Microfluidic pumps, valves, mixers
- Mirrorless, tuneable lasers
- Soft ferro-electrics

[Mark Warner (Cavendish Lab., Cambridge),
13th International Ferro-electric Liquid Crystals Conference (2011)]

I. Kundler & H. Finkelmann (1995)

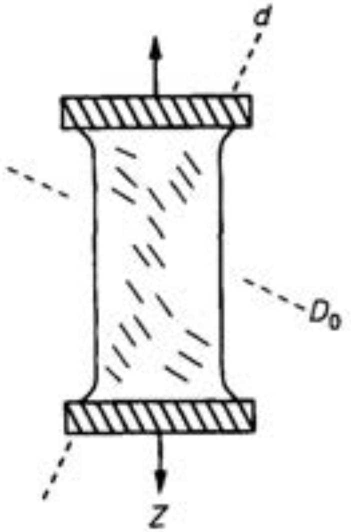


Fig. 3a.

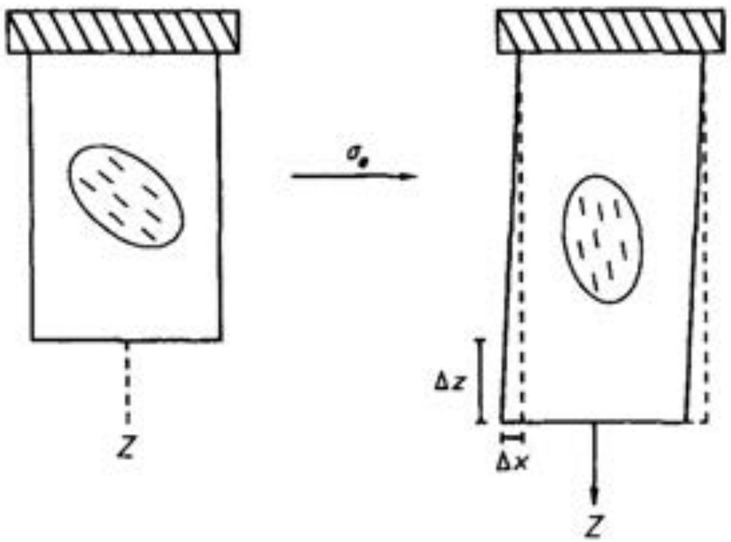


Fig. 3b.

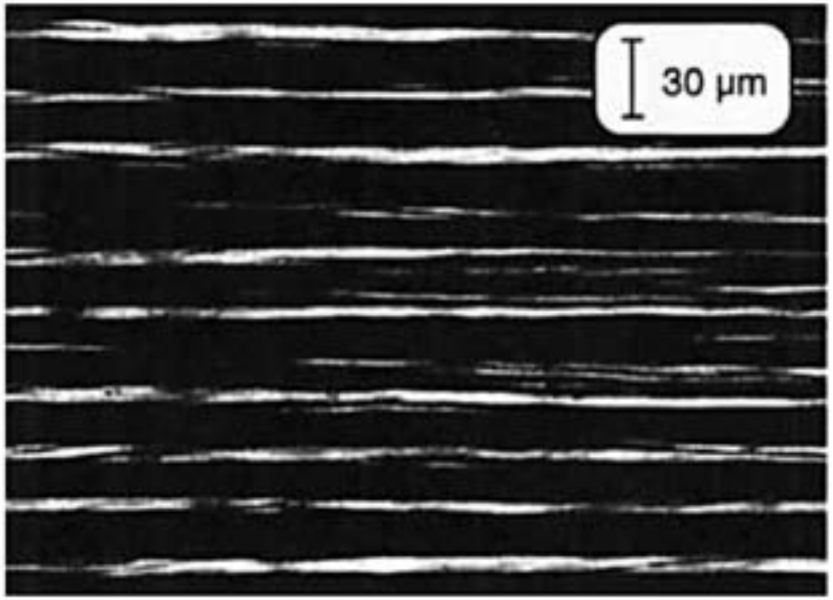
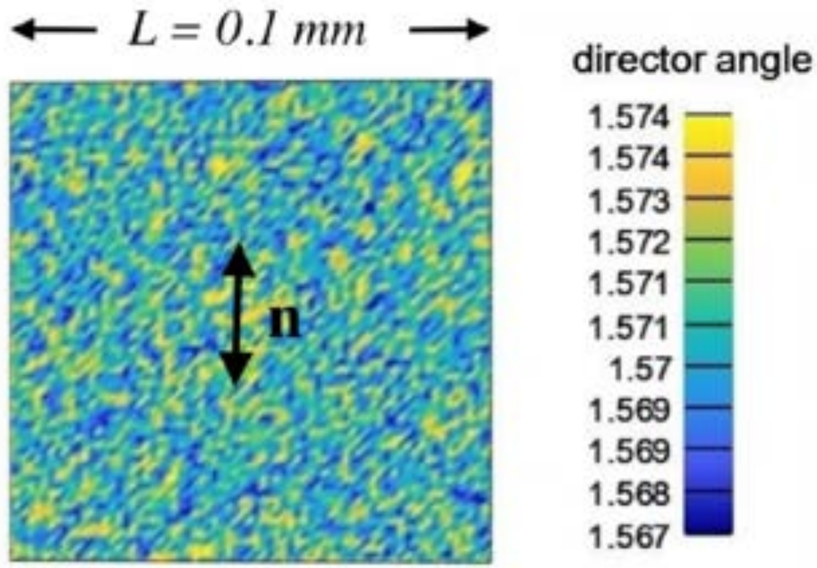
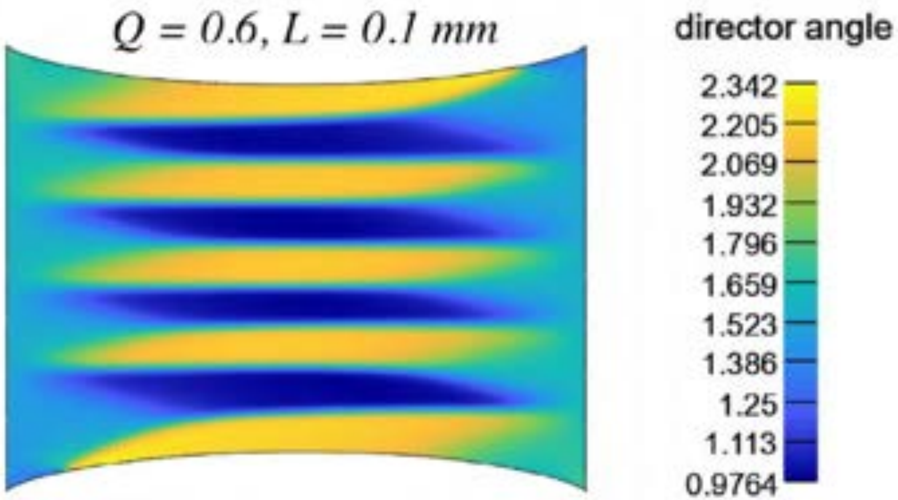


Fig. 4. Periodic pattern formation within the extended elastomer, $\Theta_0 = 90°$

R. Poudel, Y. Sengul & A. Mihai (2024)





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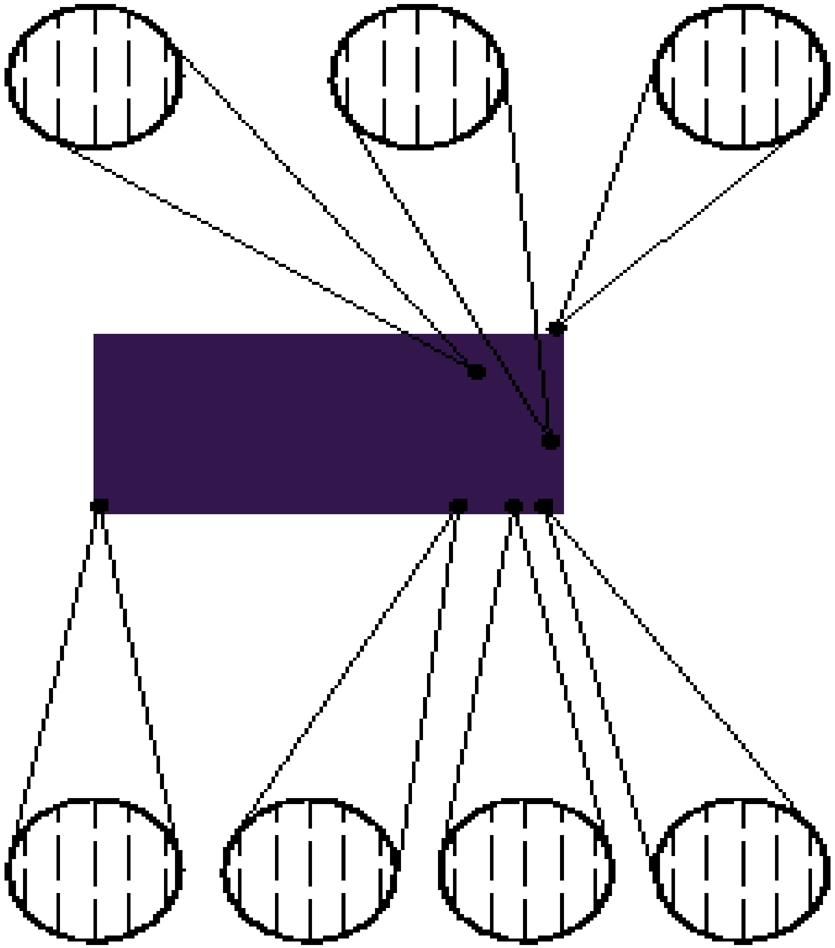

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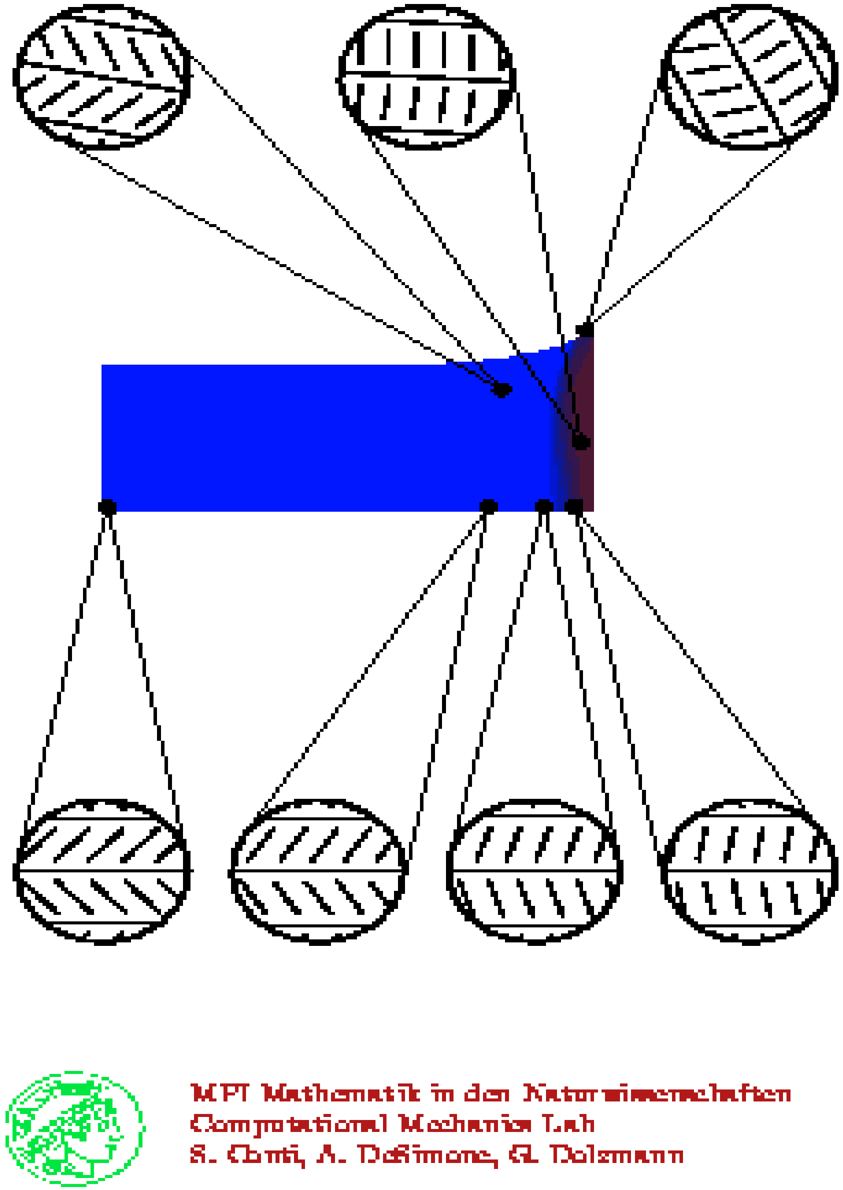

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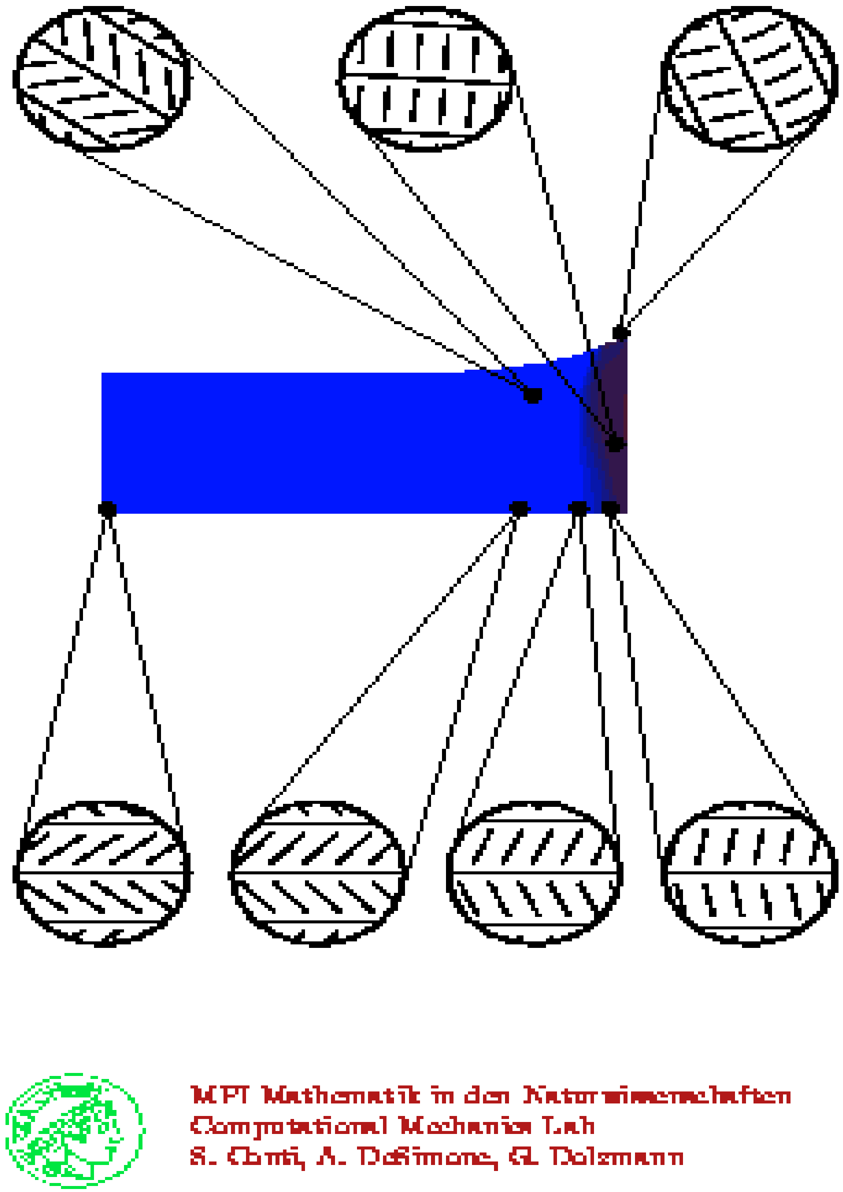

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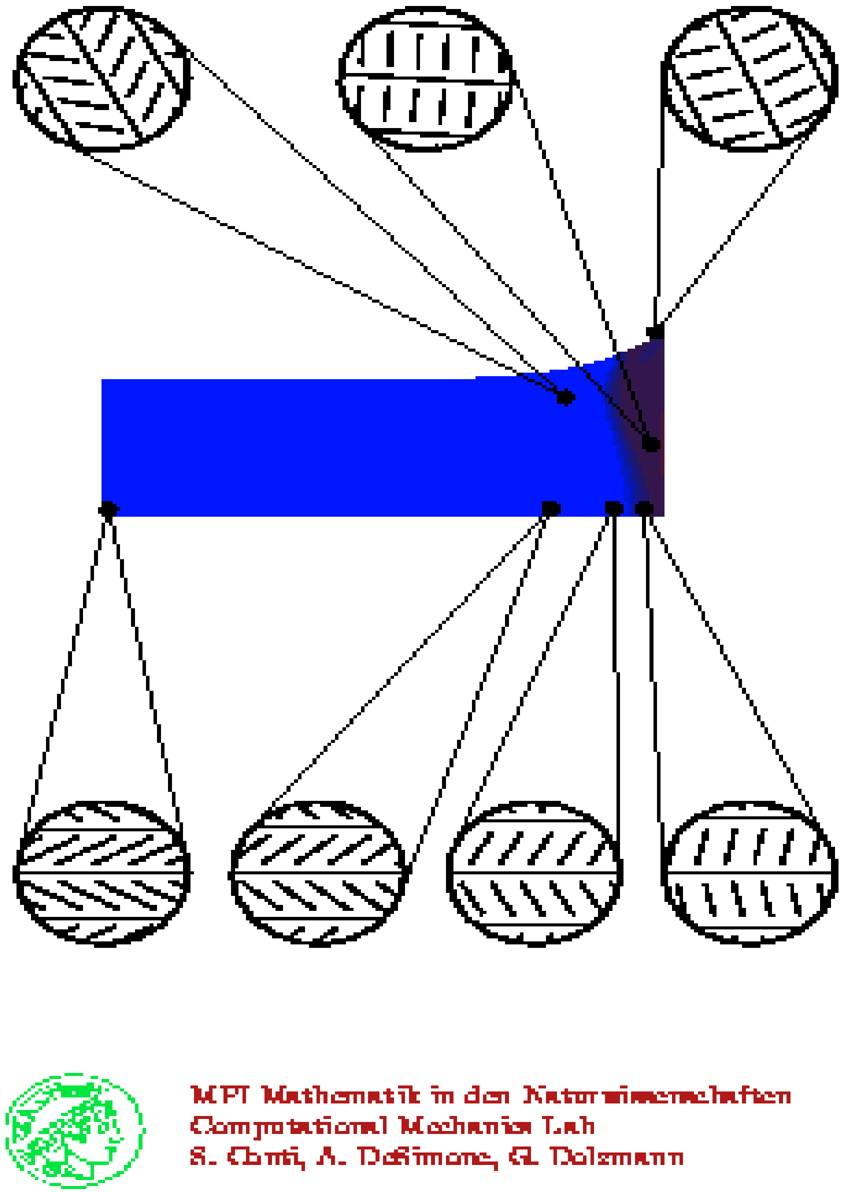

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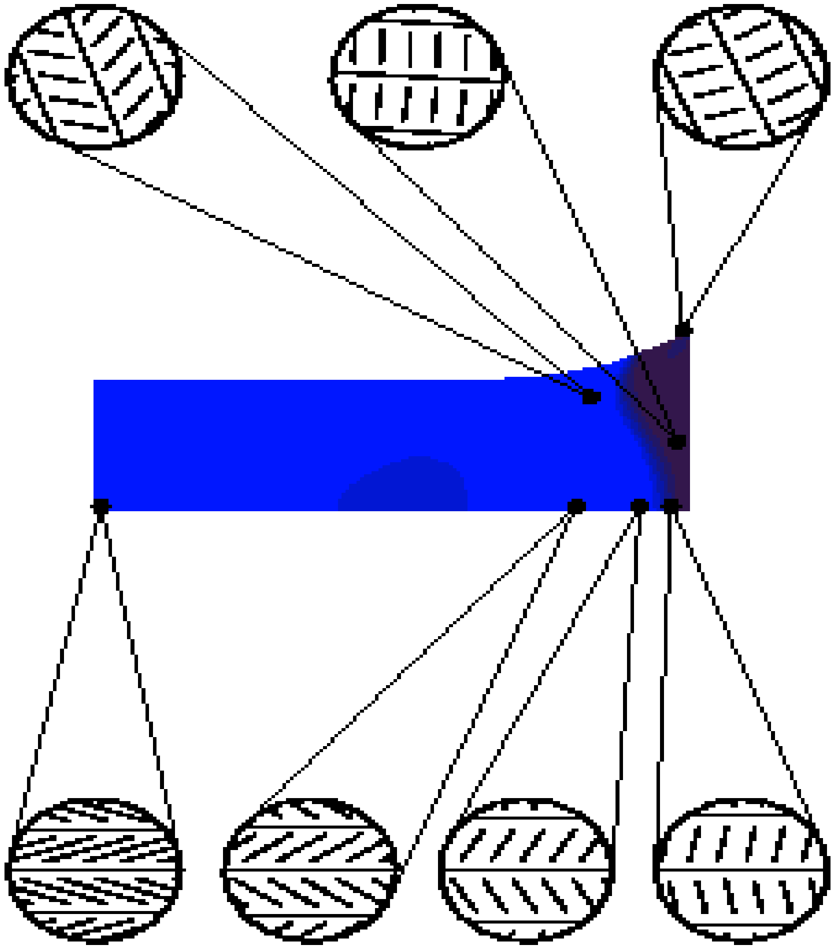

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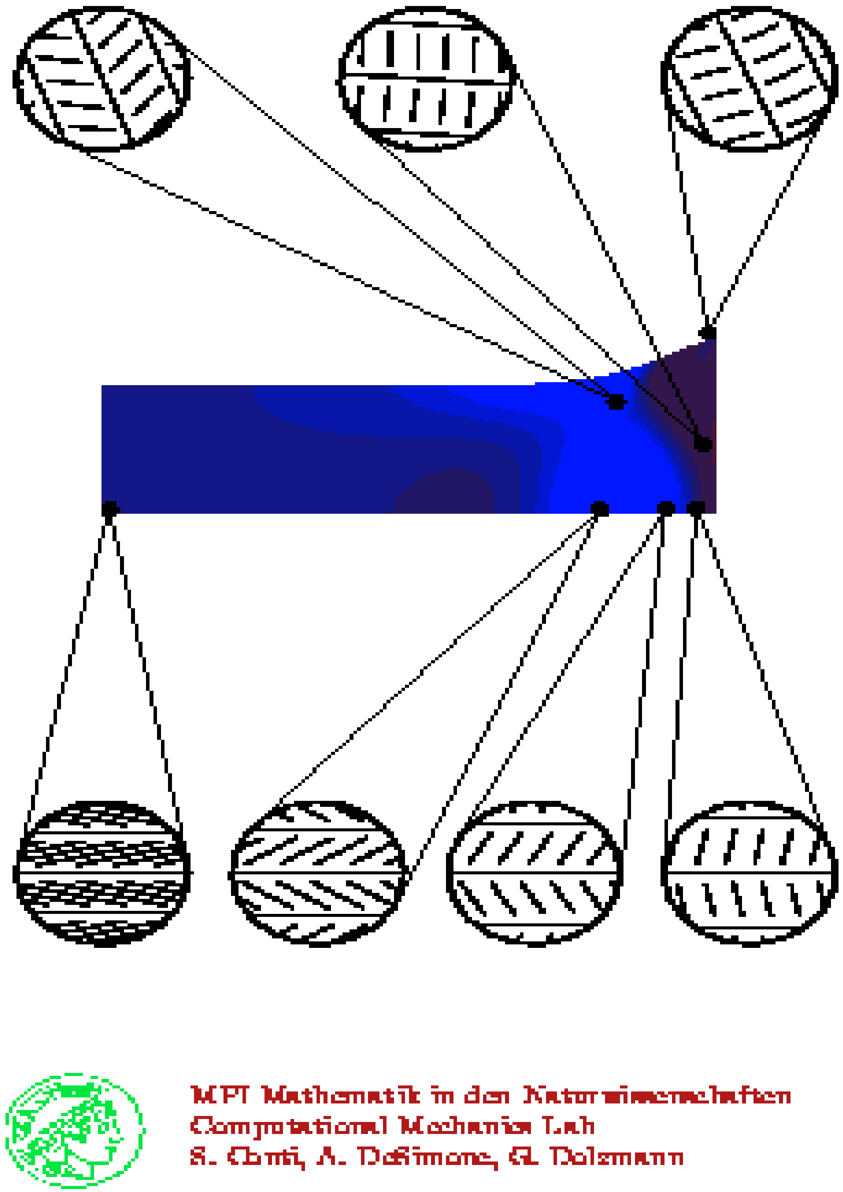

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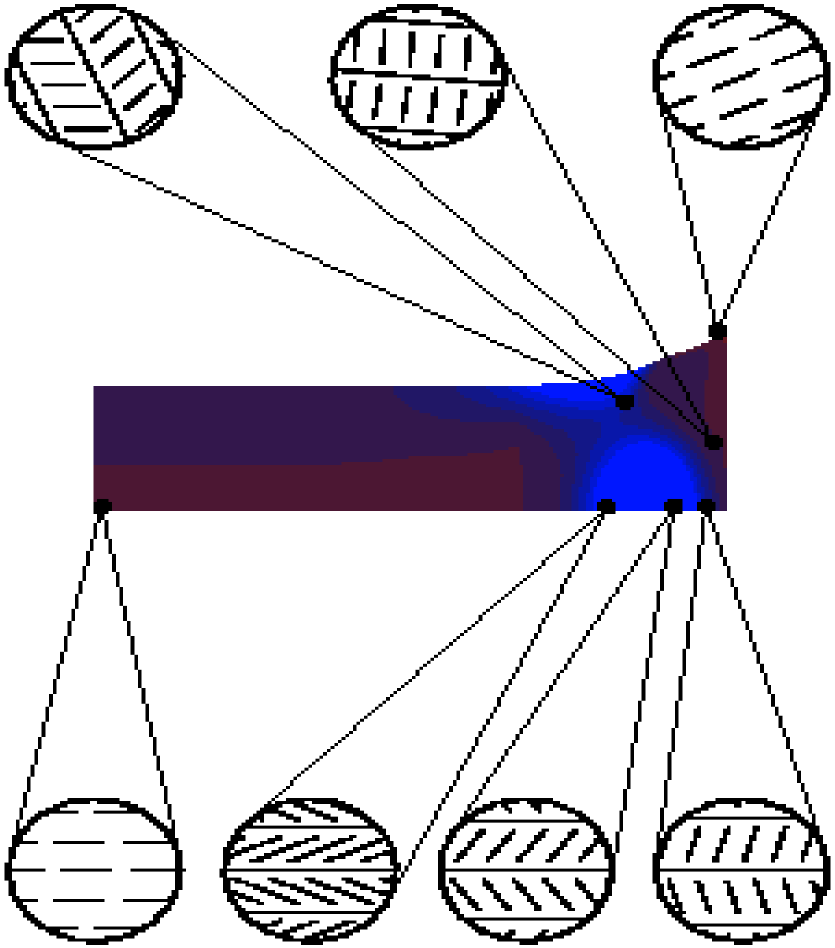

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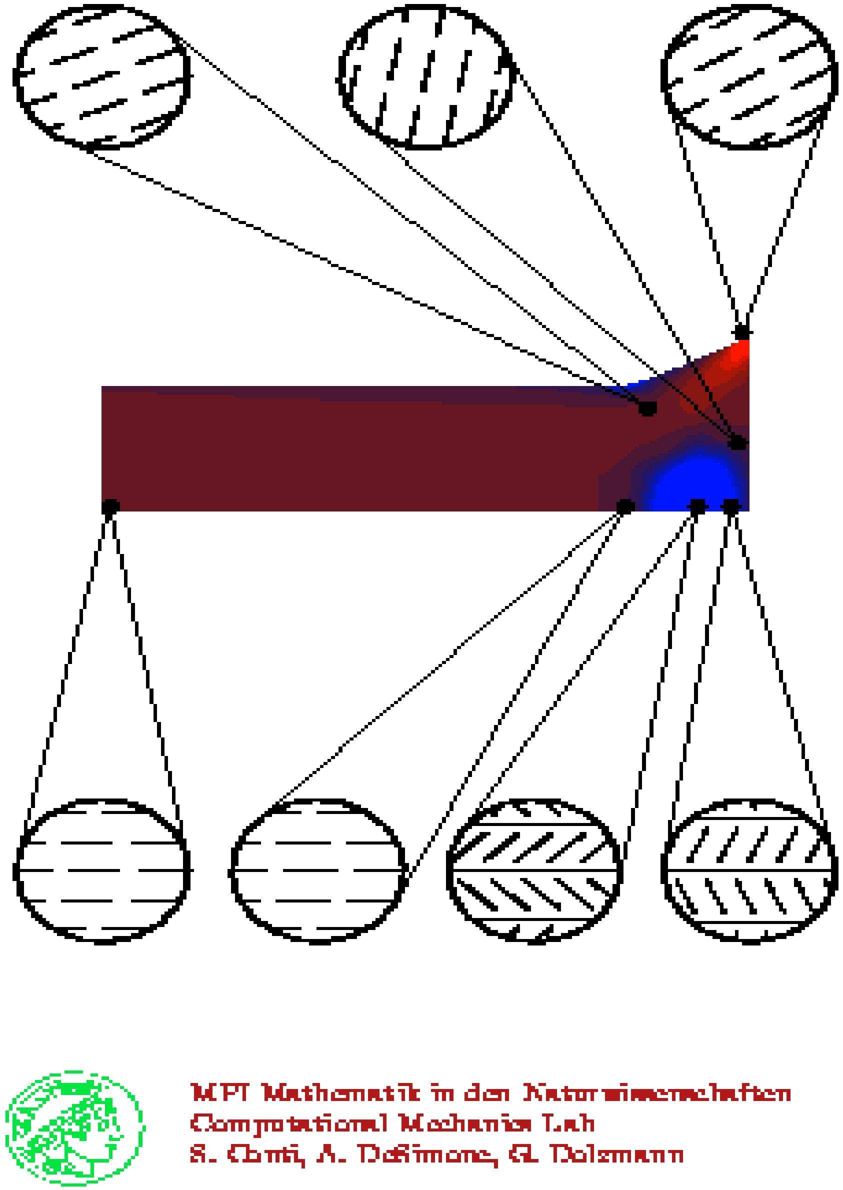

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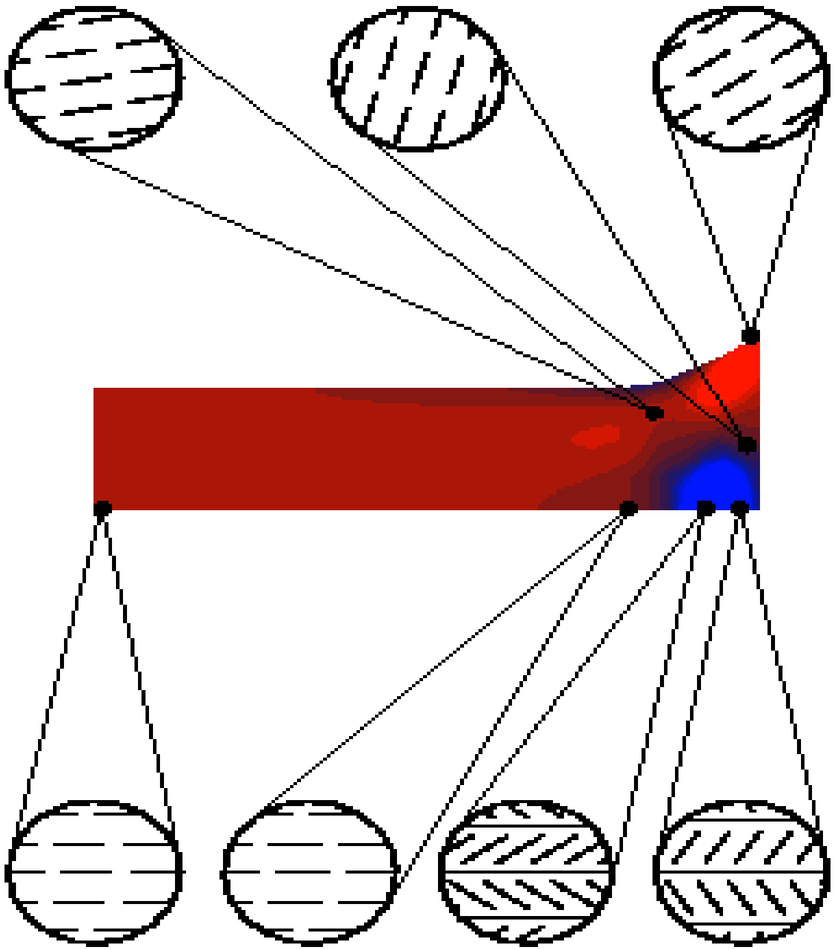

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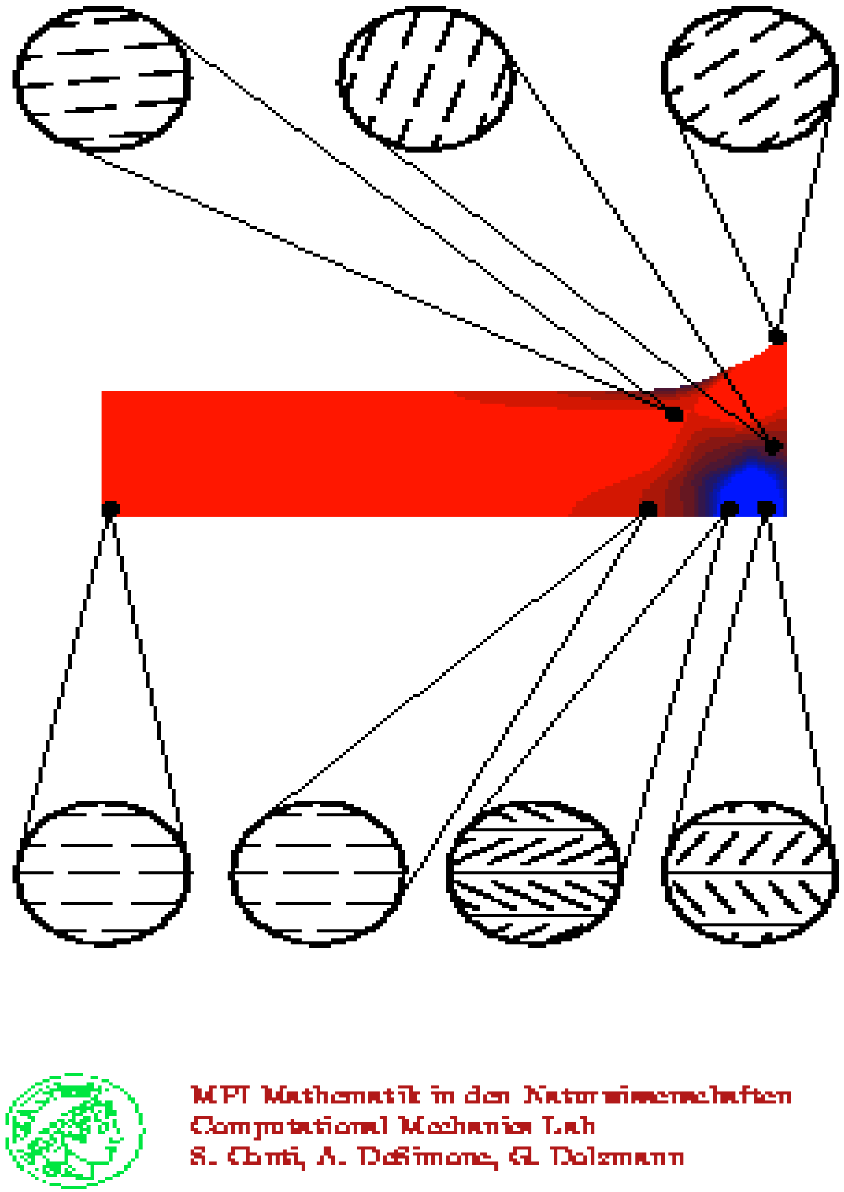

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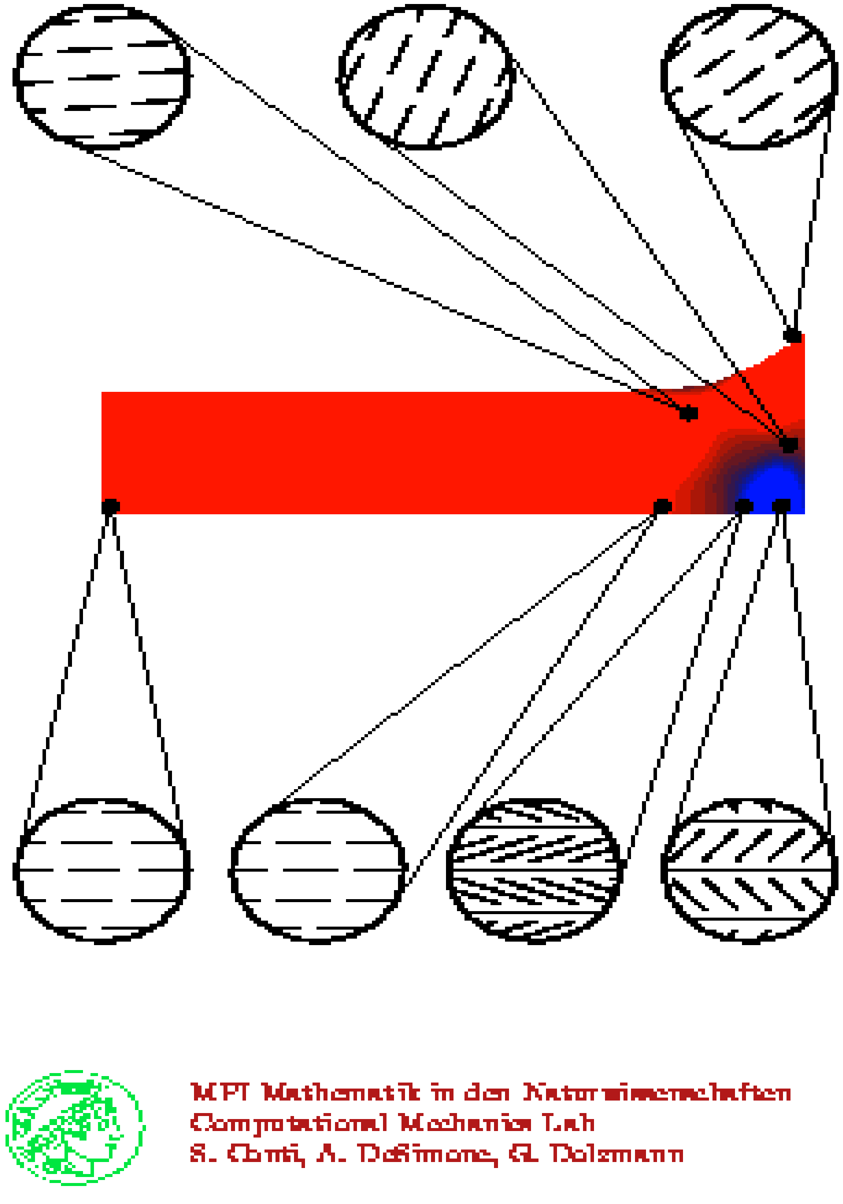

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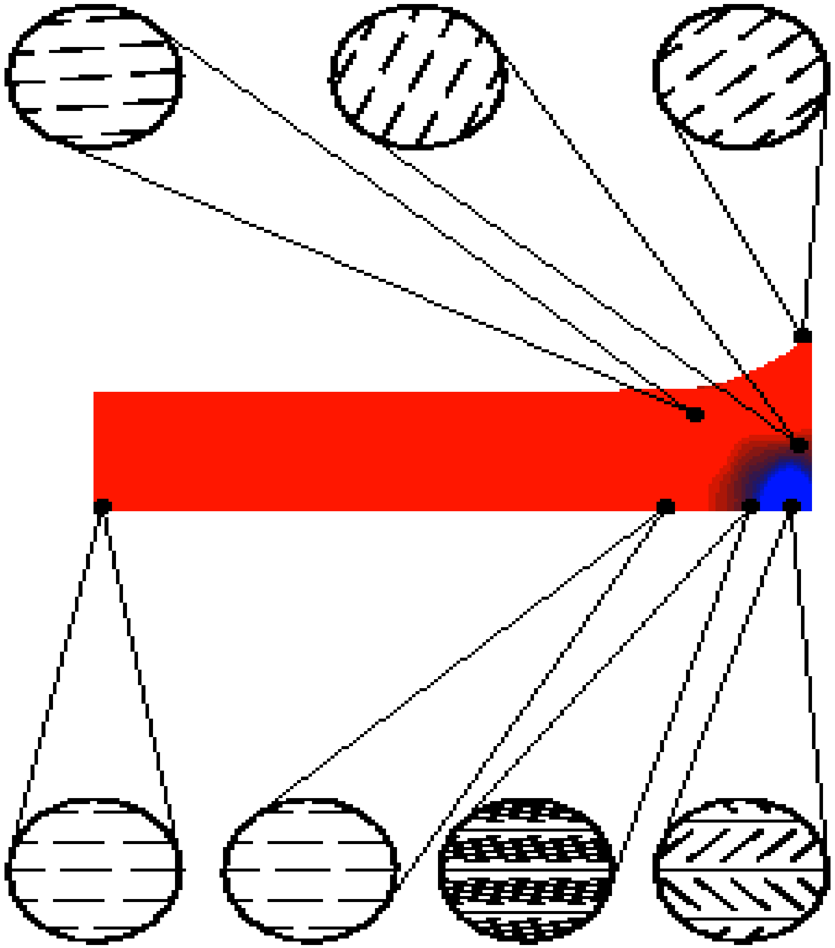

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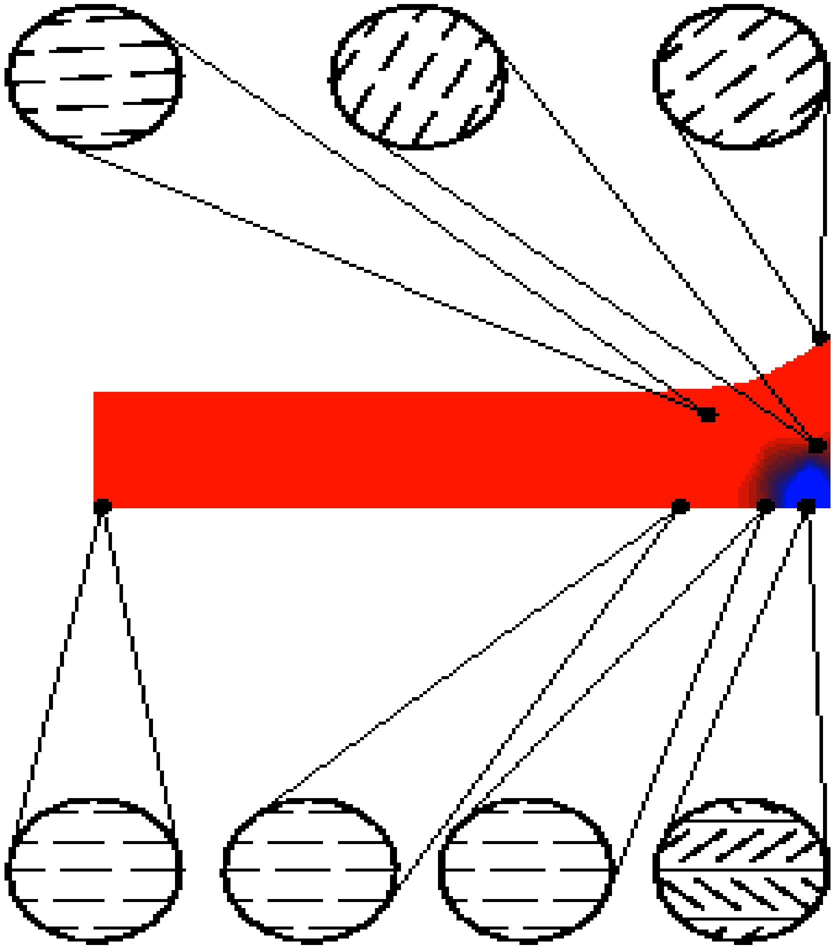

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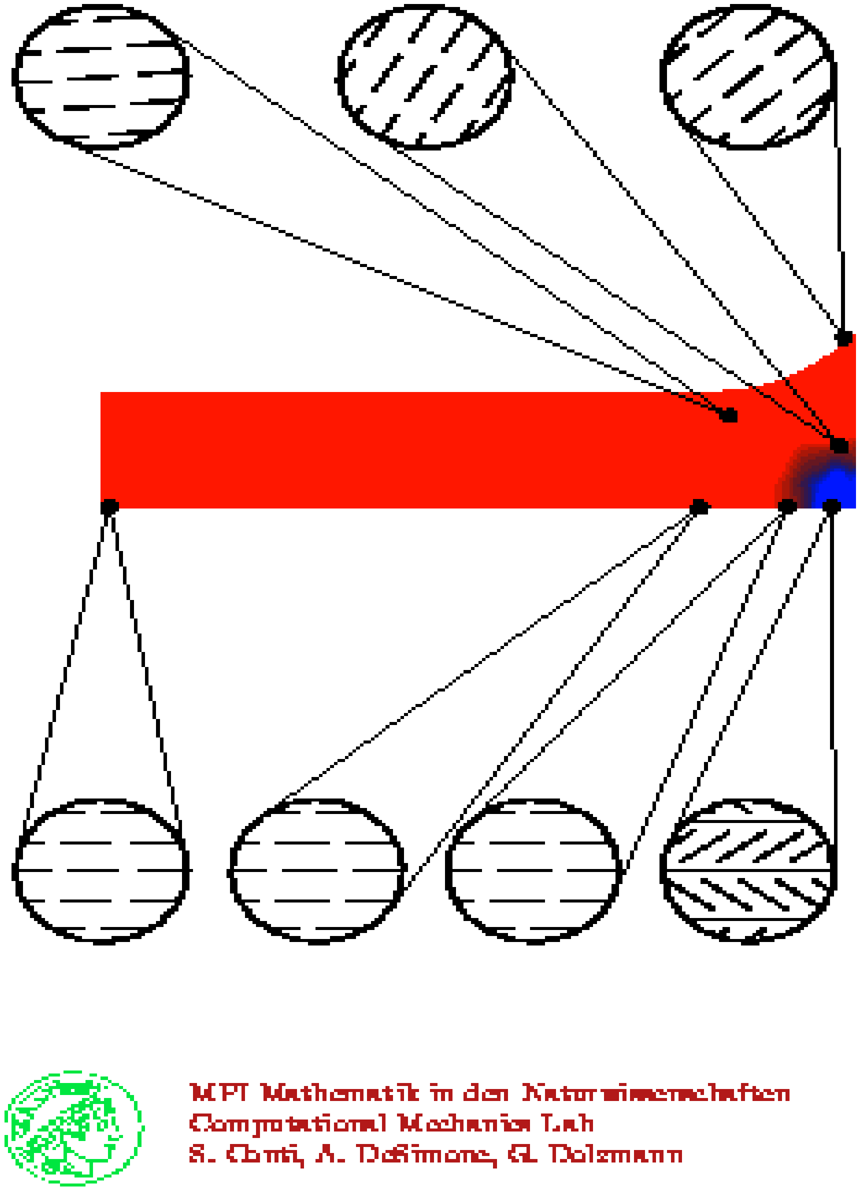

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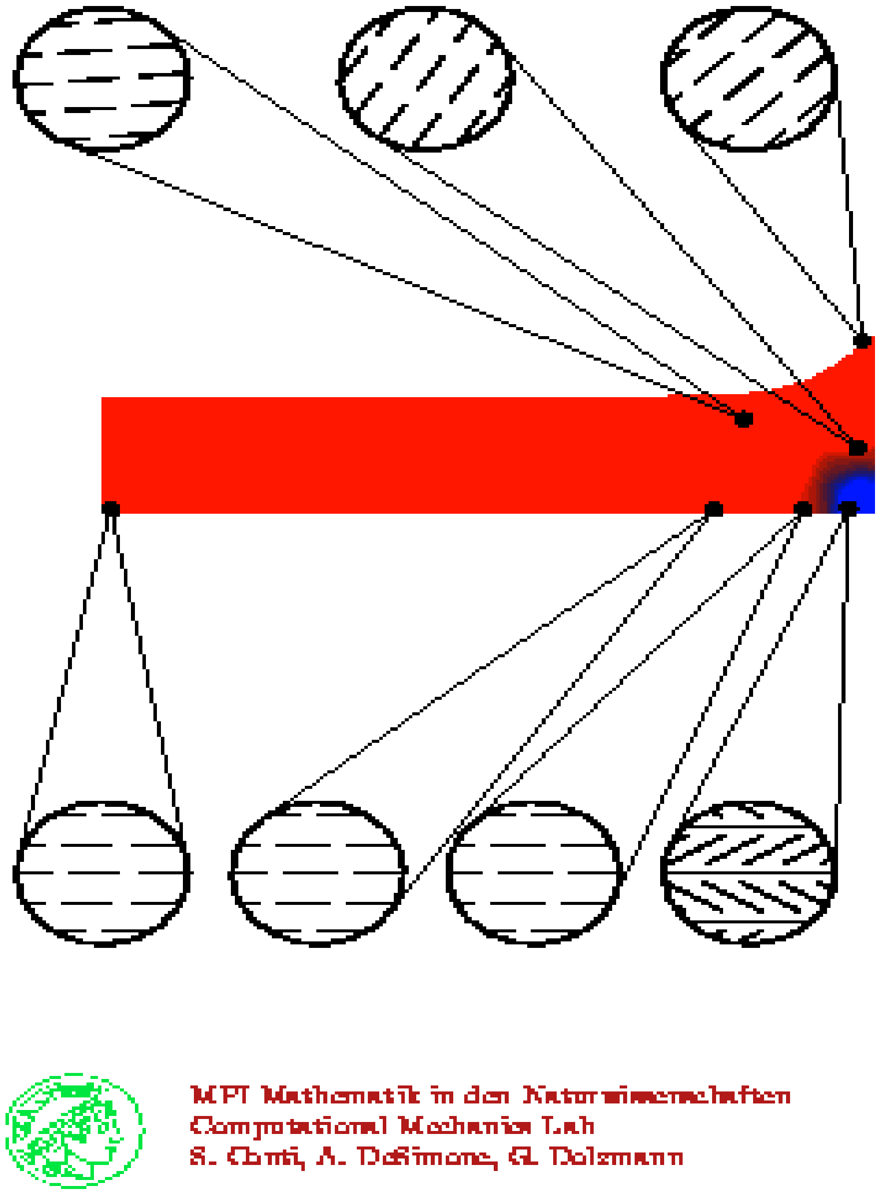

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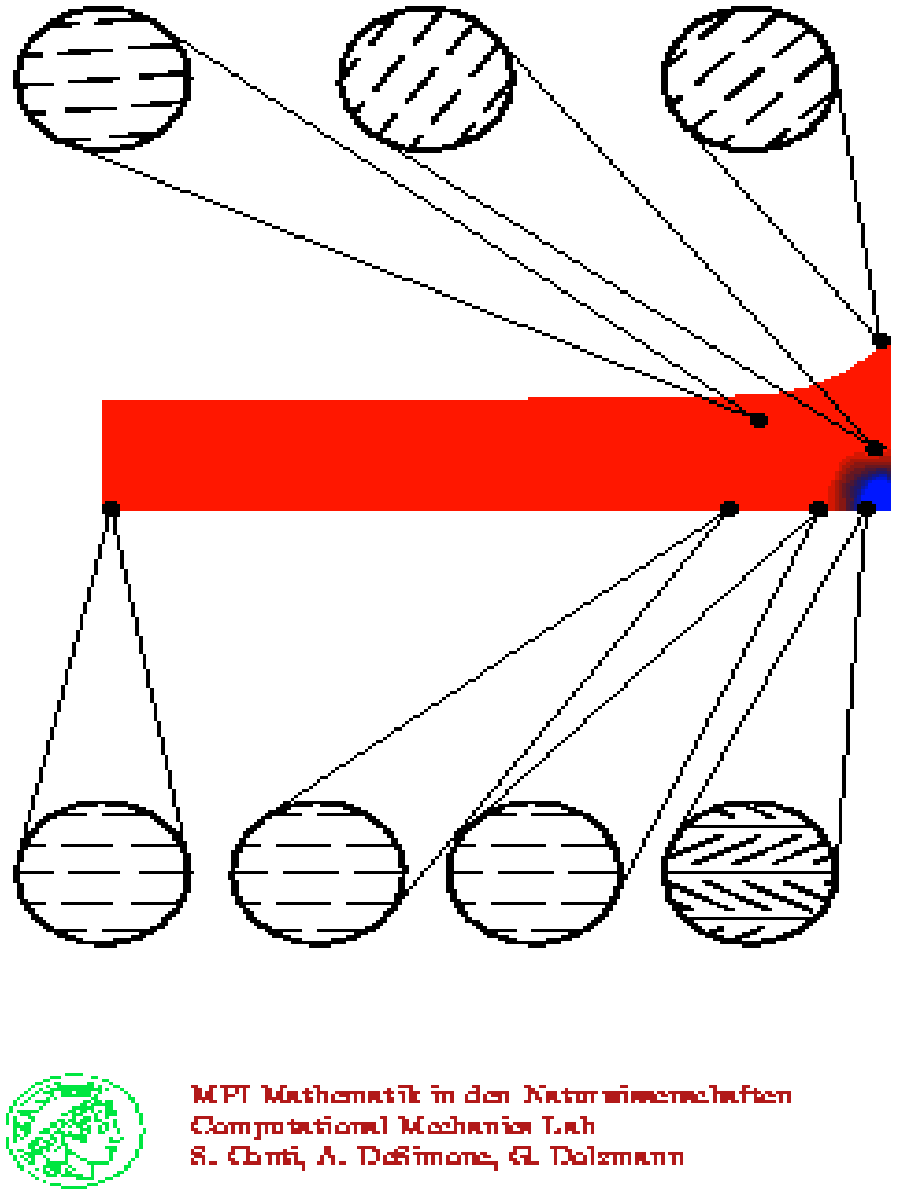

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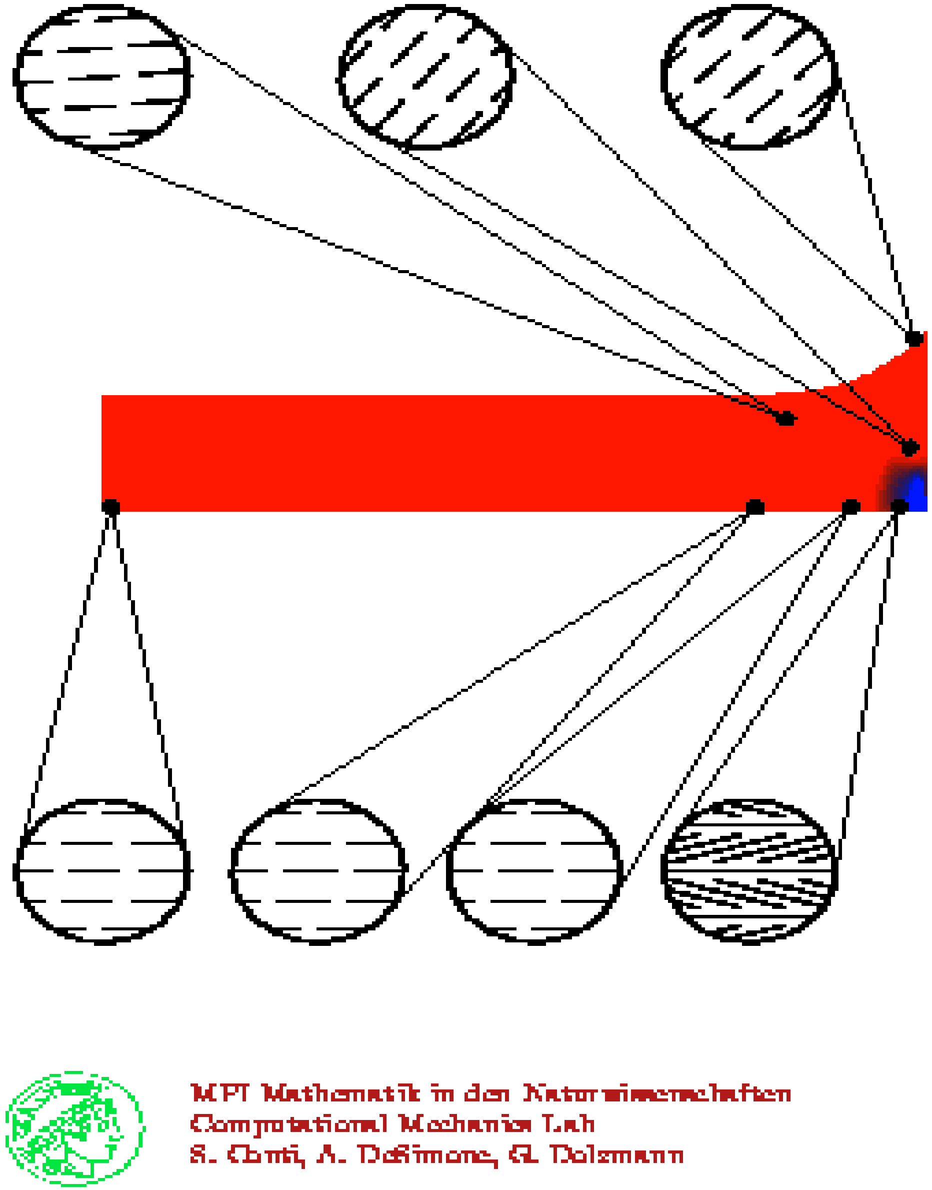

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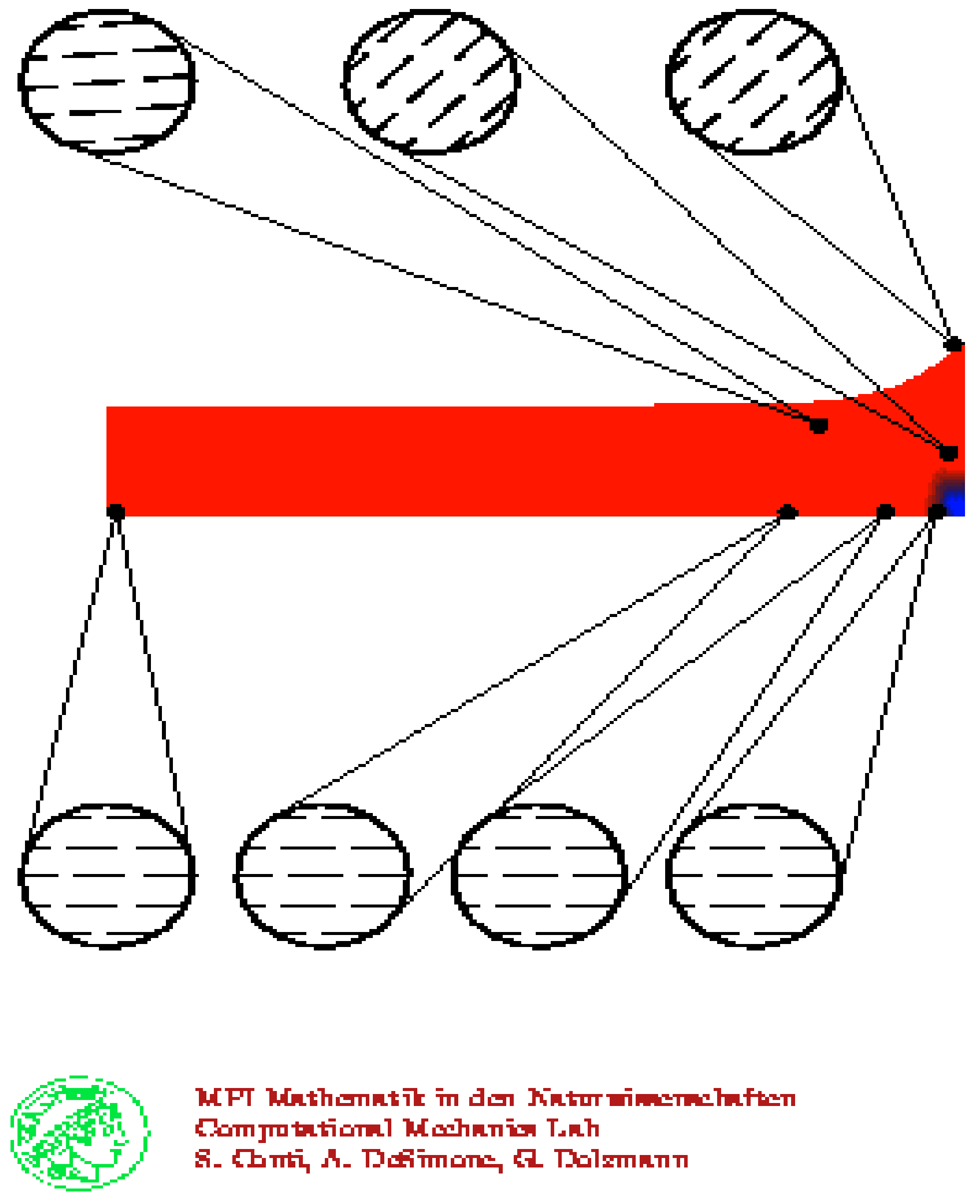

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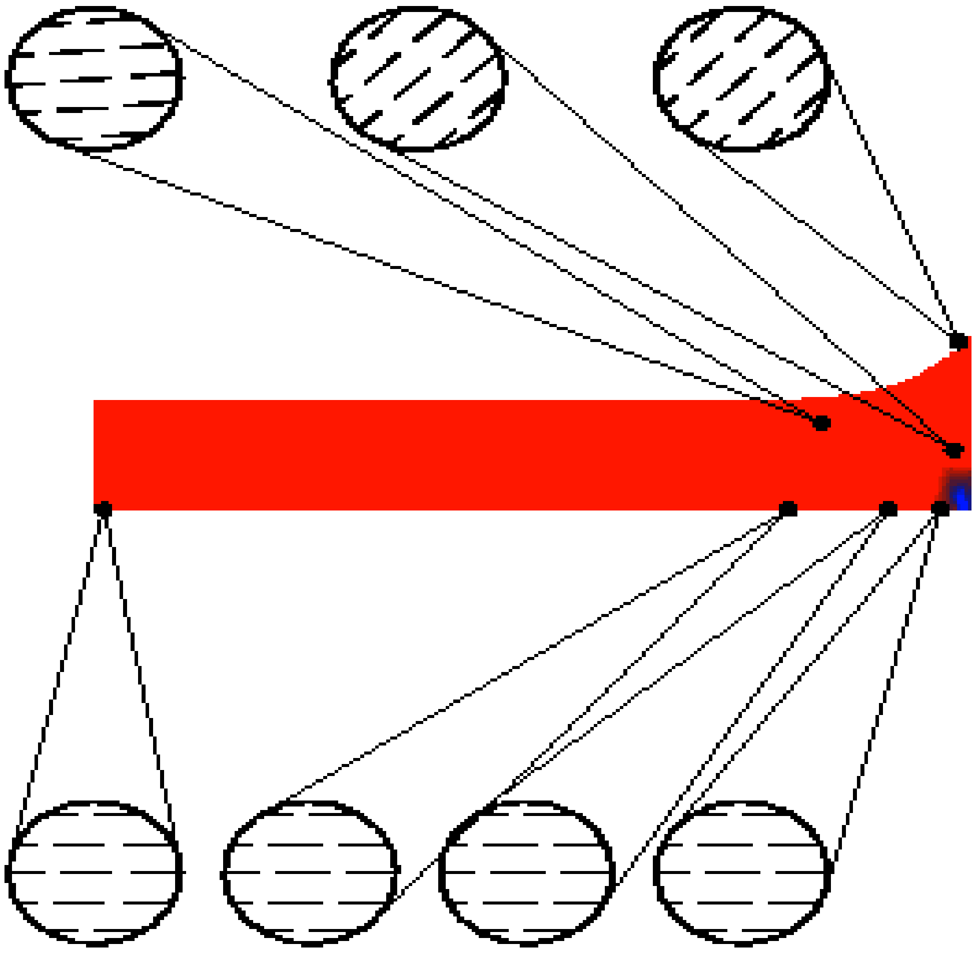

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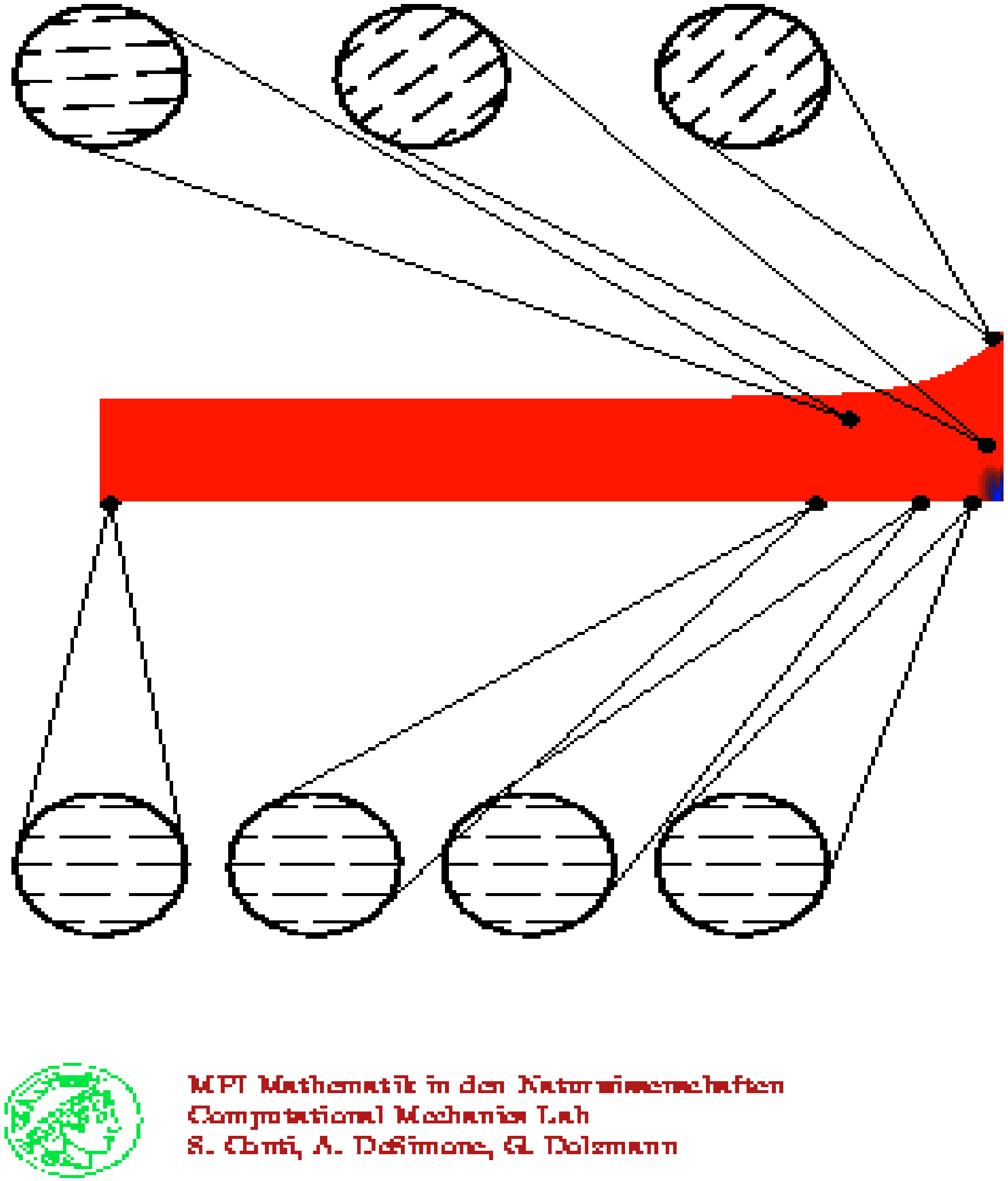

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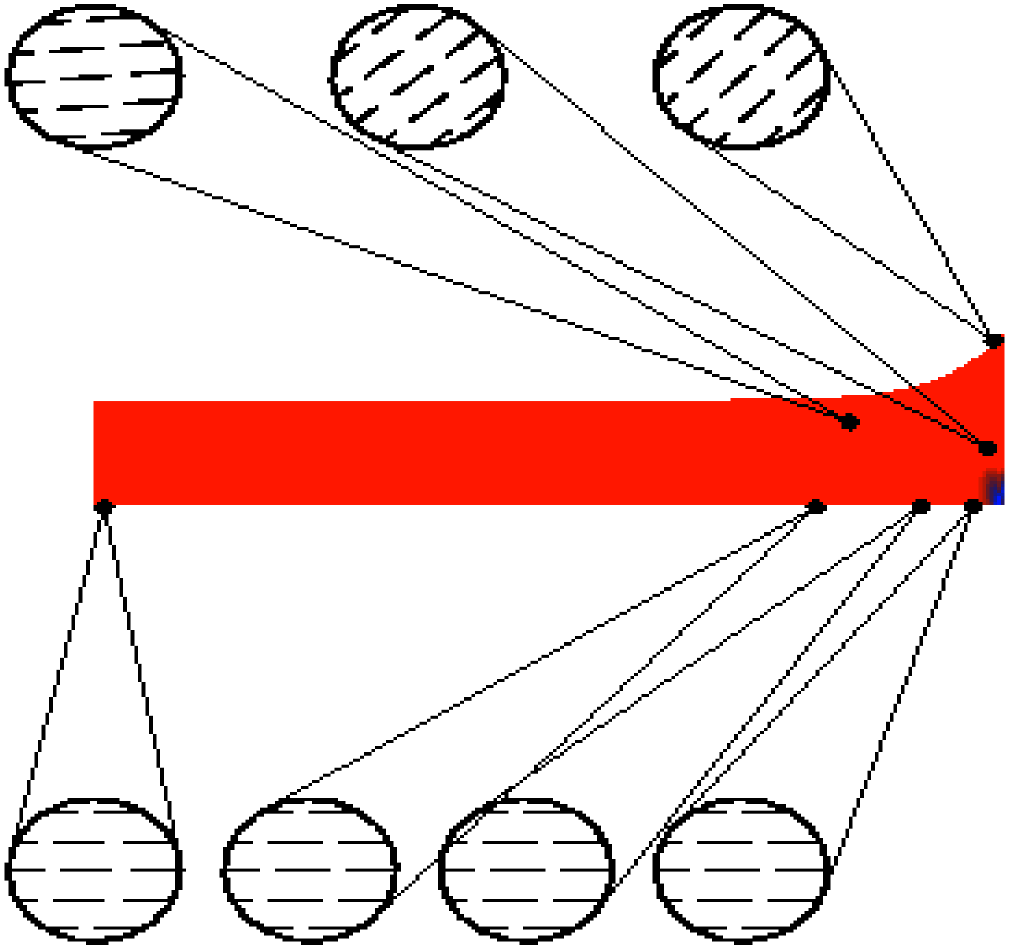

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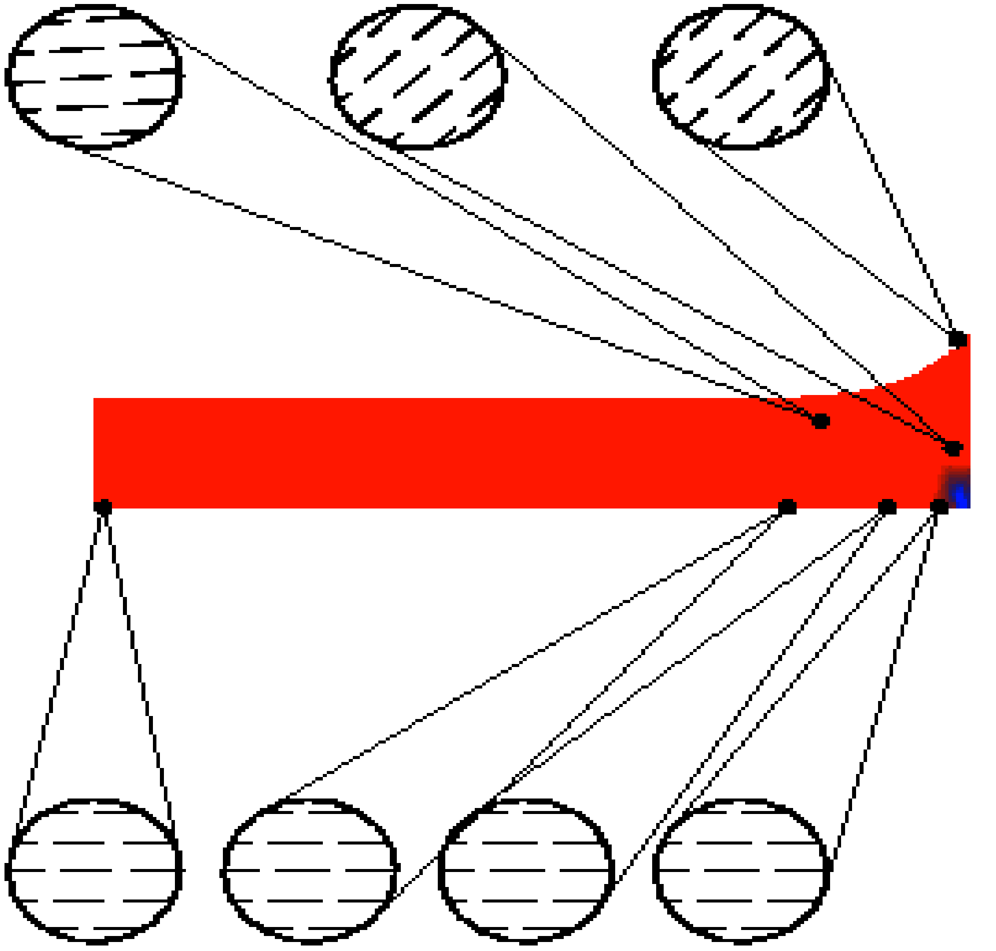

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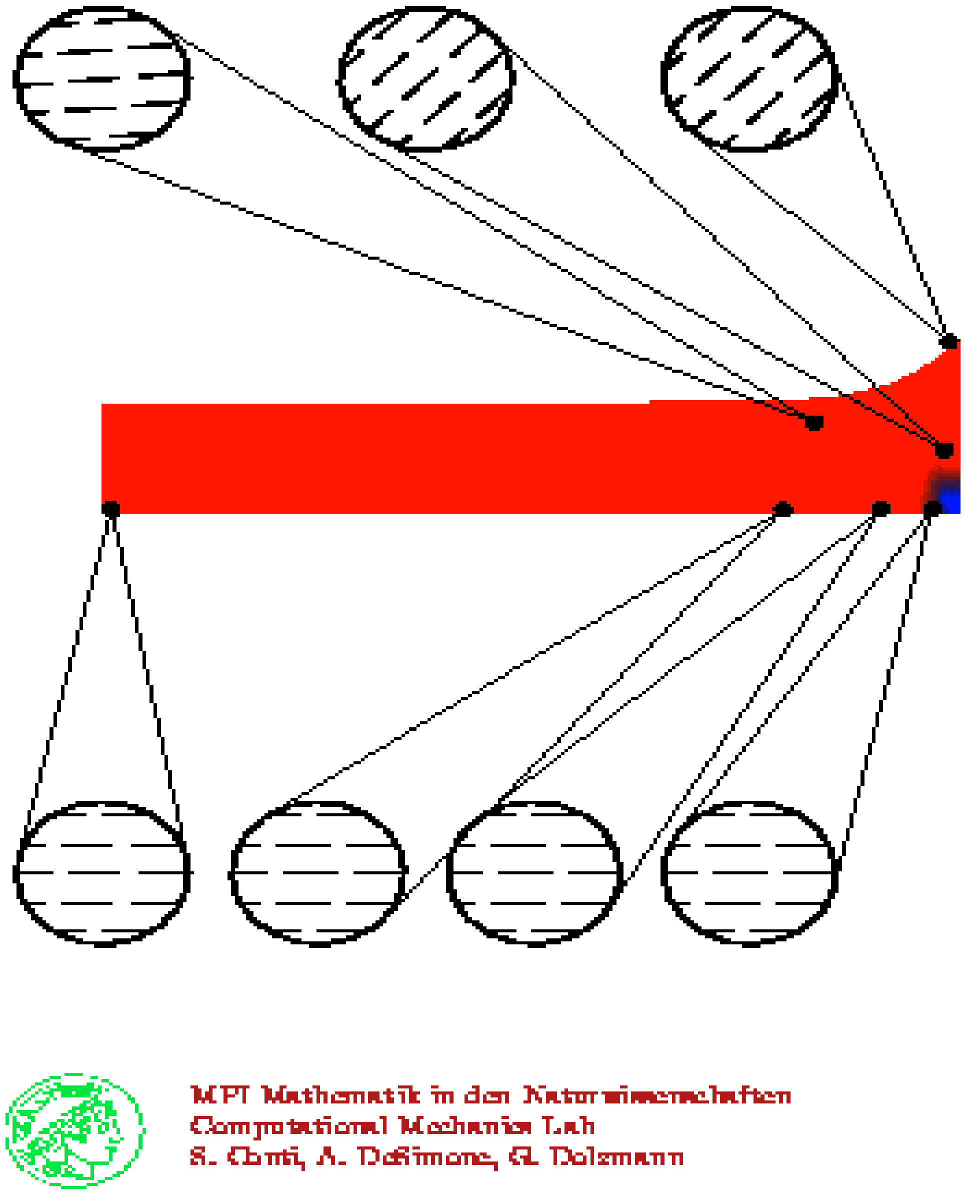

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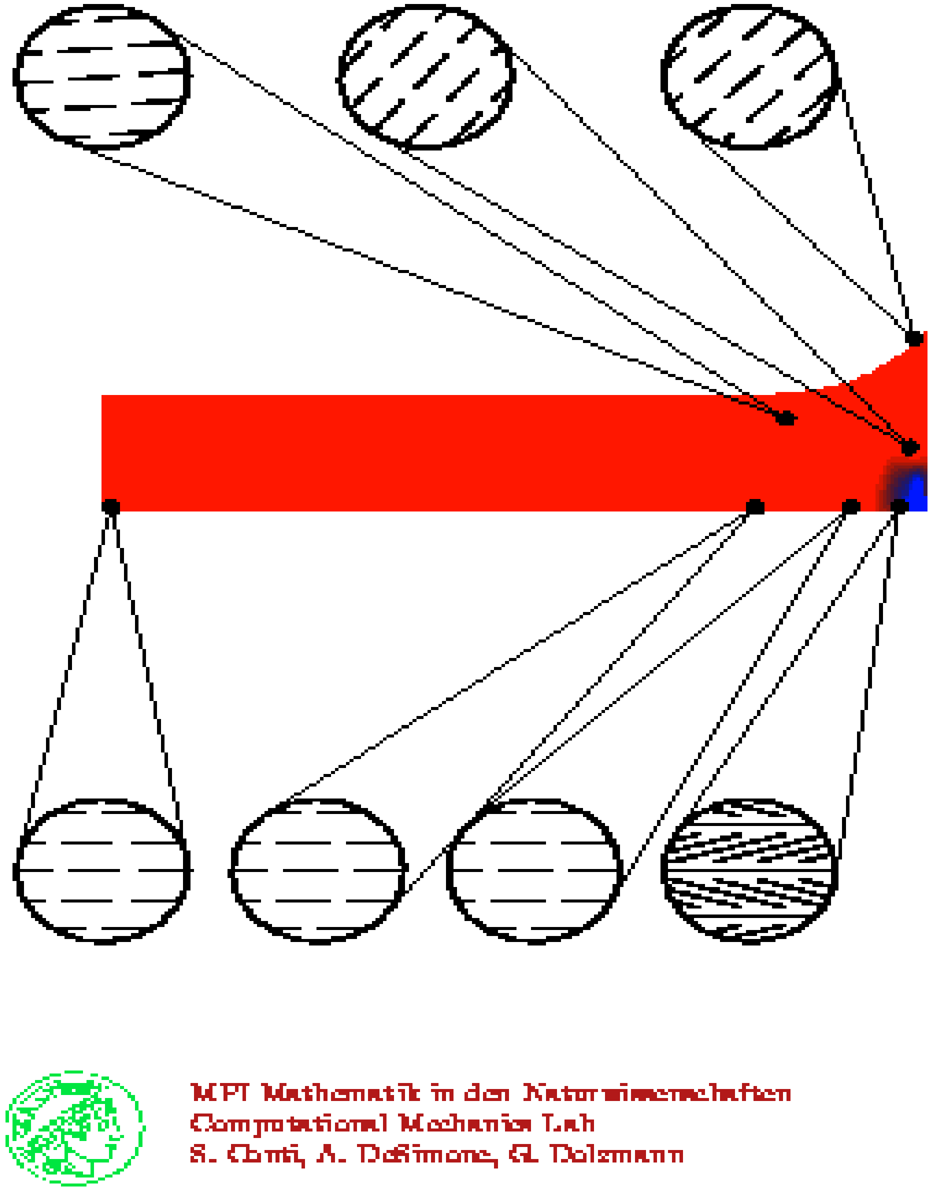

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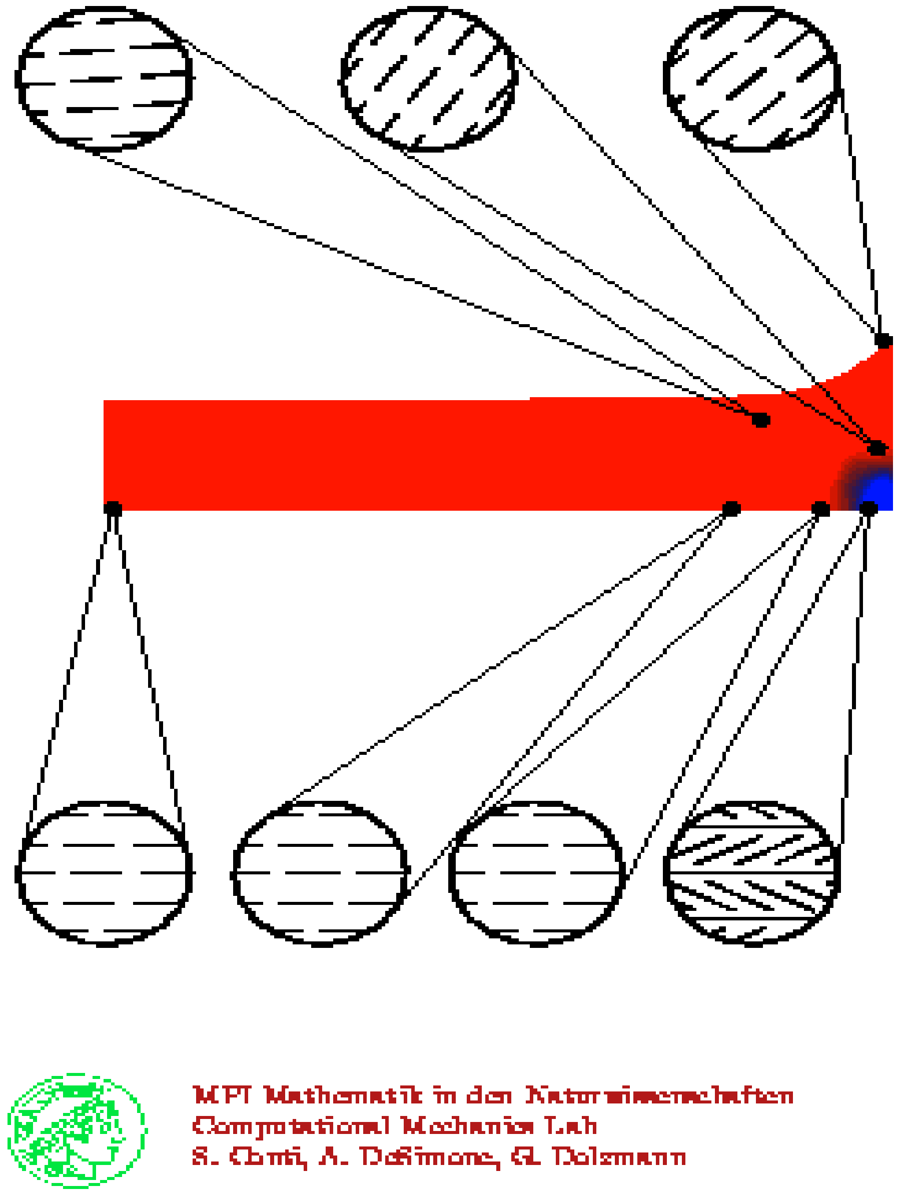

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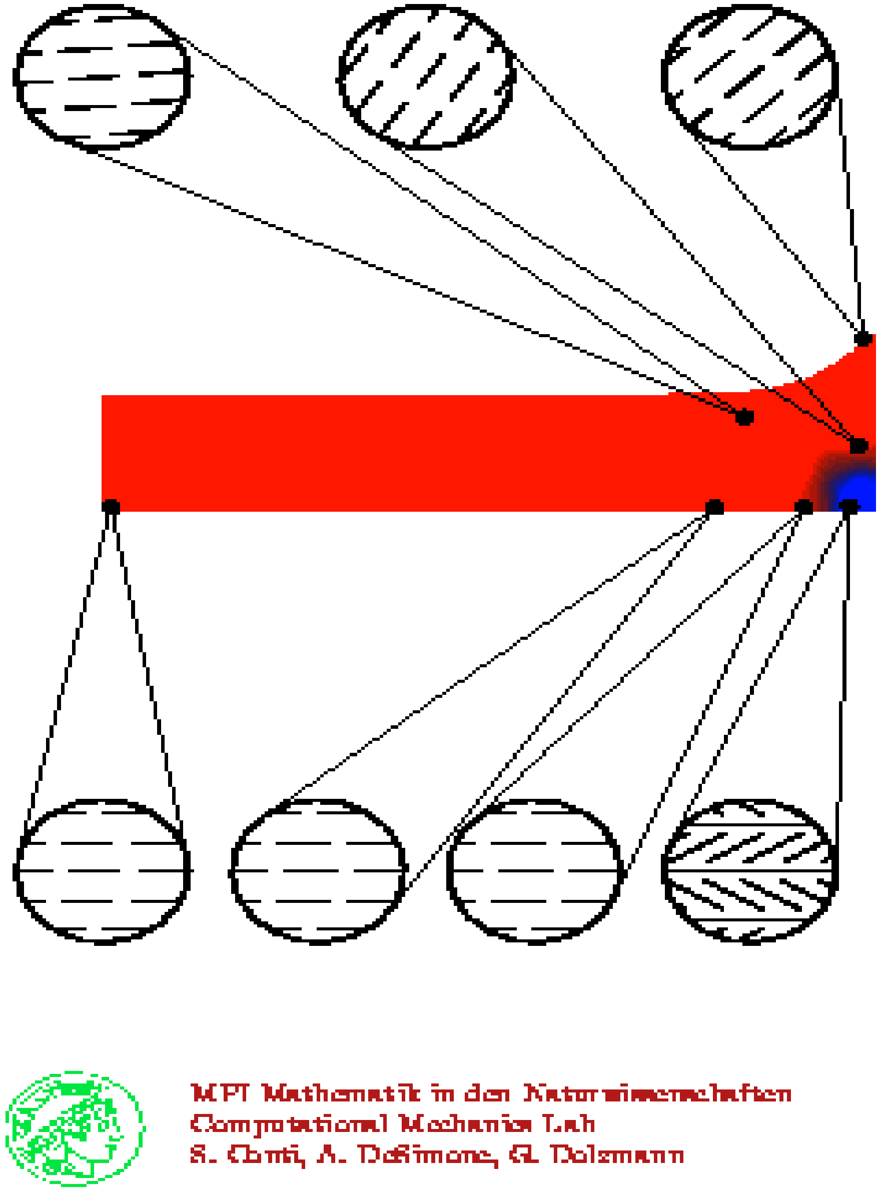

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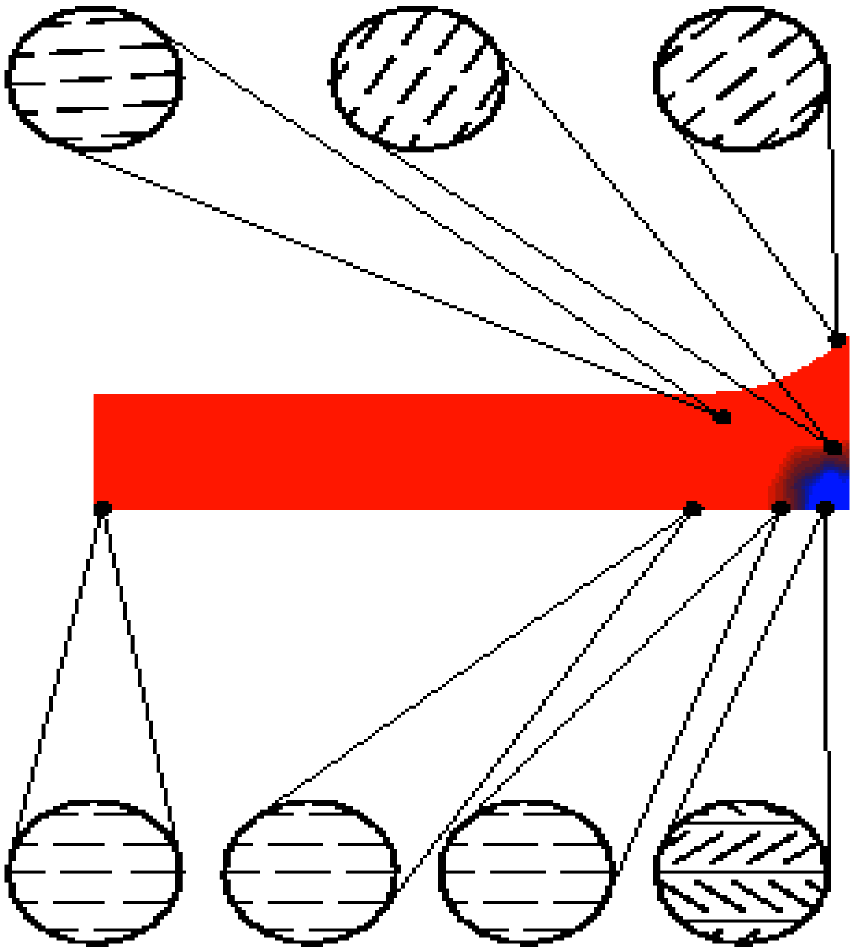

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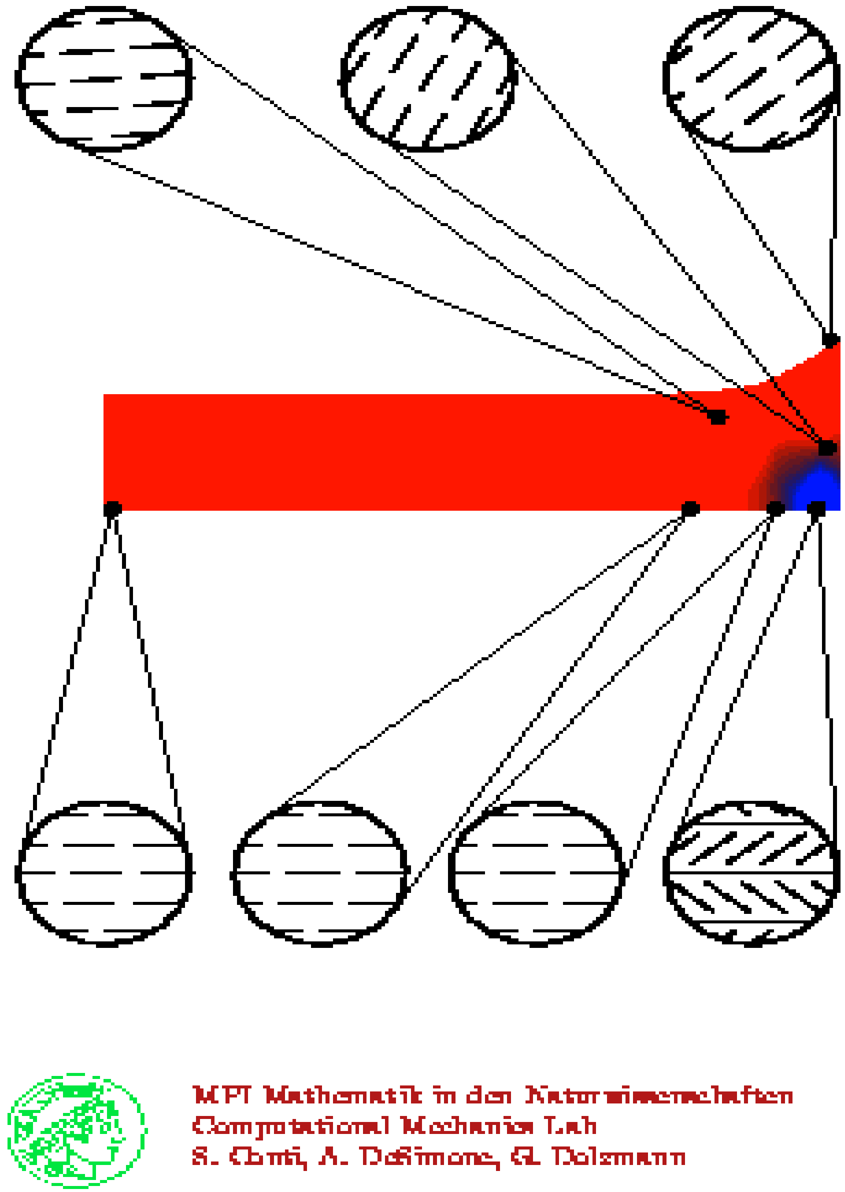

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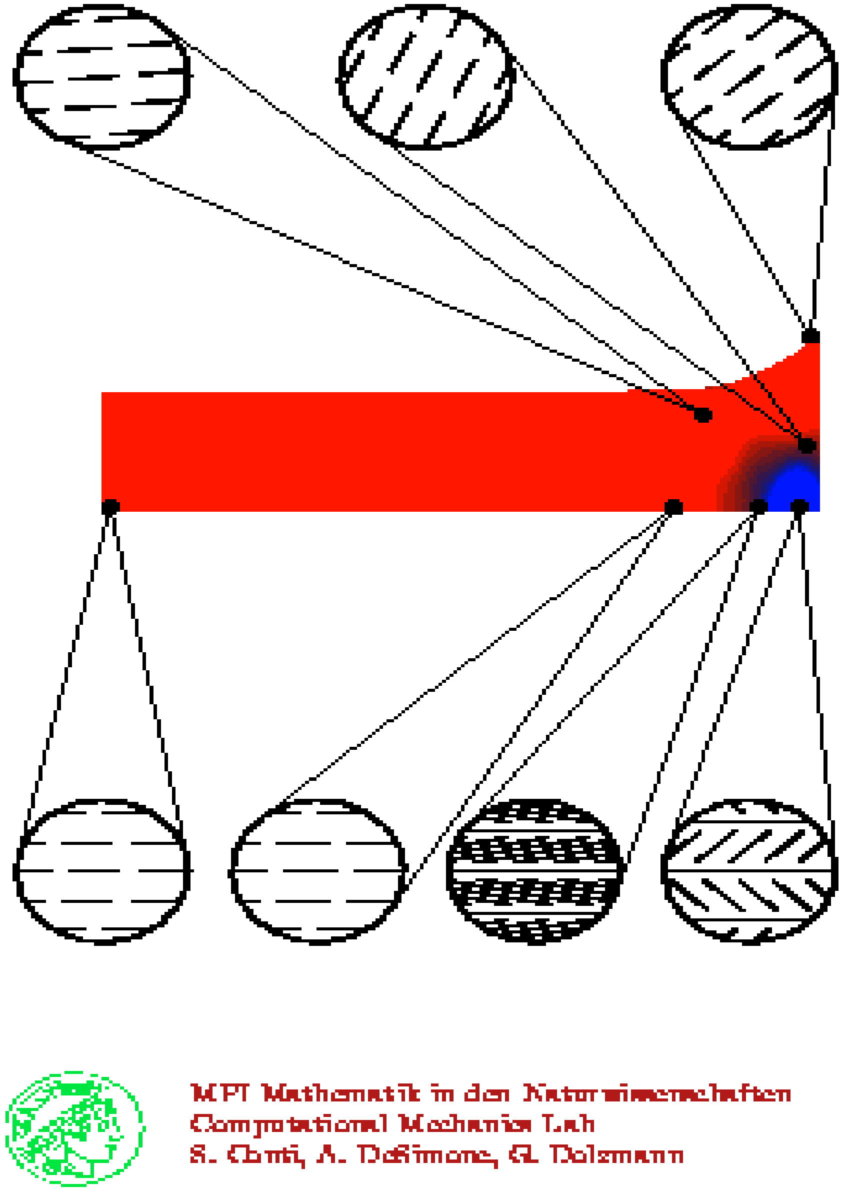

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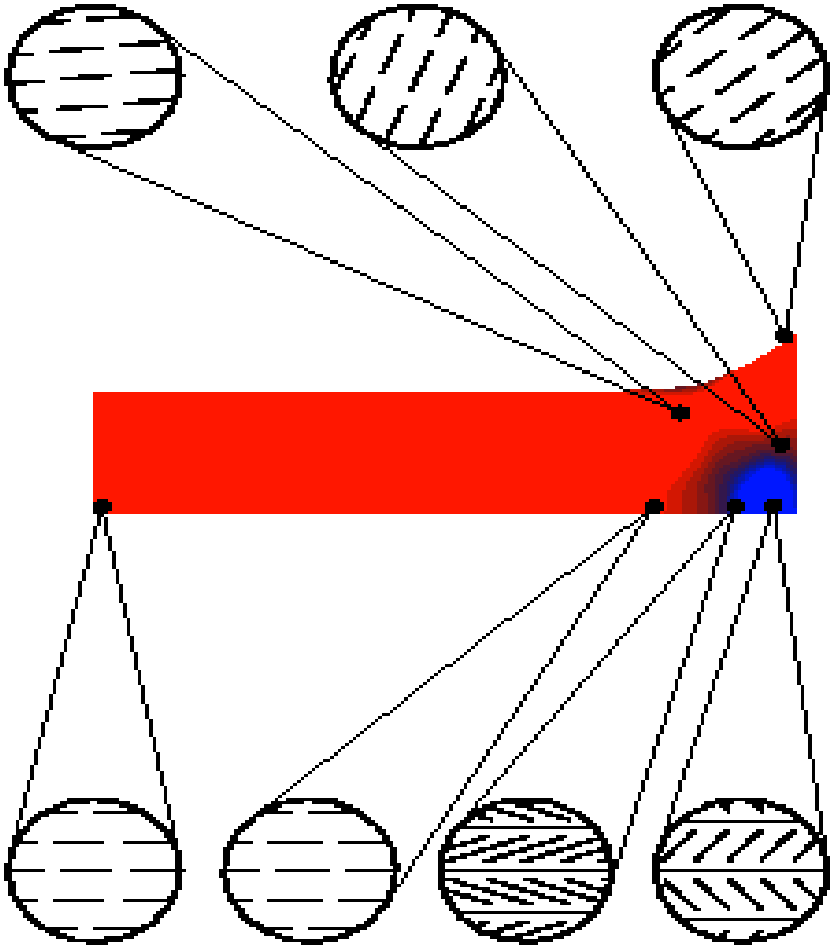

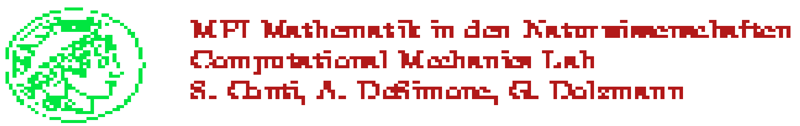

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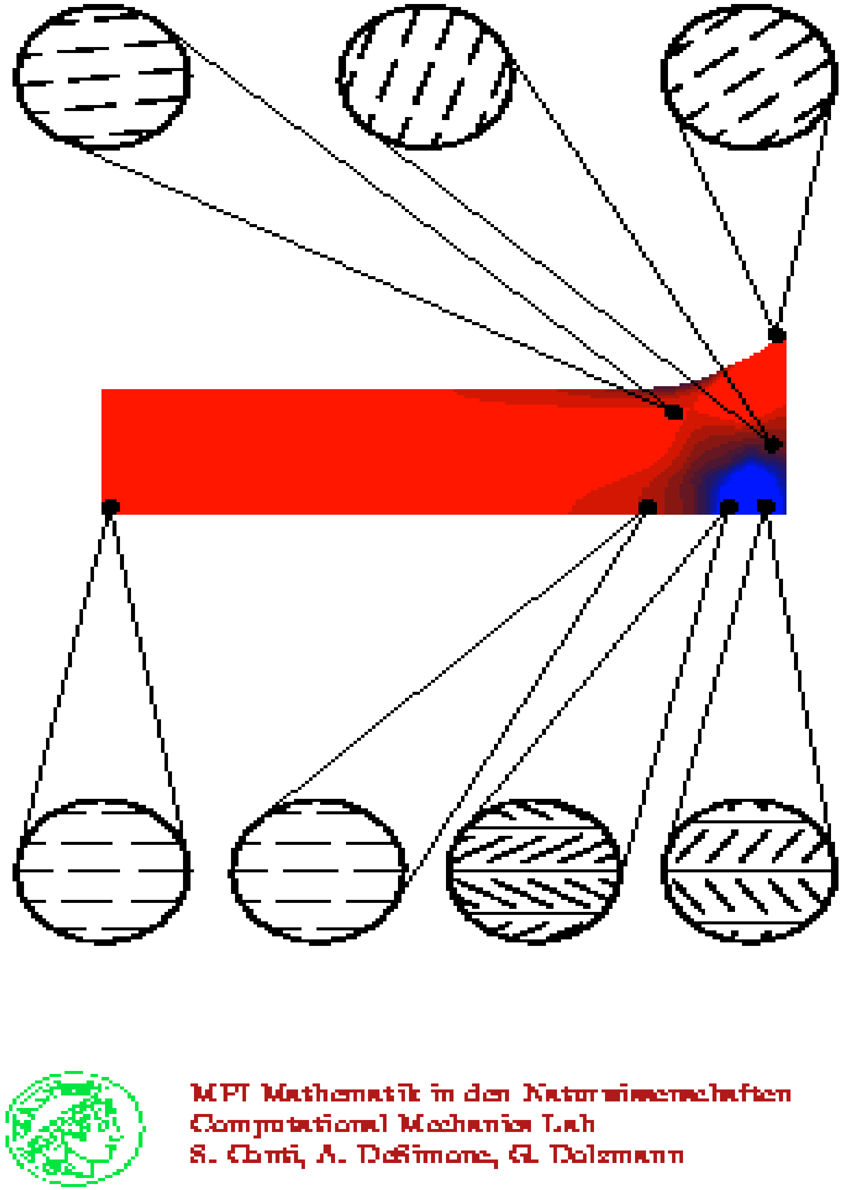

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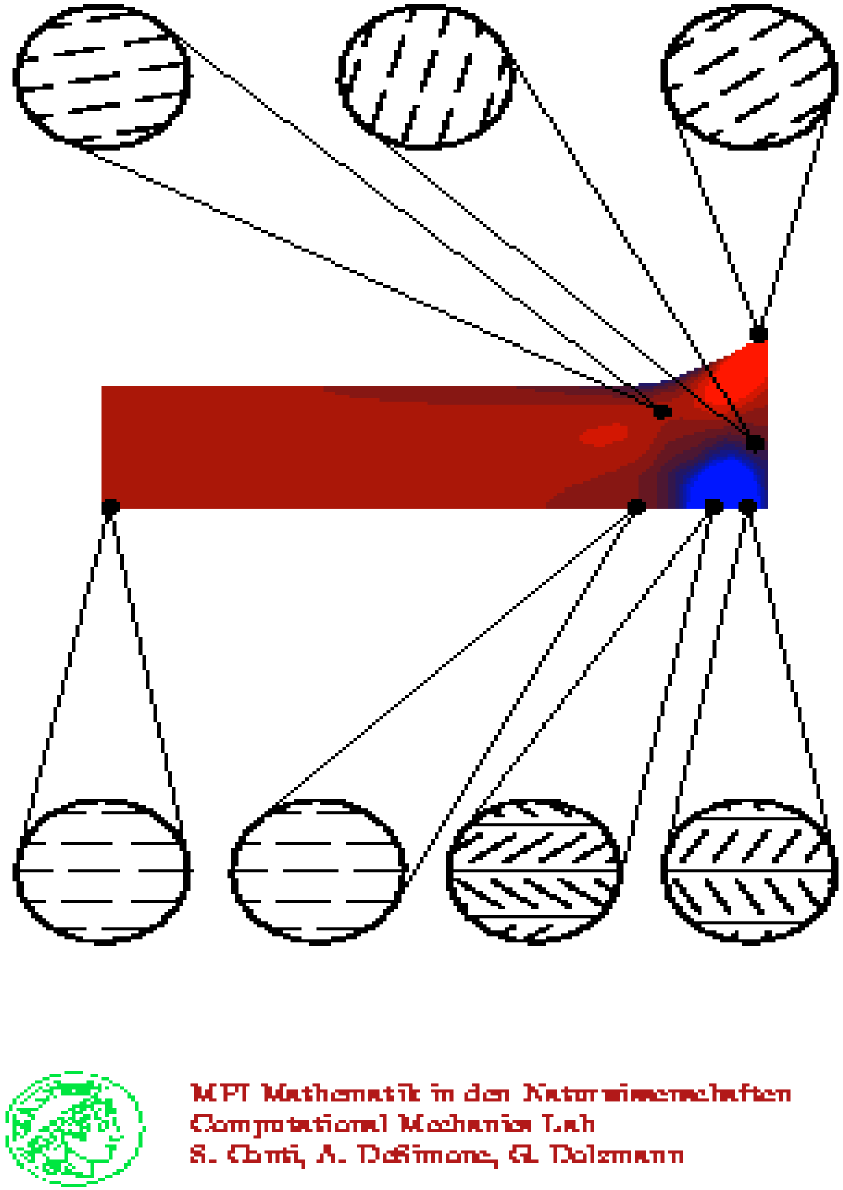

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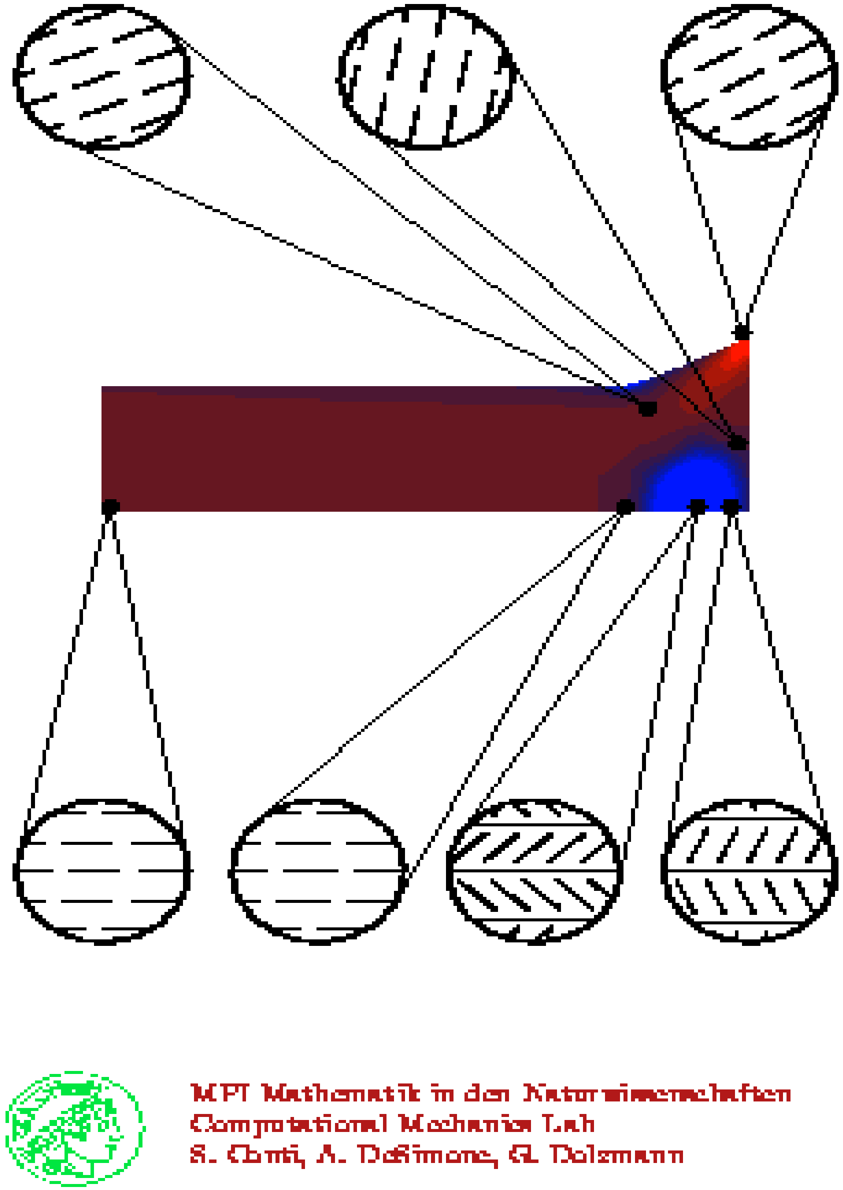

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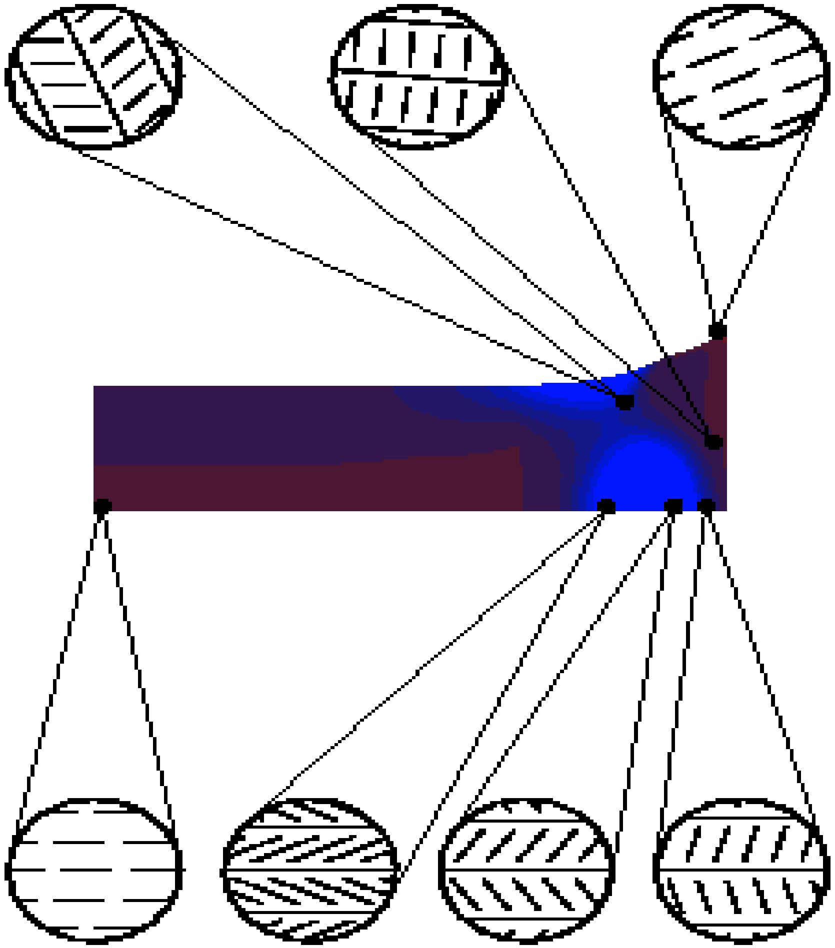

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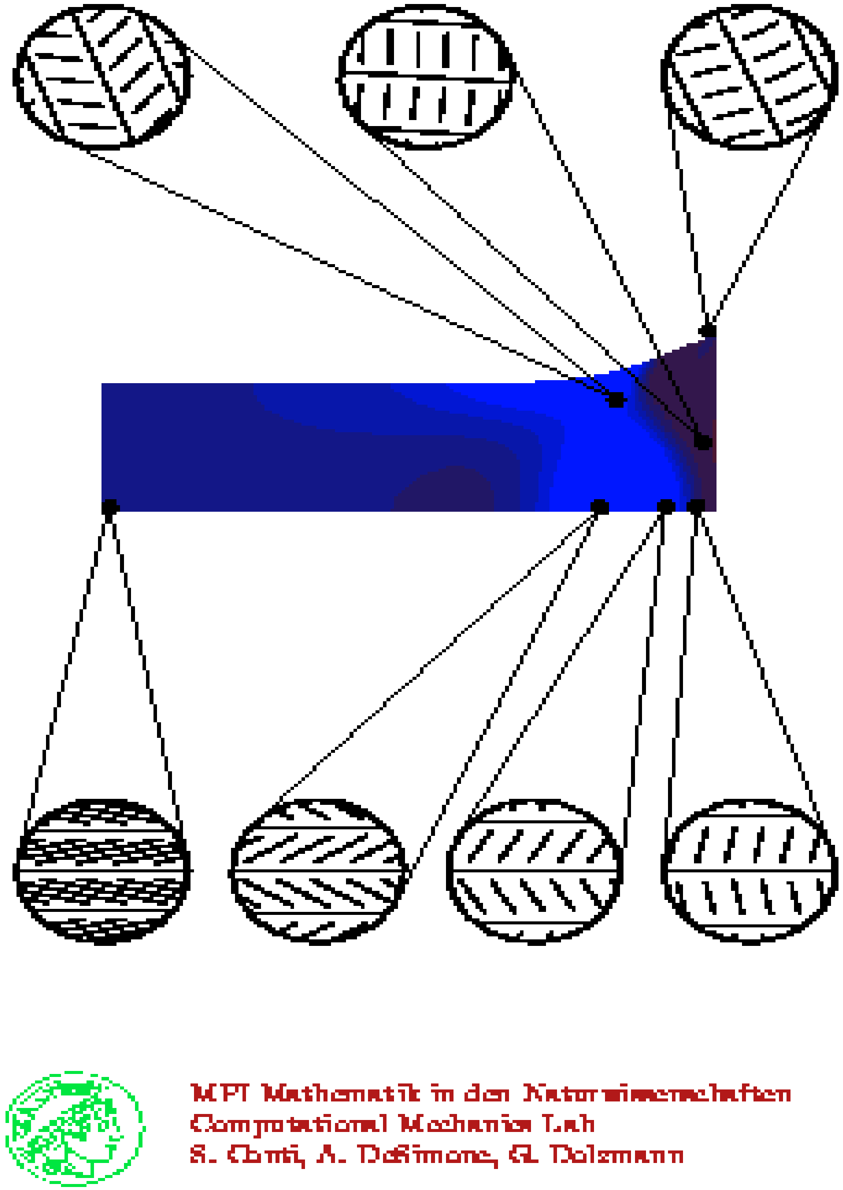

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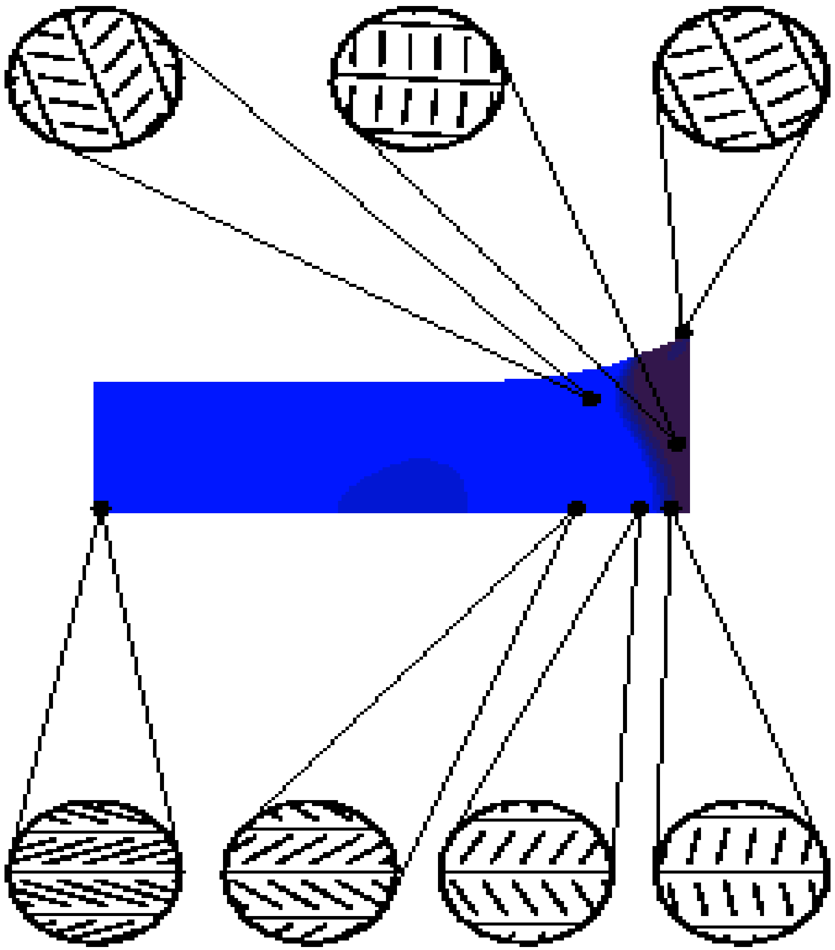

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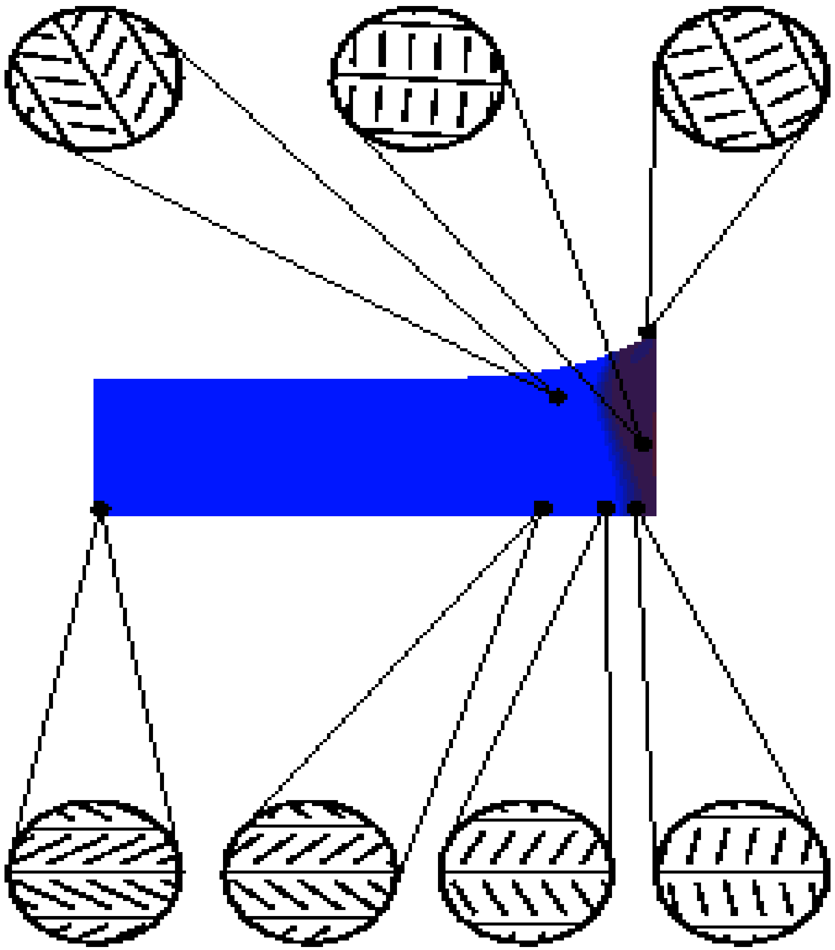

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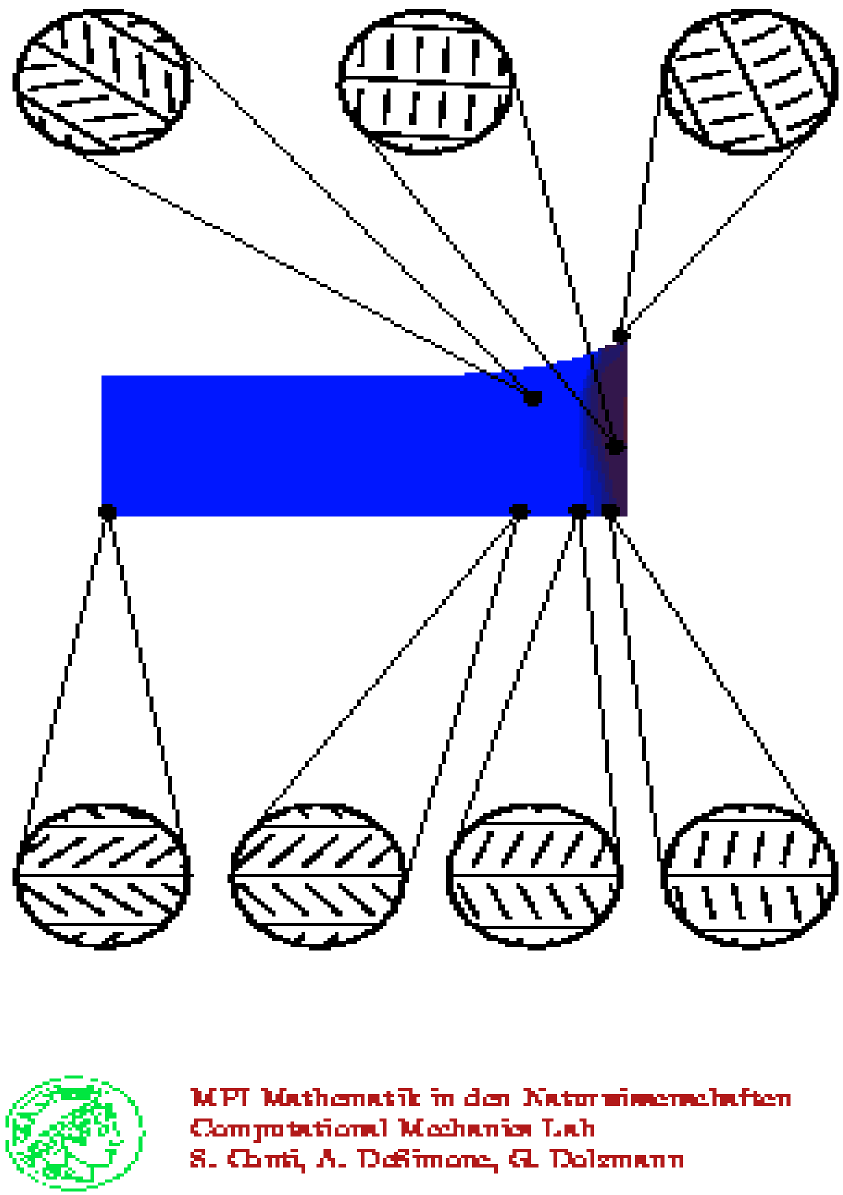

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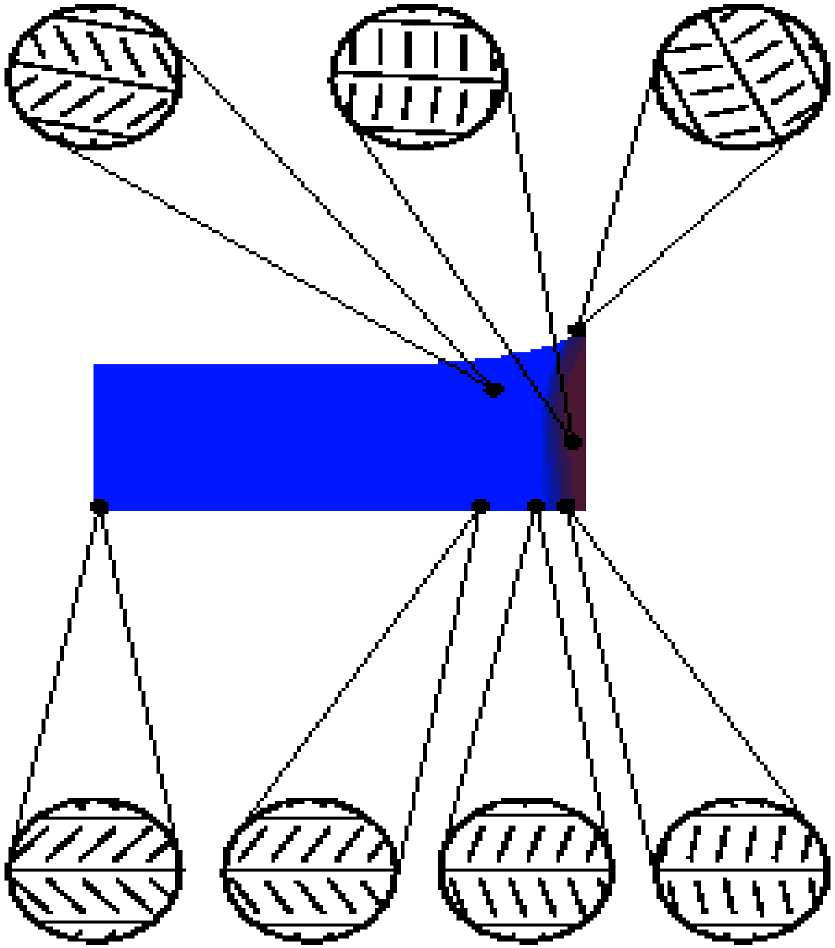

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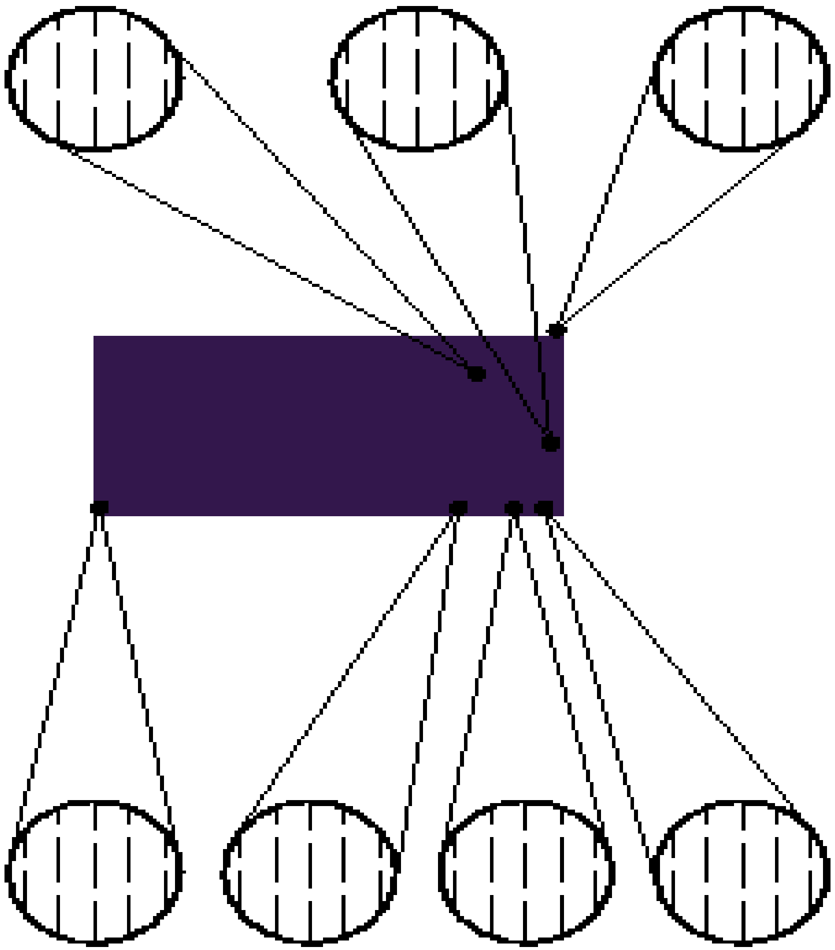

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Geometrically nonlinear models

M. Barchiesi & A. DeSimone [*ESAIM:COCV*, 2015]

$$\min_{\substack{(\boldsymbol{u},\boldsymbol{n})\\ \det D\boldsymbol{u}\equiv 1}} \int_\Omega W_{\mathrm{mec}}(D\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{n}(\boldsymbol{u}(\boldsymbol{x})))\,\mathrm{d}\boldsymbol{x} + \int_{\boldsymbol{u}(\Omega)} |D\boldsymbol{n}(\boldsymbol{y})|^2\,\mathrm{d}\boldsymbol{y}$$

- Bladon-Terentjev-Warner [*J. Phys. II*, 1994]

$$W_{\mathrm{mec}}(\boldsymbol{F}, \boldsymbol{n}) := \frac{\mu}{2}\,\mathrm{tr}\left(\boldsymbol{L}_r \boldsymbol{F}^T \boldsymbol{L}^{-1}(\boldsymbol{n})\boldsymbol{F}\right)$$

$$\boldsymbol{L}(\boldsymbol{n}) = a^{\frac{2}{3}}\boldsymbol{n}\otimes\boldsymbol{n} + a^{-\frac{1}{3}}\left(\boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n}\right)$$

- Cesana-DeSimone [*M3AS*, 2009]
- DeSimone-Teresi [*Eur. Phys. J. E*, 2009]
- Agostiniani-DeSimone [*Int. J. Nonlin. Mech.*, 2012]

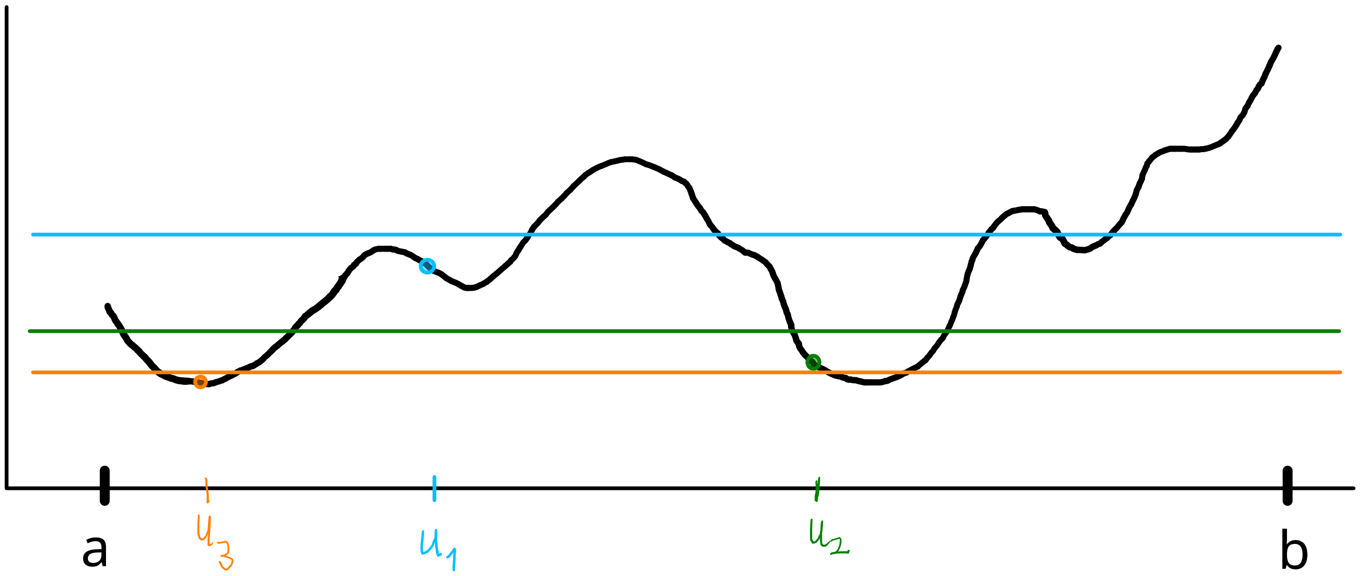
Existence of minimizers

Direct method of the calculus of variations

Minimize $F : [a,b] \to \mathbb{R}$

Weierstrass extreme value theorem:

Case in which F is bounded below.



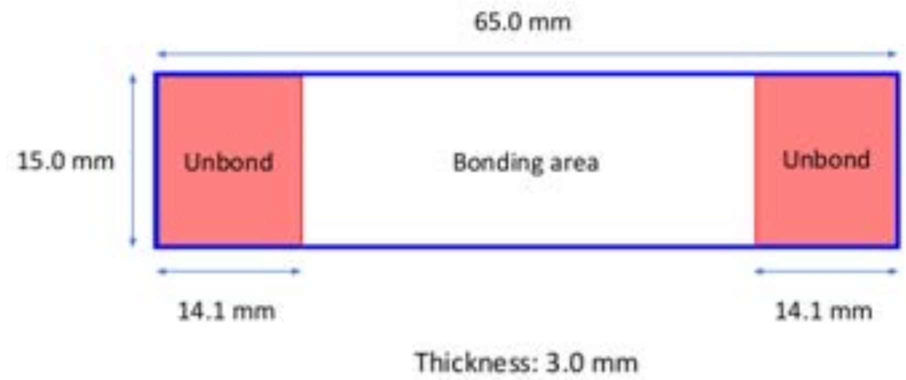
$$F(u_j) \xrightarrow[j \to \infty]{} \inf_{[a,b]} F$$

$$u_{j_k} \xrightarrow[k \to \infty]{} u_\infty$$

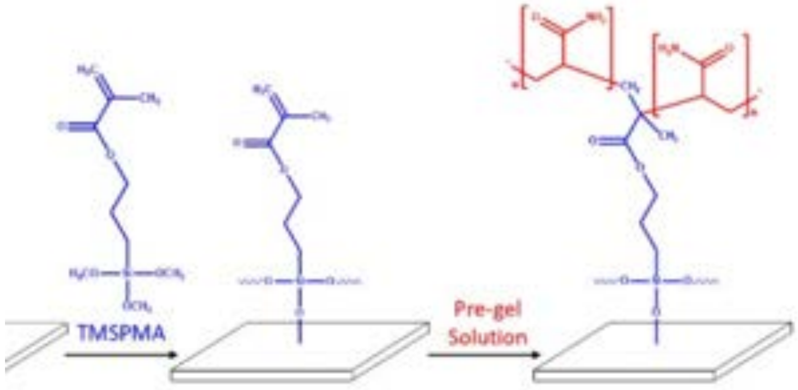
$$F[u_\infty] = \lim_{k \to \infty} F[u_k]$$

Joint with C. Calderer, M. Sánchez, R. Siegel, S. Song:

Initial state of partially bonding gel on the glass slide

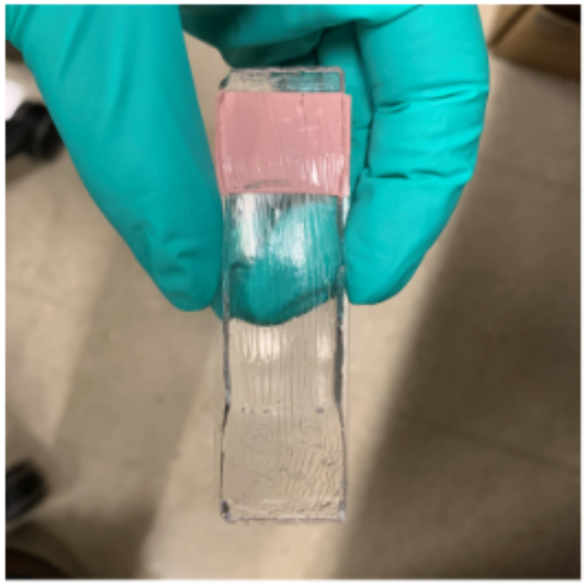


Polyacrylamide (PAAm) gel
T = 296 K
G = 0.13 MPa
$\phi_0 = 0.2035$



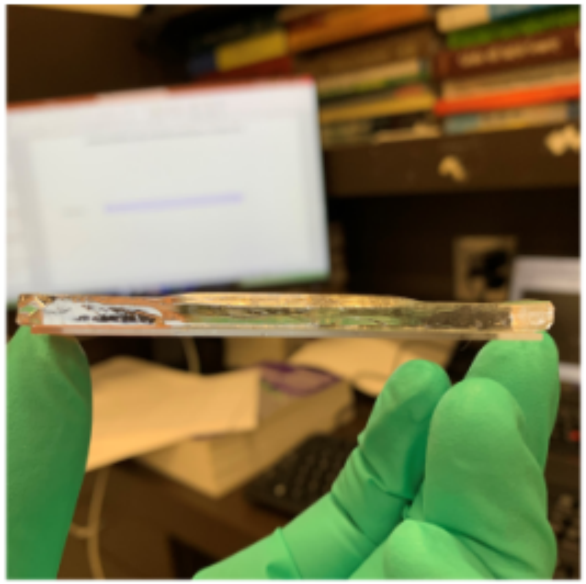
Swelling equilibrium state of partially bonding gel on the glass slide

Top view



Left view

Main view



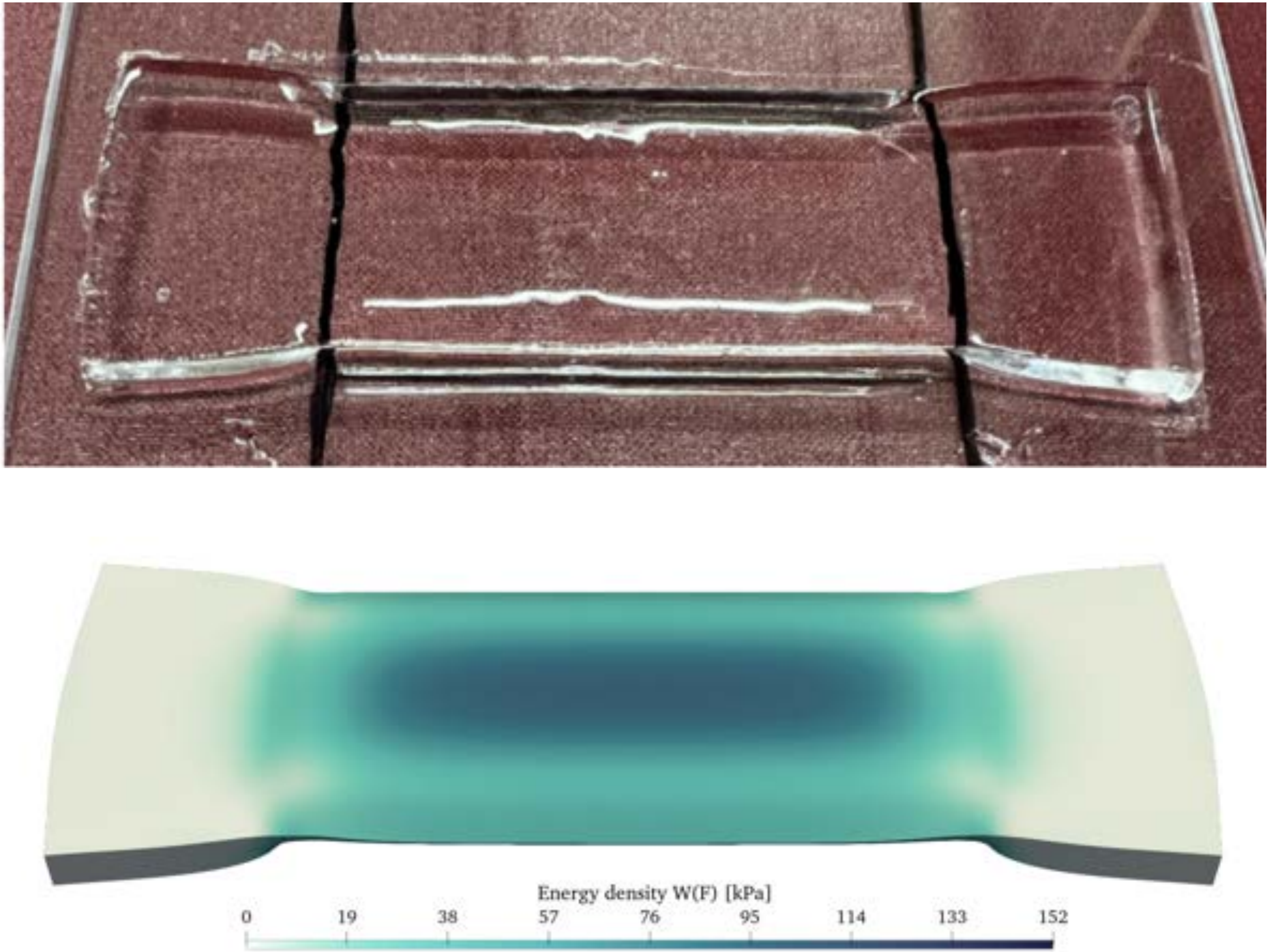
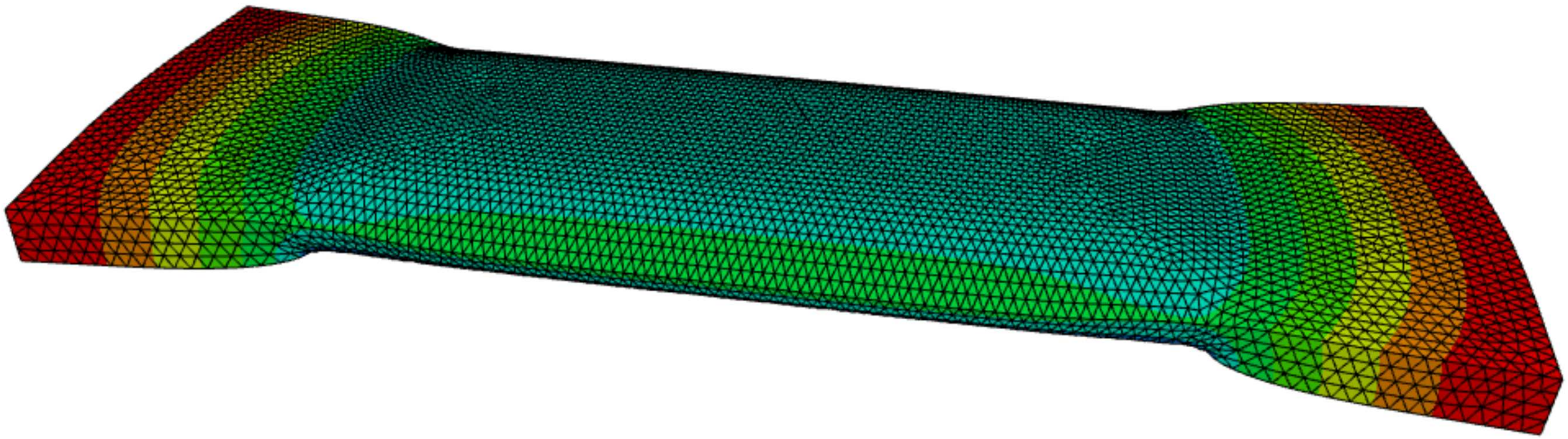
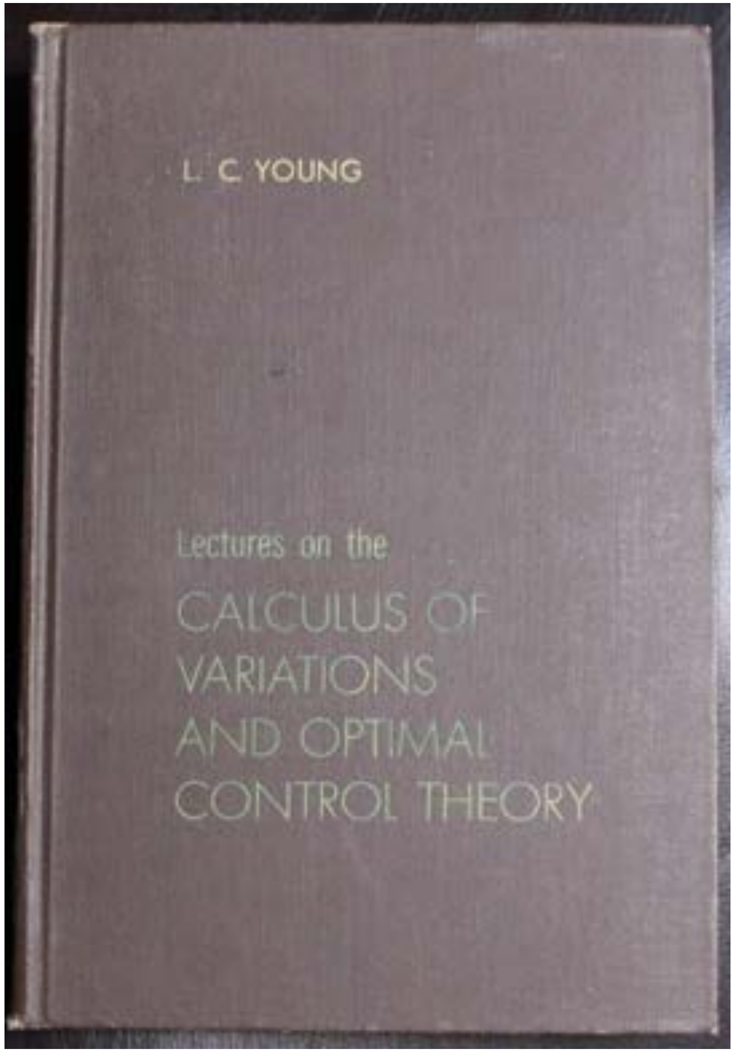


Figure 4: The upper panel is the top view of the partially bonded gel with $\delta = 0.9$ and reference configuration $90.0\,\text{mm} \times 23.5\,\text{mm} \times 3.00\,\text{mm}$ at swelling equilibrium. The lower panel illustrates the simulated deformed gel shape summarized in the third entry of Table 7, with the average energy density of 74.7 kPa.

Netgen/NGSolve



- Quadratic elements
- 418509 degrees of freedom
- $u_1 = u_2 = u_3 = 0$ on the bonded part of the interface
- $u_2 = 0$ on the debonded part of the interface
- Gravity
- Obstacle constraint: $u_2 \geq 0$
- Incremental softening



$$\varphi \in C_c^\infty(\Omega) \mapsto \int_\Omega u(x)\varphi(x)\,\mathrm{d}x$$

Weak compactness

Banach-Alaoglu-Bourbaki theorem:

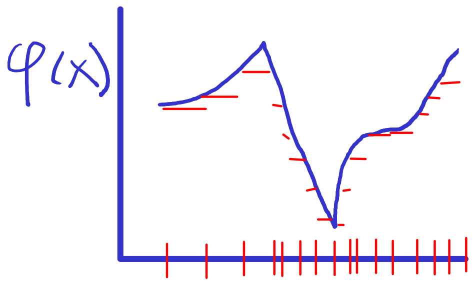
$$\int_\Omega |u_j(x)|^p \leq M \quad \text{for all } j \in \mathbb{N}$$

$\Rightarrow$ there exists $u \in L^p(\Omega)$ such that $\qquad u_j \xrightarrow{L^p} u$

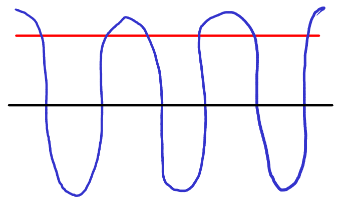
for all $\varphi \in L^q(\Omega)$ $\qquad \displaystyle\int_\Omega u_j(x)\varphi(x)\,dx \longrightarrow \int_\Omega u(x)\varphi(x)\,dx.$

$$\left(\frac{1}{p} + \frac{1}{q} = 1 \right)$$

Example: $\Omega = (0,1)$, $\quad u_j(x) = \text{sen}(\pi j x)$



$u_j \longrightarrow 0$



Weak compactness

$$u_j \rightharpoonup u, \quad v_j \rightharpoonup v \quad \not\Rightarrow \quad u_j v_j \rightharpoonup uv$$

Example: $\Omega = (0,1)$, $p = 2$, $u_j = \text{sen}(\pi j x)$

$v_j = \text{sen}(\pi j x)$

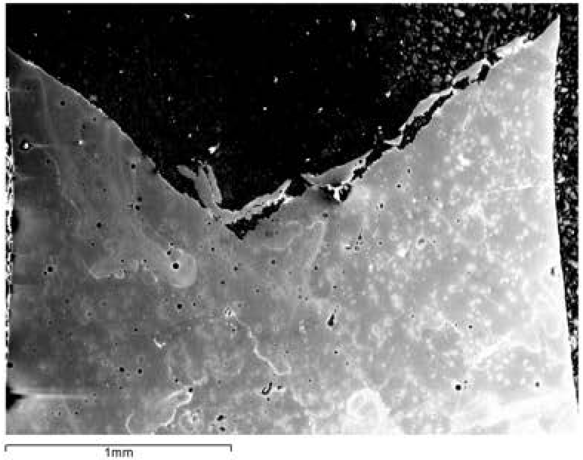
$$\int u_j v_j \varphi = \int u_j v_j = \int_0^1 \frac{1 - \cos(2\pi j x)}{2} \, dx$$

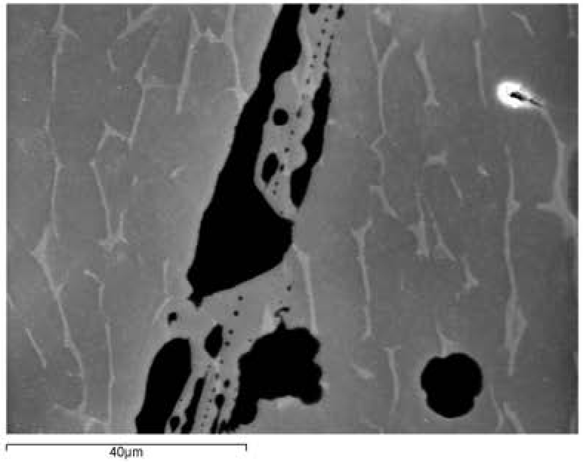
$\downarrow$

$\varphi \equiv 1$

$$\xrightarrow{j \to \infty} \frac{1}{2} \neq \int_0^1 0 \cdot \varphi$$

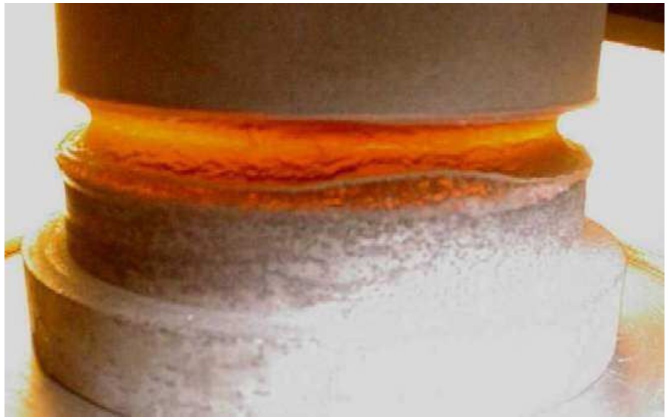
Ductile fracture

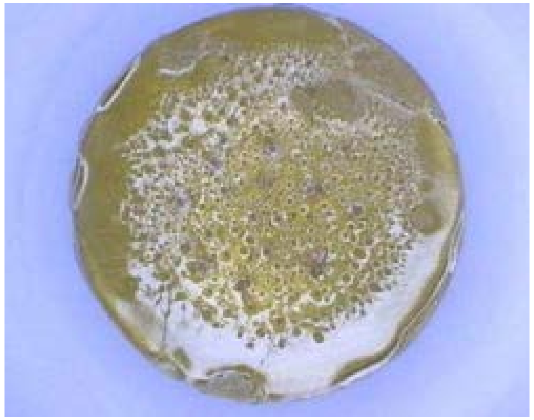




N. Petrinic, J. L. Curiel Sosa, C. R. Siviour, B. C. F. Elliot: Improved Predictive Modelling of Strain Localisation and Ductile Fracture in a Ti-64Al-4V Alloy Subjected to Impact Loading. *J. Phys. IV France* **134** (2006), 147–155.

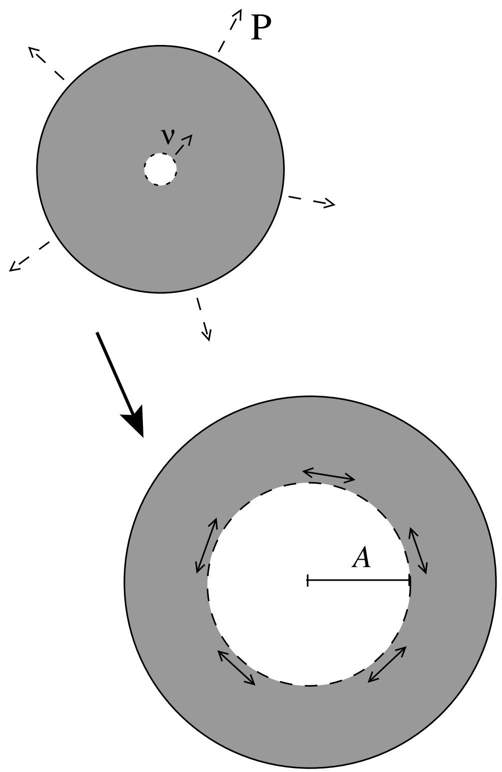
Cavitation



Hydroxyl-terminated polybutadiene (HTPB)

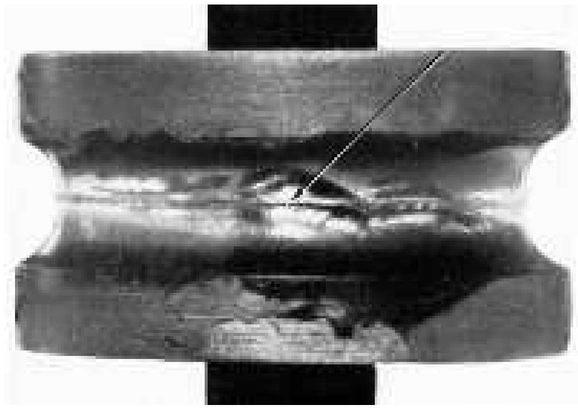
Courtesy of Robert Nevière (SNPE Matériaux Energétiques,
Centre de Recherches du Bouchet)



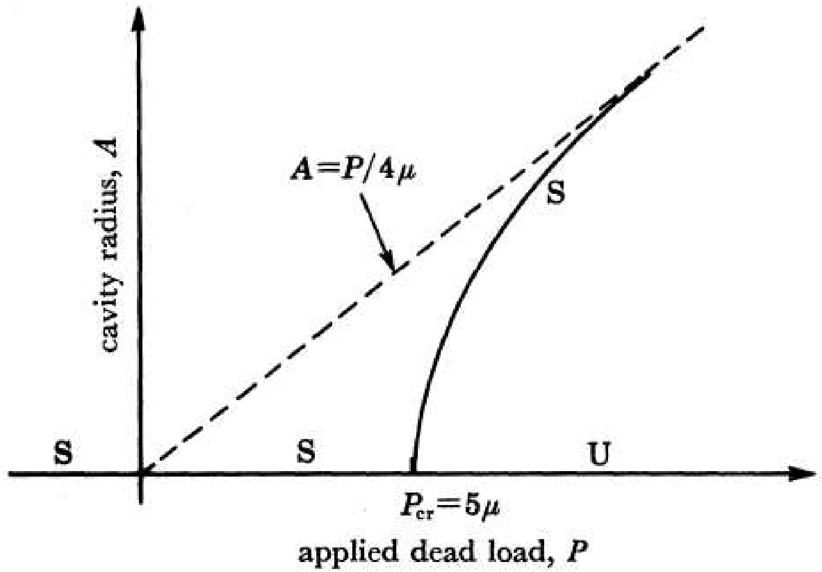
- $\boldsymbol{u}(\boldsymbol{x}) = u(r)\dfrac{\boldsymbol{x}}{r},\ \ r = |\boldsymbol{x}|$
- $u'u^{n-1} = r^{n-1} \Rightarrow u(r) = \left(A^n + r^n\right)^{\frac{1}{n}}$
- $T(r) = \displaystyle\int_{v(r)}^{v(1)} \frac{1}{v^n - 1}\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}v}\,\mathrm{d}v$

- Gent & Lindley '59, Ball '82
- $\frac{1}{v^n-1}\frac{\mathrm{d}\hat{\Phi}(v)}{\mathrm{d}v} \in L^1(1+\delta,\infty)$
- For $W(\boldsymbol{F}) = \frac{\mu}{p}|\boldsymbol{F}|^p$, this is $p < n$.

Incompressible limit: $r(R) = (A^n + R^n)^{\frac{1}{n}}$, $A > 0$ cavity radius.

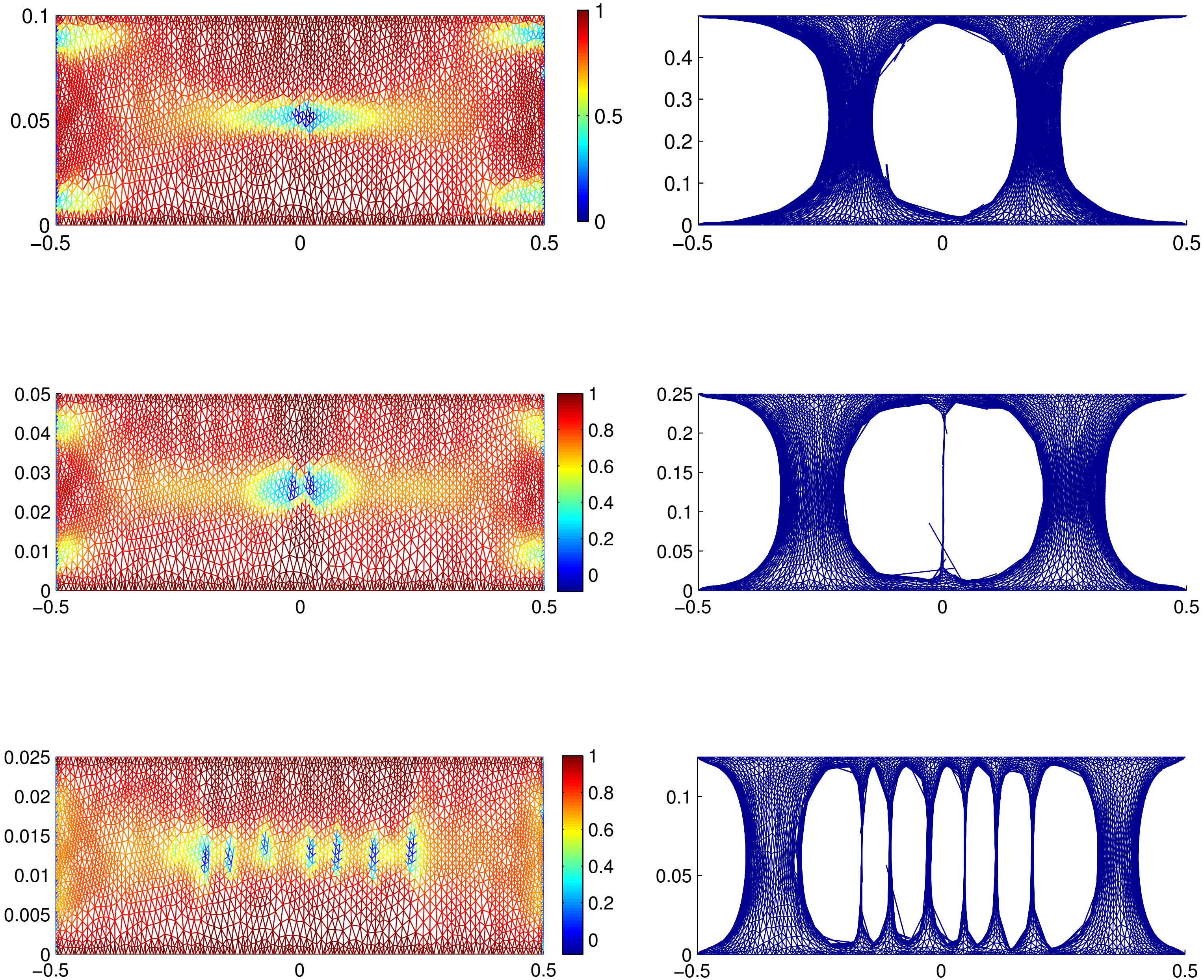


Gent & Lindley, 1959.

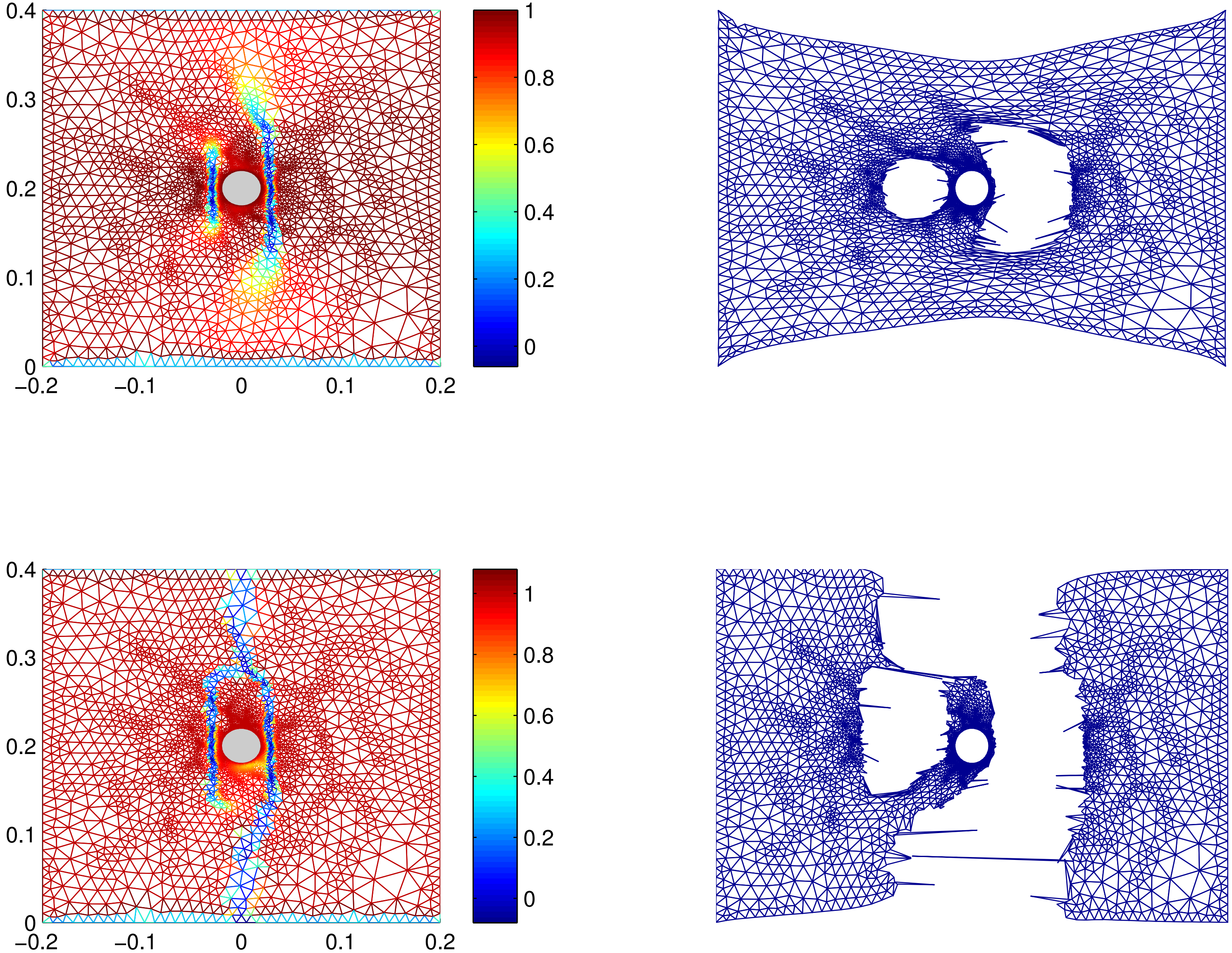


Ball, 1982.

H., Mora-Corral & Xu CMAME '16



Rigid inclusion



Kumar & Lopez-Pamies, *J. Mech. Phys. Solids* **150** (2021) 104359
Kumar, Bourdin, Francfort & Lopez-Pamies, *JMPS* **142** (2020) 104027
Francfort, Giacomini & Lopez-Pamies, *Analysis and PDE* **12** (2019)
Poulain, Lefèvre, Lopez-Pamies & Ravi-Chandar, *Int J Fract* **205** (2017) 1-21

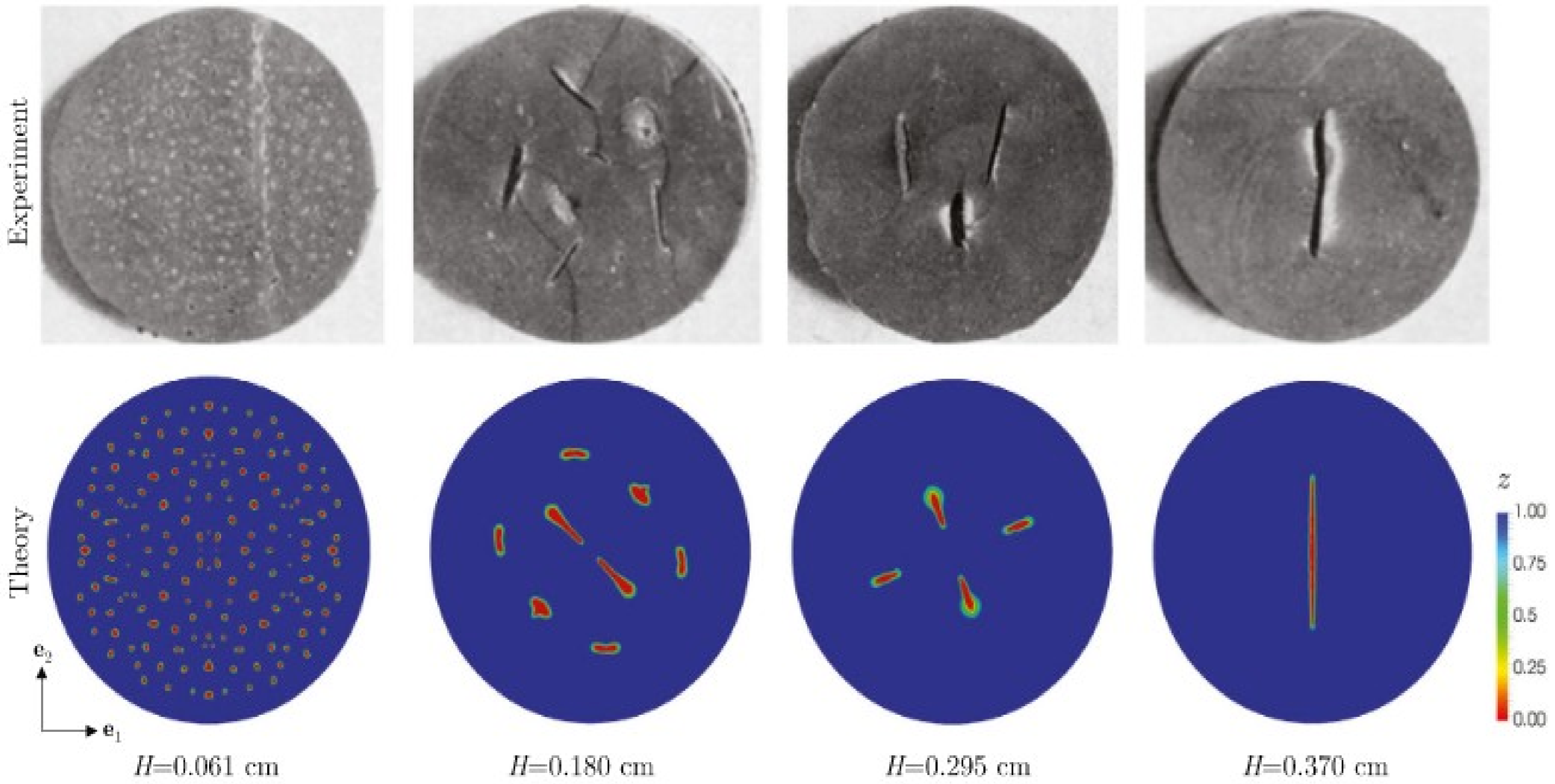
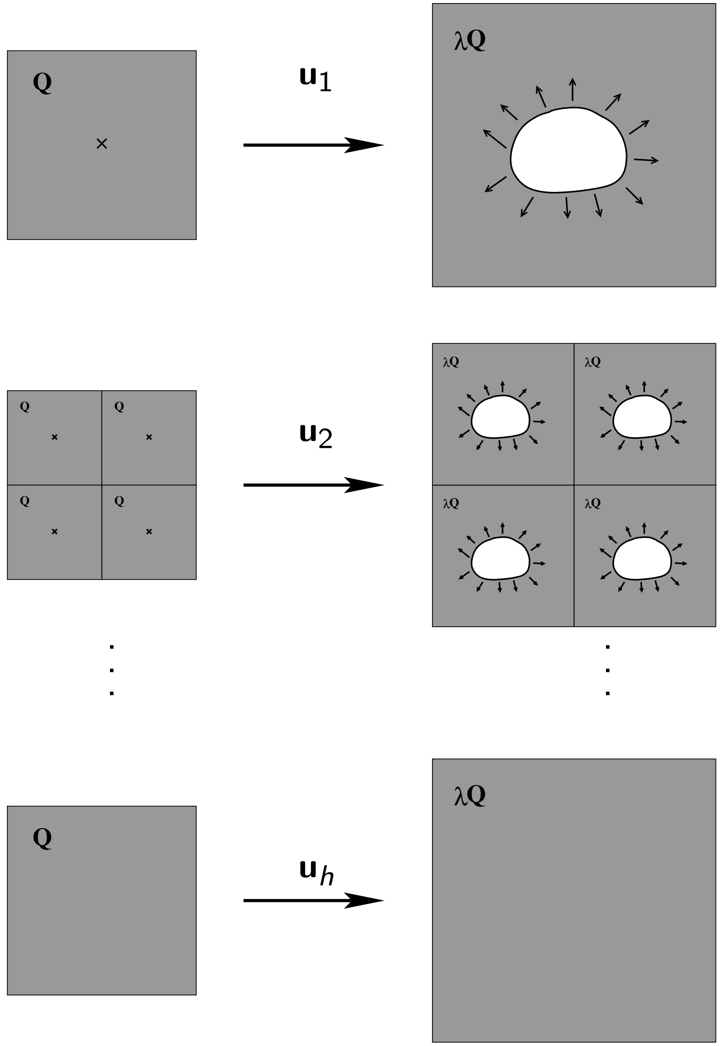


Fig. 11. Comparison between theory and experiment for the post-mortem images of the midplane of poker-chip specimens – cut open after reaching a normalized force of $S = 2.75$ MPa – with four increasing initial thicknesses H.

Ball & Murat 1984



B.C.: $\boldsymbol{u}(\boldsymbol{x}) = \lambda \boldsymbol{x}$ on ∂Q.

$$
\begin{aligned}
D\boldsymbol{u}_j \quad &\rightharpoonup \quad ⨍_Q D\boldsymbol{u}_1 \\
&= \quad \int_{\partial Q} \boldsymbol{u}_1 \otimes \boldsymbol{\nu} \, \mathrm{d}\mathcal{H}^{n-1} \\
&= \quad ⨍_Q \lambda \mathbf{1} \;=\; \lambda \mathbf{1}
\end{aligned}
$$

Hence $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $W^{1,p}$, but

$$1 = \det D\boldsymbol{u}_j \not\rightharpoonup \det D\boldsymbol{u} = \lambda^2$$

Quasiconvexity; lower semicontinuity

Classical existence theory in nonlinear elasticity

$$\min \int_\Omega W(D\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}$$

$$W(\boldsymbol{F}) = g(\boldsymbol{F}, \operatorname{cof}\boldsymbol{F}, \det\boldsymbol{F}),$$

$$W(\boldsymbol{F}) \geq C(|\boldsymbol{F}|^p + |\operatorname{cof}\boldsymbol{F}|^q - 1)$$

Ball '77: $p \geq 2$, $q \geq p'$.

Müller, Qi & Yan '94: $p = 2$, $q \geq \frac{3}{2}$.

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) - u\frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) - u\frac{\partial^2 v}{\partial x \partial y}$$

Schwarz's theorem

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) - u\frac{\partial^2 v}{\partial x \partial y}$$

Schwarz's theorem

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) - u\frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) - u\frac{\partial^2 v}{\partial x \partial y}$$

Schwarz's theorem

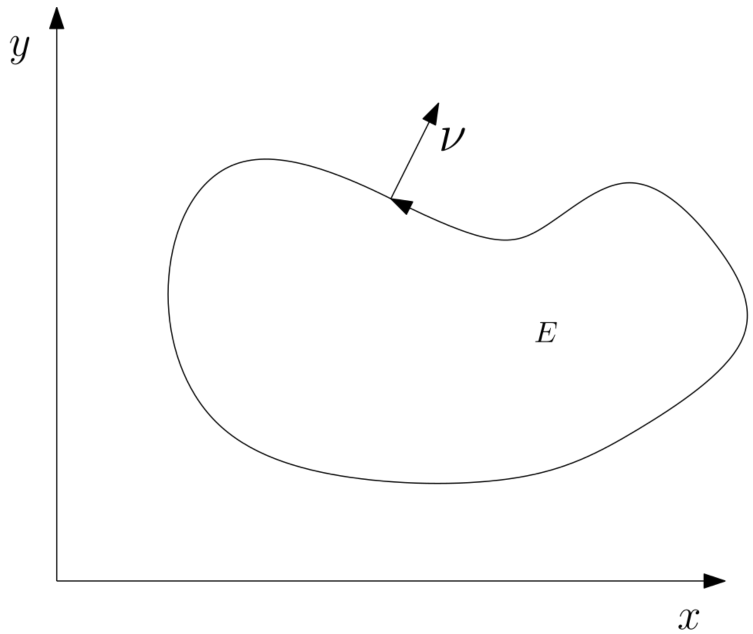
$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) - u\frac{\partial^2 v}{\partial y \partial x}$$

Distributional determinant

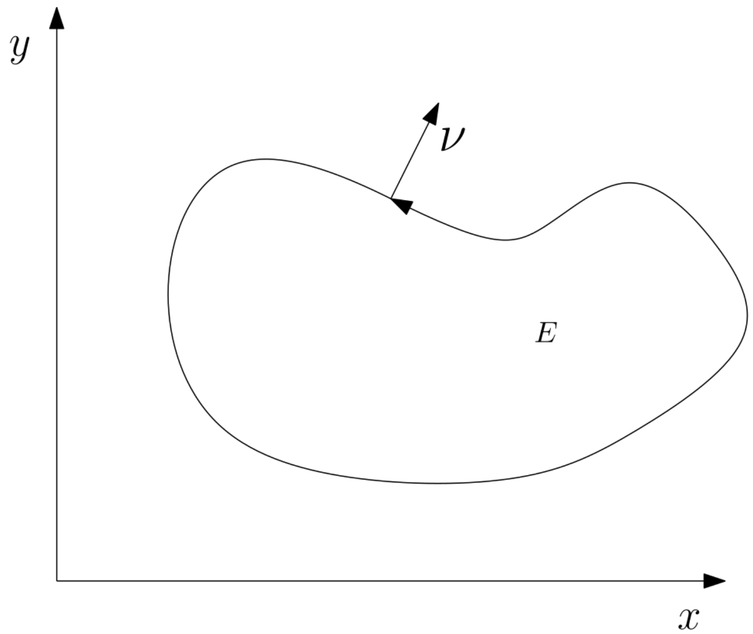
$$\int_\Omega \frac{\partial(u,v)}{\partial(x,y)}\varphi = \int_\Omega \begin{vmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{vmatrix} \varphi = \underbrace{-\int_\Omega u\frac{\partial v}{\partial y}\frac{\partial \varphi}{\partial x} + u\frac{\partial v}{\partial x}\frac{\partial \varphi}{\partial y}}_{:=\mathrm{Det}\, D(u,v)}$$

Topology. Green's theorem



$$\int_{\partial E} P\mathrm{d}x + Q\mathrm{d}y = \int_{\partial E} \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\,\mathrm{d}A$$

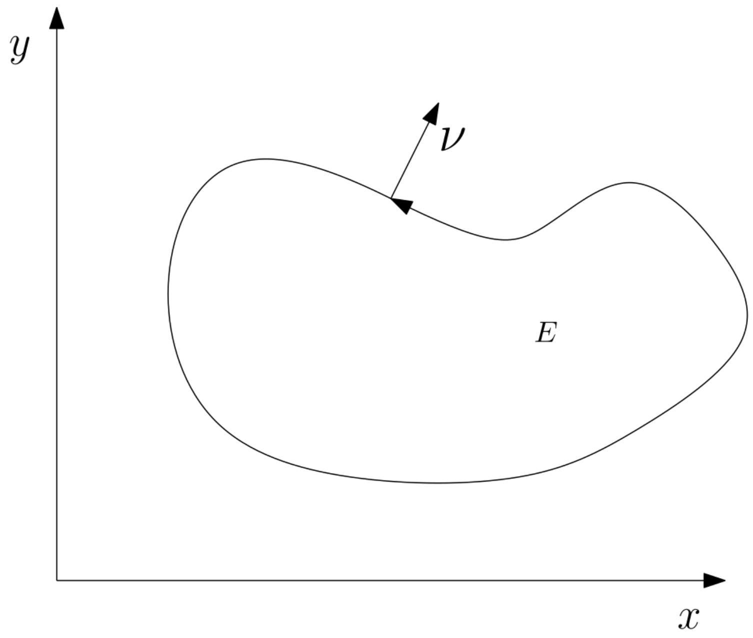
Topology. Green's theorem



$$\int_{\partial E} P\mathrm{d}x + Q\mathrm{d}y = \int_{\partial E} \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\,\mathrm{d}A$$

$$\int_{\partial E} y\mathrm{d}x - x\mathrm{d}y = \int_{E} (1 - (-1))\,\mathrm{d}A$$

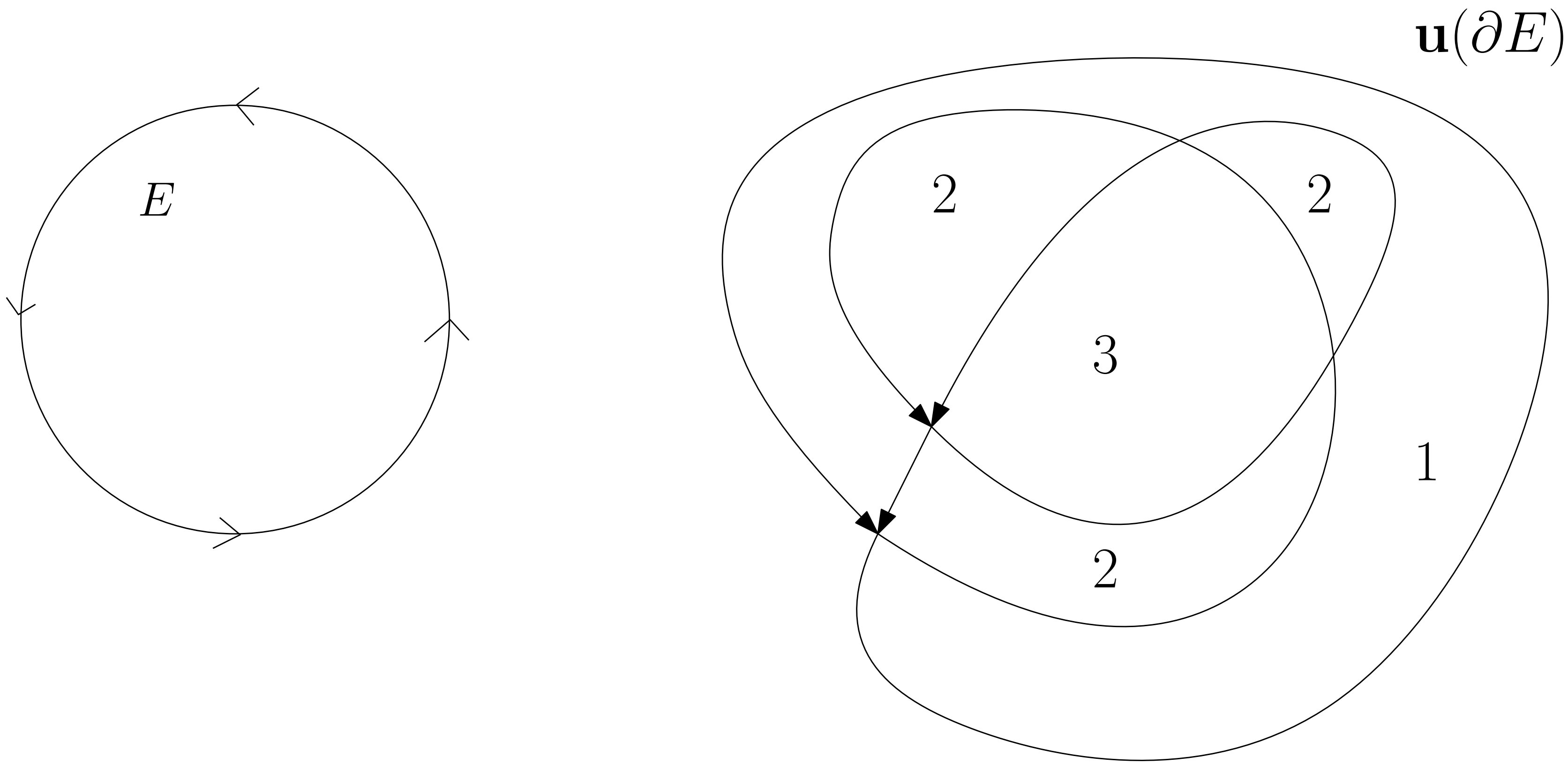
Topology. Green's theorem



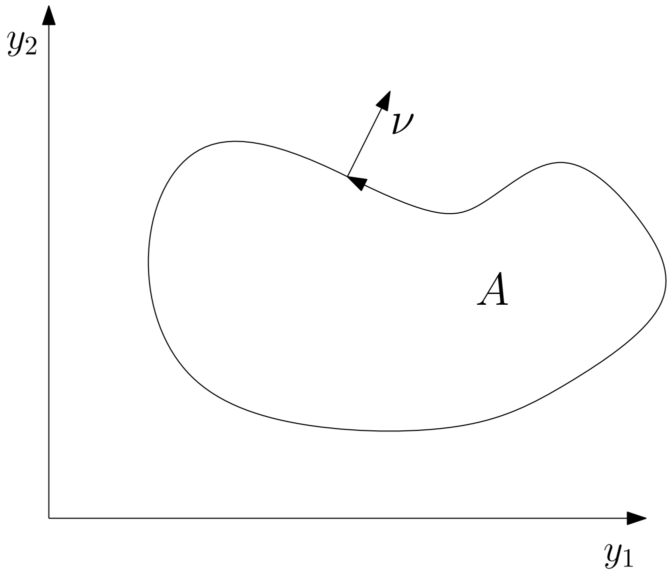
$$\int_{\partial E} P\mathrm{d}x + Q\mathrm{d}y = \int_{\partial E} \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\,\mathrm{d}A$$

$$\int_{\partial E} y\mathrm{d}x - x\mathrm{d}y = \int_{E} (1 - (-1))\,\mathrm{d}A$$

$$\int_{\partial E} \begin{vmatrix} x & y \\ \mathrm{d}x & \mathrm{d}y \end{vmatrix} = 2\mathrm{area(E)}$$


E
$\mathbf{u}(\partial E)$
2
2
3
1
2

Two dimensional interlude



$$\int_{\partial A} \boldsymbol{g}(\boldsymbol{y}) \cdot \boldsymbol{\nu}(\boldsymbol{y}) \, \mathrm{d}s = \int_A \operatorname{div} \boldsymbol{g}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}$$

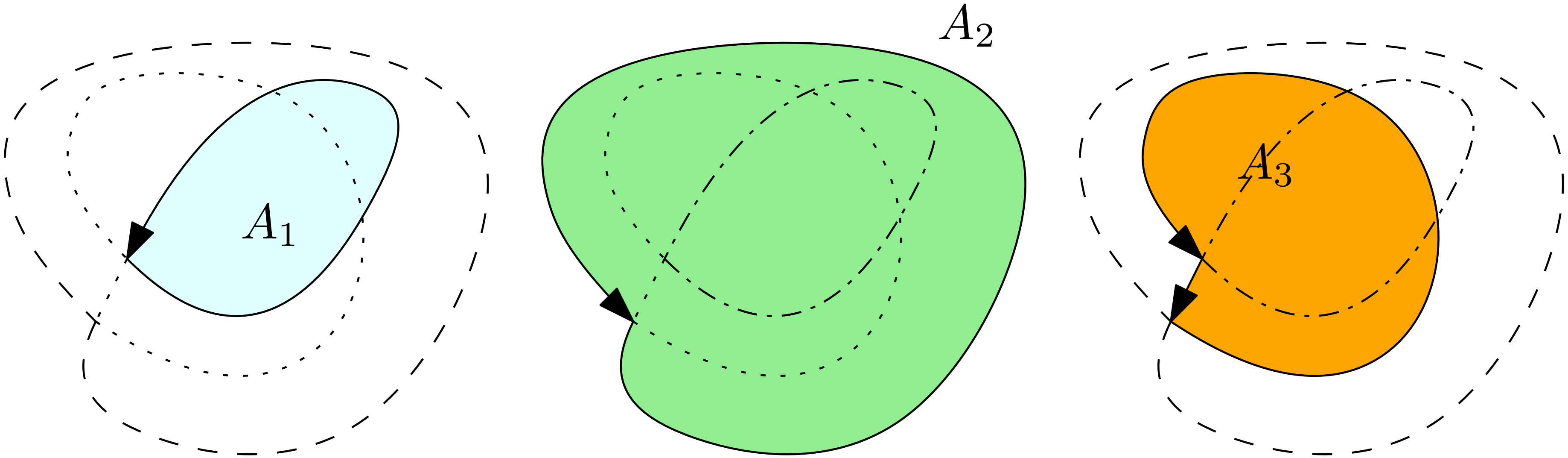
$$\int_{\boldsymbol{y} \in \partial A} g_1(\boldsymbol{y}) \, \mathrm{d}y_2 + g_2(\boldsymbol{y}) \cdot (-\, \mathrm{d}y_1)$$

$$= \int_A \frac{\partial g_1}{\partial y_1} + \frac{\partial g_2}{\partial y_2} \, \mathrm{d}\boldsymbol{y}$$

If ∂A were $\boldsymbol{u}(\partial E)$ and $\boldsymbol{u}|_{\partial E}$ were injective and orientation preserving, the line integral could be rewritten as:

$$\int_{s=a}^{b} g_1\Big(\boldsymbol{u}\big(\boldsymbol{x}(s)\big)\Big) \frac{\mathrm{d}}{\mathrm{d}s} u_2\big(\boldsymbol{x}(s)\big) - g_2\Big(\boldsymbol{u}\big(\boldsymbol{x}(s)\big)\Big) \frac{\mathrm{d}}{\mathrm{d}s} u_1\big(\boldsymbol{x}(s)\big) \; \mathrm{d}s$$

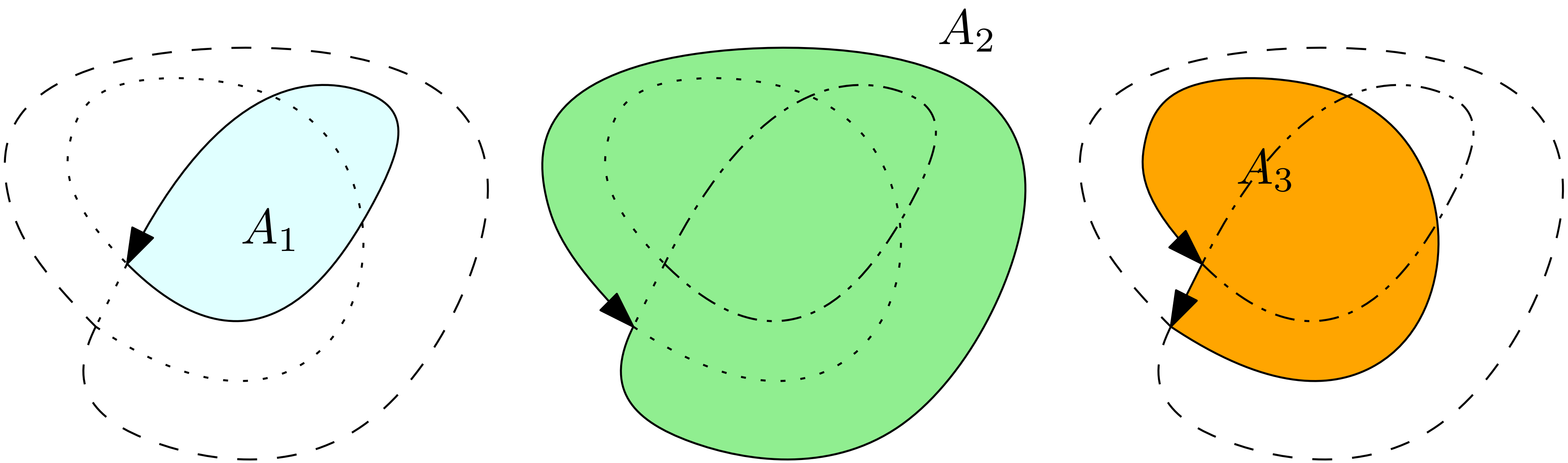
But, in general,



$$\int_{s=a}^{b} g_1\Big(\boldsymbol{u}\big(\boldsymbol{x}(s)\big)\Big)\frac{\mathrm{d}}{\mathrm{d}s}u_2\big(\boldsymbol{x}(s)\big) - g_2\Big(\boldsymbol{u}\big(\boldsymbol{x}(s)\big)\Big)\frac{\mathrm{d}}{\mathrm{d}s}u_1\big(\boldsymbol{x}(s)\big)\ \mathrm{d}s$$

$$= \left(\int_{\partial A_1} + \int_{\partial A_2} + \int_{\partial A_3}\right)\boldsymbol{g}\cdot\boldsymbol{\nu}$$

$$= \left(\int_{A_1} + \int_{A_2} + \int_{A_3}\right)\operatorname{div}\boldsymbol{g}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^3}\deg(\boldsymbol{u}, E, \boldsymbol{y})\operatorname{div}\boldsymbol{g}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$



Both in 2D and in 3D, the formula can be written as:

$$\int_{\partial E} \boldsymbol{g}\big(\boldsymbol{u}(\boldsymbol{x})\big) \cdot \big(\operatorname{cof} D\boldsymbol{u}(\boldsymbol{x})\big)\boldsymbol{\nu}(\boldsymbol{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x}) = \int_{\mathbb{R}^n} \deg(\boldsymbol{u}, E, \boldsymbol{y}) \operatorname{div} \boldsymbol{g}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

Now,

$$\int_{\mathbb{R}^n} \deg(\boldsymbol{u}, E, \boldsymbol{y}) \operatorname{div} \boldsymbol{g}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = \int_{\partial E} \boldsymbol{g}\big(\boldsymbol{u}(\boldsymbol{x})\big) \cdot \big(\operatorname{cof} D\boldsymbol{u}(\boldsymbol{x})\big)\boldsymbol{\nu}(\boldsymbol{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{x})$$

$$= \int_{\partial E} \Big((\operatorname{adj} D\boldsymbol{u})\boldsymbol{g} \circ \boldsymbol{u} \Big) \cdot \boldsymbol{\nu} \, \mathrm{d}A$$

$$= \int_E \operatorname{Div} \Big(\big(\operatorname{adj} D\boldsymbol{u}\big)\boldsymbol{g} \circ \boldsymbol{u} \Big) \, \mathrm{d}\boldsymbol{x}$$

$$= \int_E \underbrace{\Big(\operatorname{Div}(\operatorname{adj} D\boldsymbol{u})^T \Big)}_{Piola} \cdot \boldsymbol{g} \circ \boldsymbol{u} + (\operatorname{adj} D\boldsymbol{u}) \cdot \Big(D_{\boldsymbol{y}}\boldsymbol{g}\big(\boldsymbol{u}(\boldsymbol{x})\big) D\boldsymbol{u}(\boldsymbol{x}) \Big) \, \mathrm{d}\boldsymbol{x}$$

$$= \int_E (\operatorname{div}_{\boldsymbol{y}} \boldsymbol{g})\Big(\boldsymbol{u}(\boldsymbol{x})\Big) \cdot \det D\boldsymbol{u}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

$$= \int_E \big(\operatorname{sgn} \det D\boldsymbol{u}(\boldsymbol{x})\big) \cdot (\operatorname{div} \boldsymbol{g})\big(\boldsymbol{u}(\boldsymbol{x})\big) |\det D\boldsymbol{u}(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x}$$

$$= \int_{\boldsymbol{y} \in \boldsymbol{u}(E)} \underbrace{\left(\sum_{\substack{\boldsymbol{x} \in E \\ \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{y}}} \operatorname{sgn} \det D\boldsymbol{u}(\boldsymbol{x}) \right)}_{= \deg(\boldsymbol{u}, E, \boldsymbol{y})} \operatorname{div} \boldsymbol{g}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}.$$

Classical existence theory in nonlinear elasticity

$$\min \int_\Omega W(D\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}$$

$$W(\boldsymbol{F}) = g(\boldsymbol{F}, \operatorname{cof}\boldsymbol{F}, \det\boldsymbol{F}),$$

$$W(\boldsymbol{F}) \geq C(|\boldsymbol{F}|^p + |\operatorname{cof}\boldsymbol{F}|^q - 1)$$

Ball '77: $p \geq 2$, $q \geq p'$.

Müller, Qi & Yan '94: $p = 2$, $q \geq \frac{3}{2}$.

Growth at infinity

$$\min \int_\Omega |D\boldsymbol{u}(\boldsymbol{x})|^p + H(\det D\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}$$

Both Ball '77 and Müller, Qi & Yan '94: $p \geq 3$.

Growth at infinity

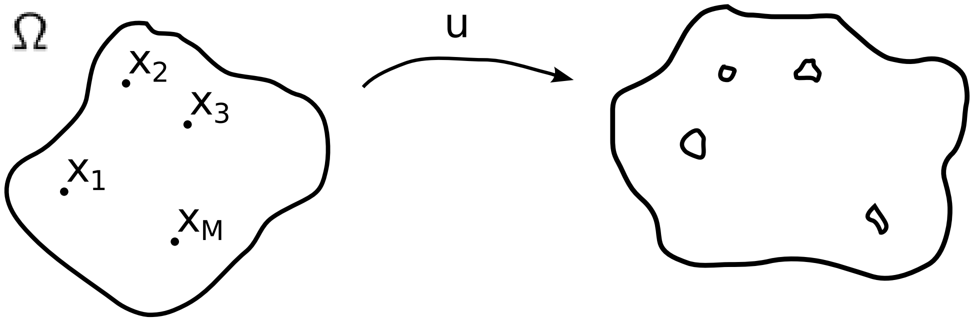
$$\min \int_\Omega |D\boldsymbol{u}(\boldsymbol{x})|^p + H(\det D\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}$$

Both Ball '77 and Müller, Qi & Yan '94: $p \geq 3$.

NeoHookean materials

$$W(\boldsymbol{F}) = \frac{\mu}{2}(|\boldsymbol{F}|^2 - 3) + \mu \ln\left(\frac{1}{\det \boldsymbol{F}}\right) + \frac{\lambda}{2}(\det \boldsymbol{F} - 1)^2$$

The distributional determinant



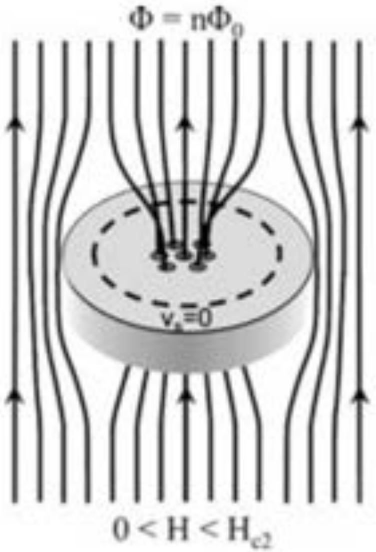
For every $\phi \in C_c^\infty(\Omega)$

$$\langle \operatorname{Det} D\boldsymbol{u} - \det D\boldsymbol{u}, \phi \rangle = -\frac{1}{3}\int_\Omega (\boldsymbol{u} \cdot (\operatorname{cof} D\boldsymbol{u}) D\phi + \phi \det D\boldsymbol{u}) \, \mathrm{d}\boldsymbol{x}$$

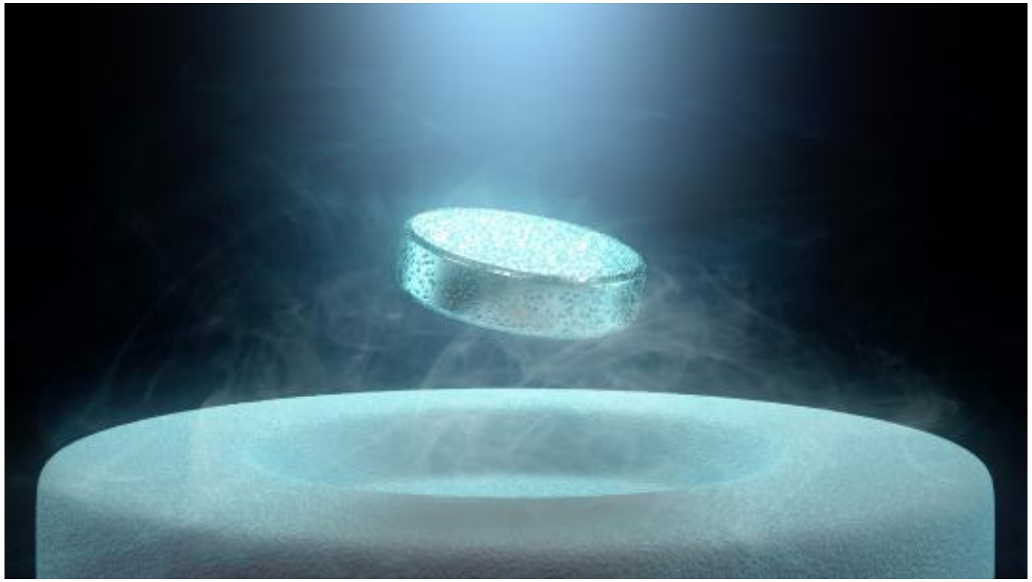
$$= \sum_{i=1}^{M} \phi(\boldsymbol{x}_i) \int_{\partial C_i} \frac{\boldsymbol{y}}{3} \cdot \boldsymbol{\nu}(\boldsymbol{y}) \, \mathrm{d}\mathcal{H}^2(\boldsymbol{y})$$

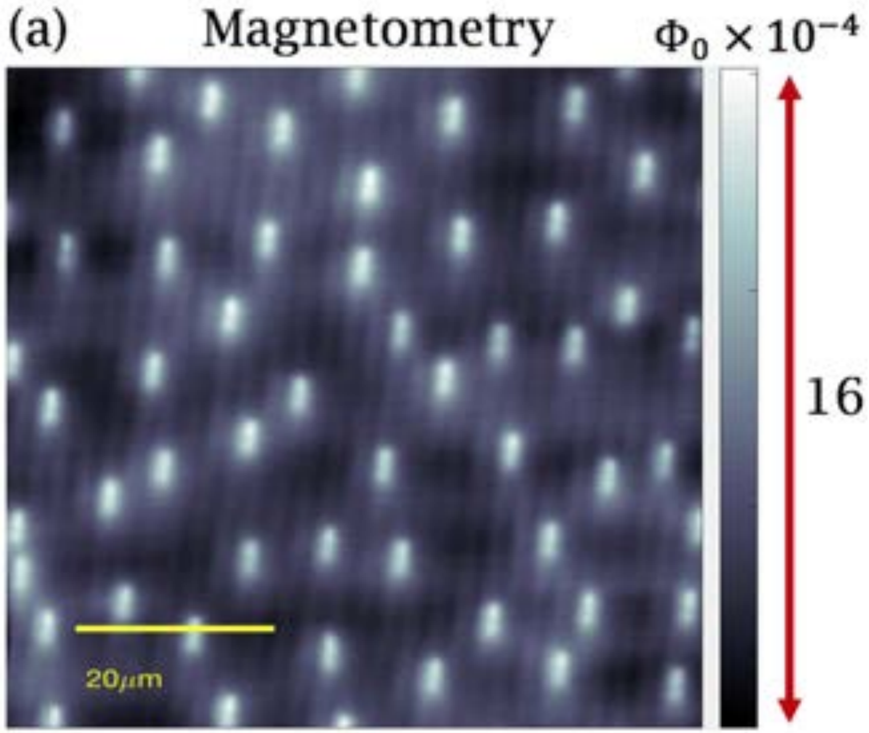
Cavitation points

$$\operatorname{Det} D\boldsymbol{u} = (\det D\boldsymbol{u})\mathcal{L}^3 + \sum_{i=1}^{M} \alpha_i \delta_{\boldsymbol{x}_i}$$

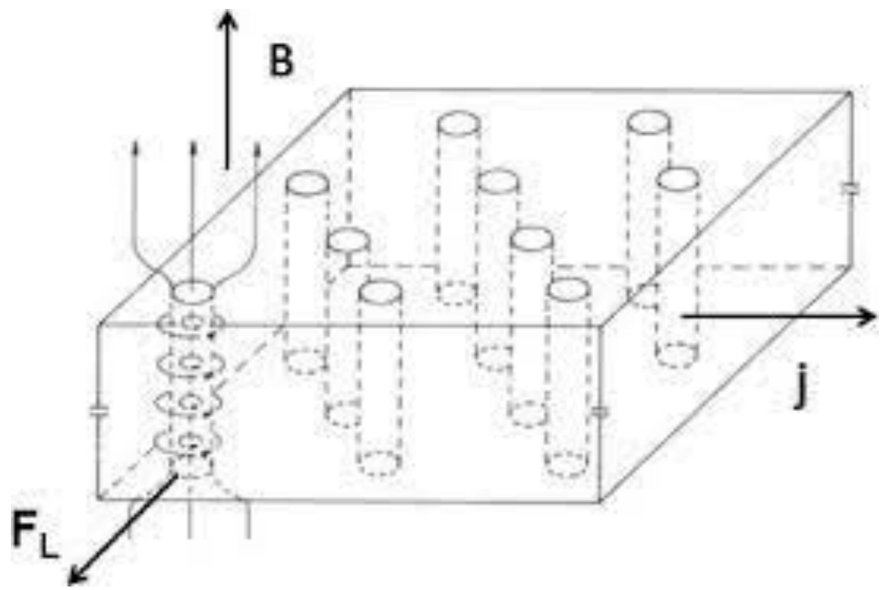


Flükiger [Rev. Acc. Sci. Tech. 5, 2012]


Lukyanchuk et al. [Nature Physics 11, 2015]



Davis et al [Phys Rev B 98, 2018]



Flükiger [Rev. Acc. Sci. Tech. 5, 2012]

Calc. Var. 14, 151–191 (2002) **Calculus of Variations**

DOI (Digital Object Identifier) 10.1007/s005260100093

Robert L. Jerrard · Halil Mete Soner

The Jacobian and the Ginzburg-Landau energy

Received: 15 December 2000 / Accepted: 23 January 2001 /
Published online: 25 June 2001 –

Abstract. We study the Ginzburg-Landau functional

$$I_\epsilon(u) := \frac{1}{\ln(1/\epsilon)} \int_U \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}\left(1-|u|^2\right)^2 dx \,,$$

for $u \in H^1(U;\mathbb{R}^2)$, where U is a bounded, open subset of $\mathbb{R}^2$. We show that if a sequence of functions u^ϵ satisfies $\sup I_\epsilon(u^\epsilon) < \infty$, then their Jacobians Ju^ϵ are precompact in the dual of $C_c^{0,\alpha}$ for every $\alpha \in (0,1]$. Moreover, any limiting measure is a sum of point masses. We also characterize the Γ-limit $I(\cdot)$ of the functionals $I_\epsilon(\cdot)$, in terms of the function space $B2V$ introduced by the authors in [16,17]: we show that $I(u)$ is finite if and only if $u \in B2V(U;S^1)$, and for $u \in B2V(U;S^1)$, $I(u)$ is equal to the total variation of the Jacobian measure Ju. When the domain U has dimension greater than two, we prove if $I_\epsilon(u^\epsilon) \le C$ then the Jacobians Ju^ϵ are again precompact in $\left(C_c^{0,\alpha}\right)^*$ for all $\alpha \in (0,1]$, and moreover we show that any limiting measure must be integer multiplicity rectifiable. We also show that the total variation of the Jacobian measure is a lower bound for the Γ limit of the Ginzburg-Landau functional.

arXiv:2407.08285v1 [math.AP] 11 Jul 2024

Approximation of topological singularities through free discontinuity functionals: the critical and super-critical regimes

V. CRISMALE, L. DE LUCA, AND R. SCALA

Theorem 3.3. *The following Γ-convergence result holds true.*

(i) *(Compactness) Let $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ be such that*

$$\sup_{\varepsilon>0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|^2} \le C, \tag{3.11}$$

for some $C>0$. Then there exist a measure $\mu \in \mathcal{M}(\Omega)\cap H^{-1}(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$ and a map $T^D \in L^2(\Omega;\mathbb{R}^2)$ with $-\operatorname{Div} T^D = \pi\mu$ such that, up to a subsequence,

$$\left\|\frac{Ju_\varepsilon}{\pi|\log\varepsilon|} - \mu\right\|_{\mathrm{flat},\Omega} \to 0 \tag{FJ}$$

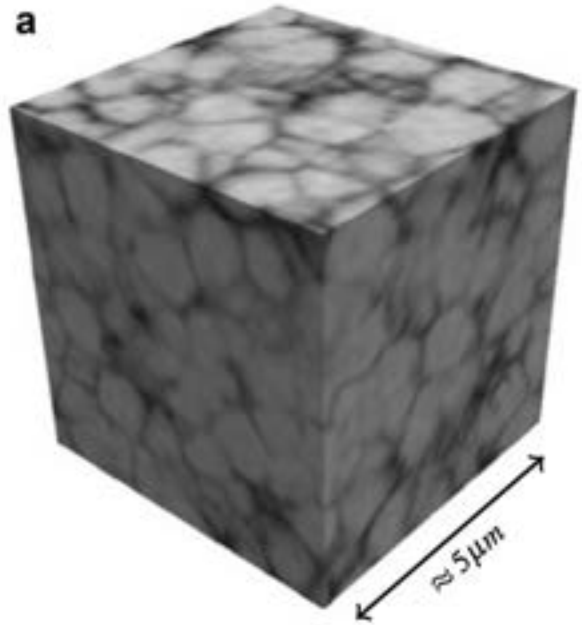
$$\frac{T^D_{u_\varepsilon}}{|\log\varepsilon|} \rightharpoonup T^D \text{ in } L^2(\Omega;\mathbb{R}^2). \tag{ACJ}$$

(ii) *(Γ-liminf inequality) For every $(\mu, T^D) \in (\mathcal{M}(\Omega)\cap H^{-1}(\Omega)) \times L^2(\Omega;\mathbb{R}^2)$ as in (i) and for every $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ satisfying (FJ) and (ACJ), it holds*

$$\pi|\mu|(\Omega) + 2\int_\Omega |T^D|^2 \,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|^2}. \tag{3.12}$$

(iii) *(Γ-limsup inequality) For every $(\mu, T^D) \in (\mathcal{M}(\Omega)\cap H^{-1}(\Omega)) \times L^2(\Omega;\mathbb{R}^2)$ as in (i) there exists $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ satisfying (FJ) and (ACJ), such that*

$$\pi|\mu|(\Omega) + 2\int_\Omega |T^D|^2 \,\mathrm{d}x \ge \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|^2}. \tag{3.13}$$



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- Garroni, Marziani & Scala (2021) SIAM J. Math. Anal.

arXiv:2405.13953

DECAY OF EXCESS FOR THE ABELIAN HIGGS MODEL

GUIDO DE PHILIPPIS, ARIA HALAVATI, AND ALESSANDRO PIGATI

ABSTRACT. In this article we prove that entire critical points (u, ∇) of the self-dual $U(1)$-Yang–Mills–Higgs functional E_1, with energy

$$E_1(u,\nabla;B_R) := \int_{B_R}\left[|\nabla u|^2 + \frac{(1-|u|^2)^2}{4} + |F_\nabla|^2\right] \le (2\pi+\tau(n))\omega_{n-2}R^{n-2}$$

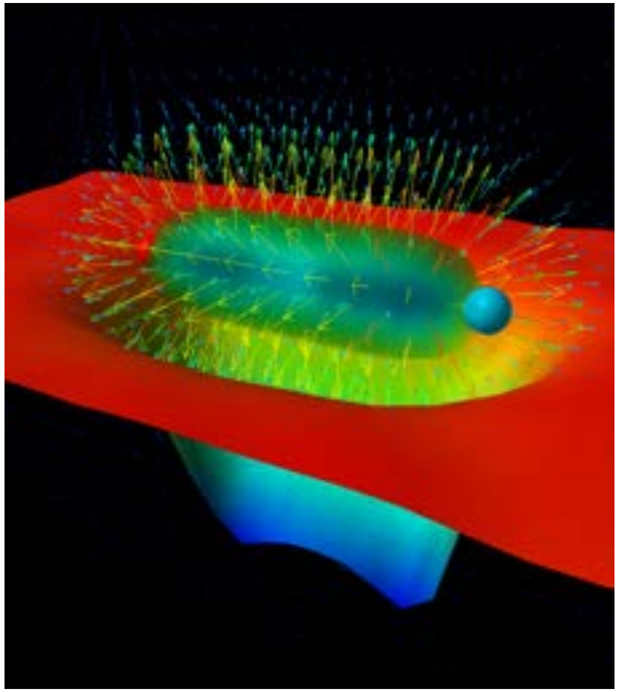
for all $R > 0$, have unique blow-down. Moreover, we show that they are two-dimensional in ambient dimension $2 \le n \le 4$, or in any dimension $n \ge 2$ assuming that (u, ∇) is a local minimizer, thus establishing a co-dimension-two analogue of Savin's theorem. The main ingredient is an Allard-type improvement of flatness.

Next, we define the gauge-invariant Jacobian, which plays an important role in the Γ-convergence theory [41], similar to the classical Jacobian in the Γ-convergence for the Ginzburg–Landau energy with no magnetic field, see [1, 8, 37]. It is the two-form given by

$$J(u,\nabla) := \psi(u) + (1-|u|^2)\omega. \tag{3.8}$$

$$\mathbf{E}(u,\nabla,B_r(x),S) := \frac{r^{2-n}}{2\pi}\int_{B_r(x)}[e_\varepsilon(u,\nabla) - J(u,\nabla)\wedge e_S^*] \tag{5.1}$$
$$= \mu_\varepsilon(B_r(x)) - \langle \Gamma_\varepsilon, \mathbf{1}_{B_r(x)}e_S^*\rangle$$

Theorem 1.9. *The previous conjecture holds for critical points in dimension* $2 \le n \le 4$, *as well as for local minimizers in all dimensions* $n \ge 2$, *even without the second assumption that* $\lim_{|y|\to\infty}|u(y,z)| = 1$ *uniformly in* z*: the pair* (u,∇) *is two-dimensional, up to rotation and change of gauge.*

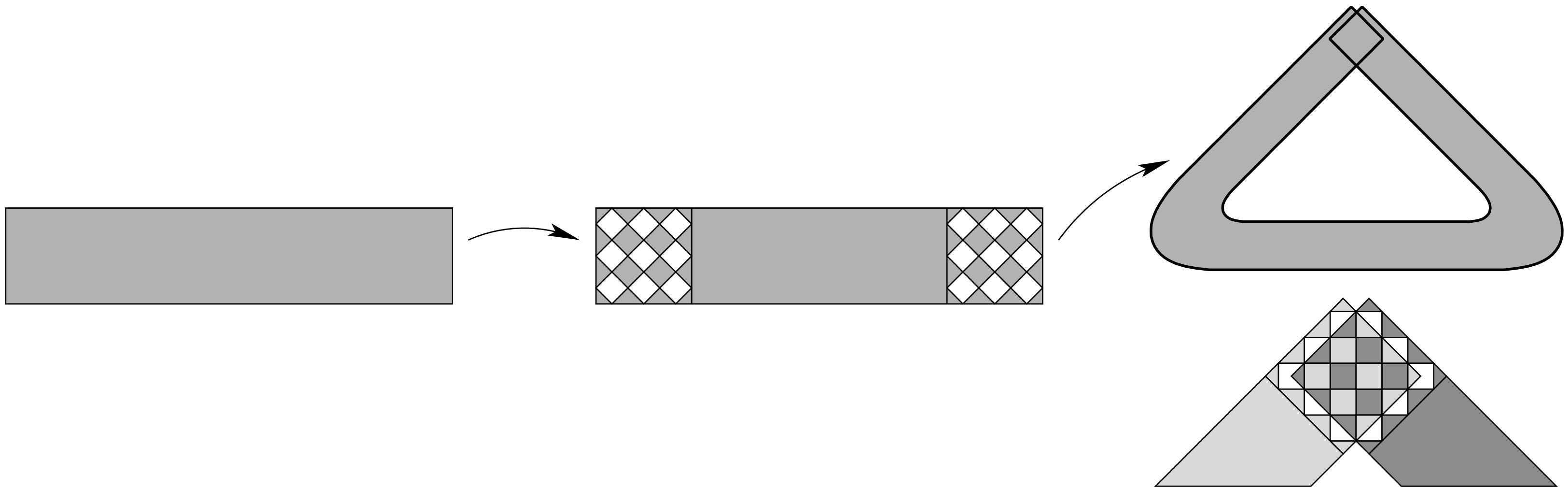


Rajarshi Day (2021)
soulofmathematics.com

Müller & Spector '95: if

- Per $\boldsymbol{u}_j(\Omega)$ is uniformly bounded; and
- $\boldsymbol{u}_j$, for every $j \in \mathbb{N}$, and $\boldsymbol{u}$ itself, are one-to-one a.e.

then $\det D\boldsymbol{u}_j \rightharpoonup \det D\boldsymbol{u}$ in $L^1(\Omega)$.



Invertibility

- J. Ball [PRSE, 1981]
- P. Ciarlet & Nečas [ARMA, 1987]
- V. Šverák [ARMA, 1988]
- S. Müller & S. Spector [ARMA, 1995]
- J. Sivaloganathan & S. Spector [J. Elast, 2000]
- H. & Mora-Corral (2010): if $(\boldsymbol{u}_j)_j$ in SBV, $\sup_j \mathcal{E}(\boldsymbol{u}_j) < \infty$ then

$$\boldsymbol{u} \text{ is one-to-one a.e.,}$$
$$(\boldsymbol{u}_j|_B)^{-1} \rightharpoonup (\boldsymbol{u}|_B)^{-1},$$
$$\chi_{\boldsymbol{u}_j(B)} \overset{*}{\rightharpoonup} \chi_{\boldsymbol{u}(B)}.$$

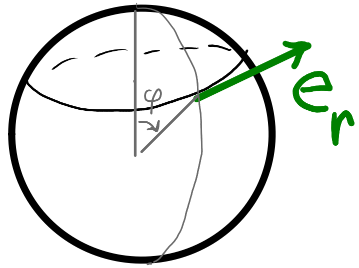
If $(\operatorname{cof} D\boldsymbol{u}_j)_j$ is equiintegrable, then $\det D\boldsymbol{u}_j \rightharpoonup \det D\boldsymbol{u}$.

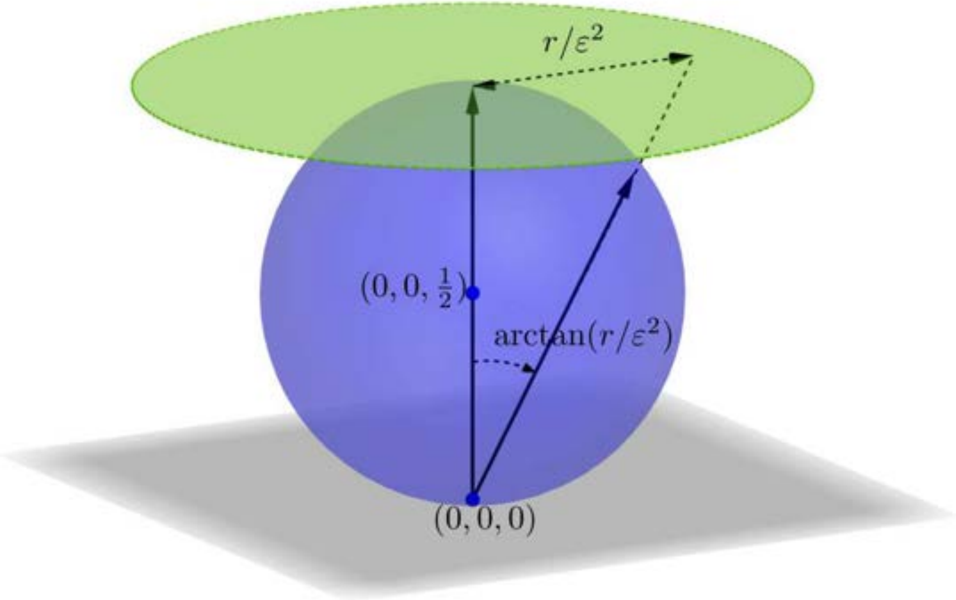
$$\min \int_\Omega |D\boldsymbol{u}(\boldsymbol{x})|^p + H(\det D\boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}, \qquad p > 2$$

$$C_\varepsilon := \{ x_1^2 + x_2^2 < \varepsilon ,\ 0 < x_3 < 1 \}$$

$$\underset{\sim}{x}(r, \theta, x_3) = r\, \underset{\sim}{e}_r + x_3\, \underset{\sim}{e}_3 ,$$

$$0 < r < \varepsilon ,\quad 0 \le \theta \le 2\pi ,\quad 0 \le x_3 \le 1$$



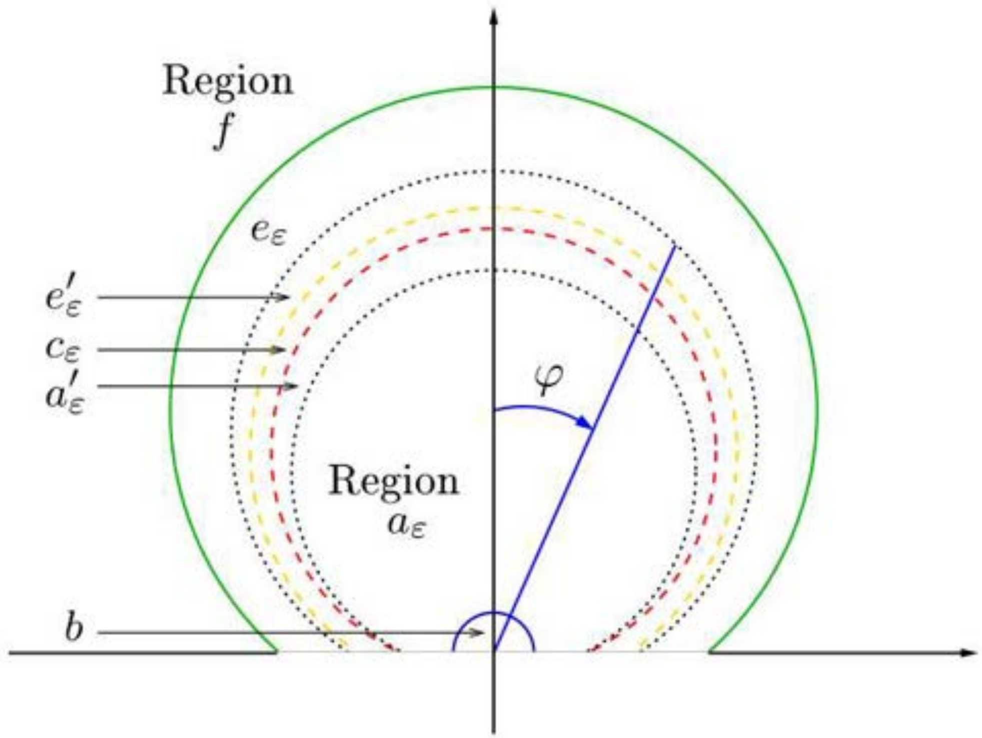


$$u_\varphi^\varepsilon = f_\varepsilon(r) := \arctan\left(\frac{r}{\varepsilon^2}\right) + \alpha_\varepsilon \frac{r}{\varepsilon} ,$$

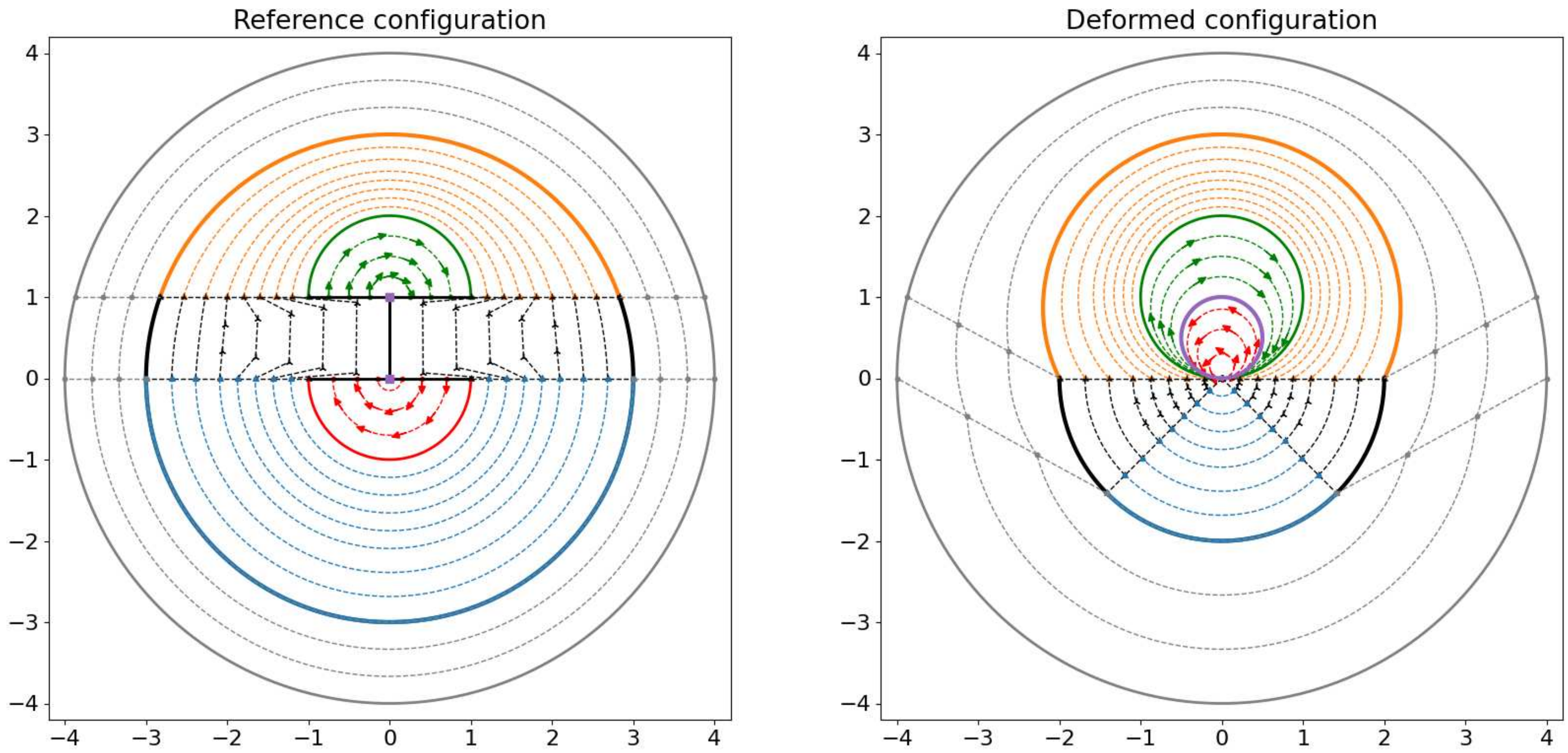
$$\alpha_\varepsilon = \arctan(\varepsilon)$$

$$f_\varepsilon(0) = 0 ,\quad f_\varepsilon(\varepsilon) = \frac{\pi}{2} , \text{ and}$$

$$f_\varepsilon'(r) > 0 \quad \text{for all } r$$

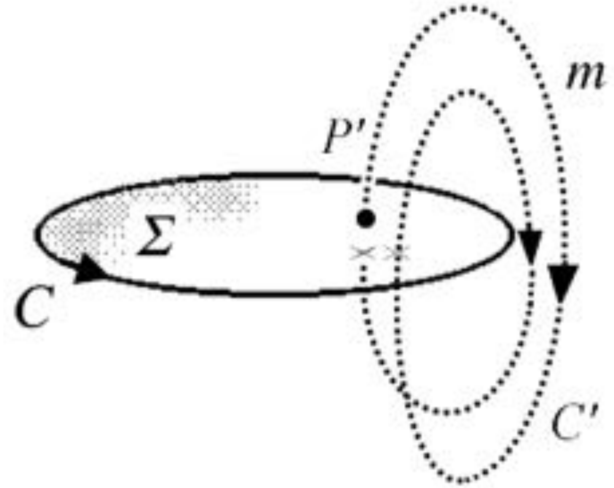


Harmonic dipoles

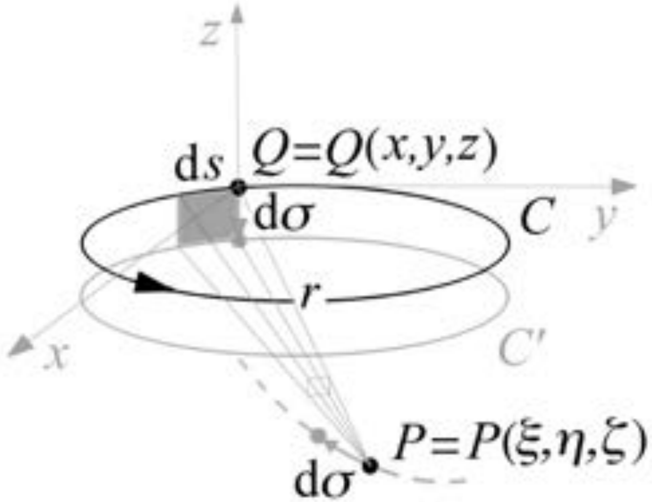


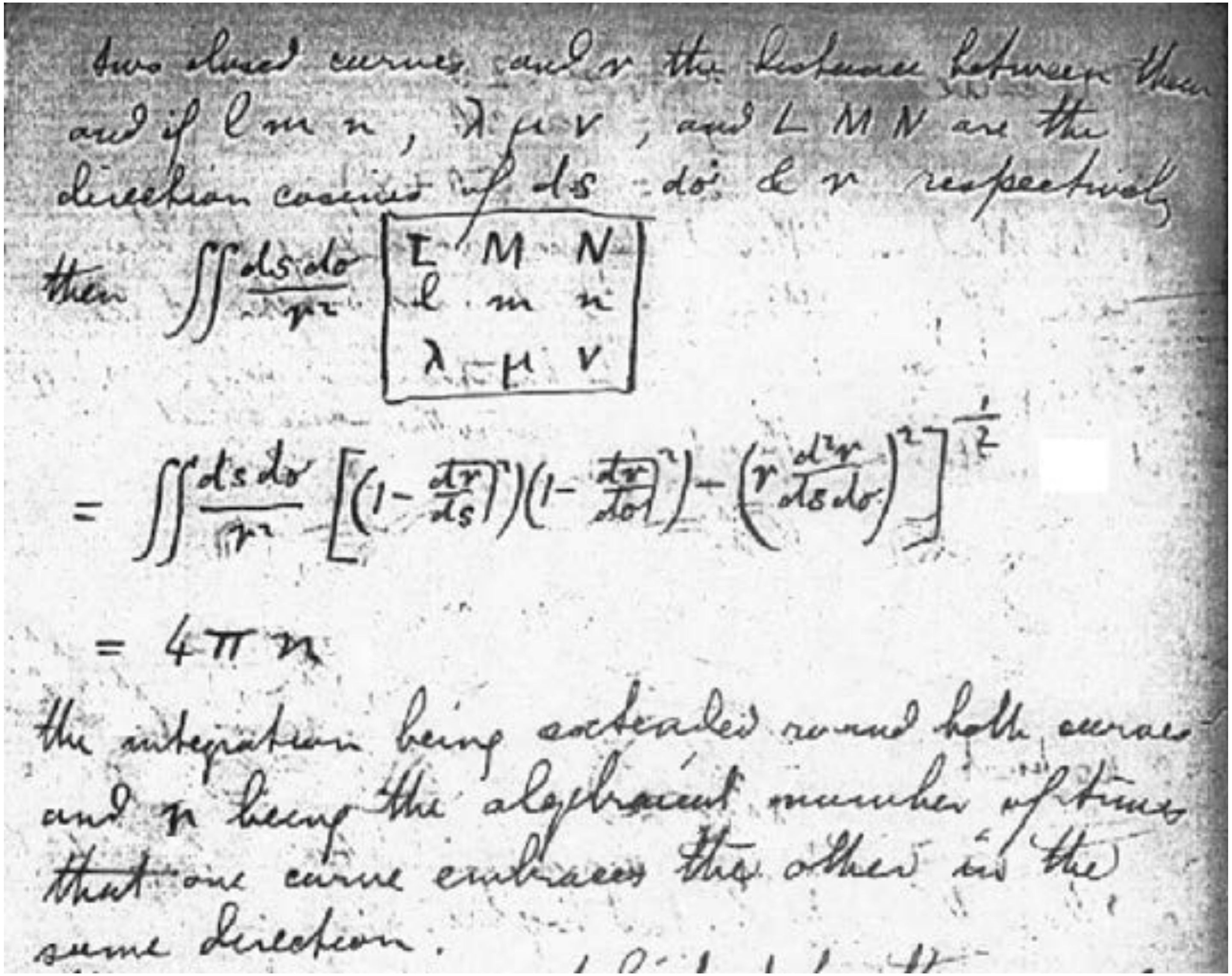
The limit map $\mathbf{u}$ by Conti & De Lellis:

- $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u} + \frac{\pi}{6}(\delta_P - \delta_N)$
- Does not satisfy INV or the divergence identities
- $J_{\mathbf{u}^{-1}} = B\Big((0,0,\tfrac{1}{2}), \tfrac{1}{2}\Big)$



(b)




two closed curves and r the distance between them and if $l\ m\ n$, $\lambda\ \mu\ \nu$, and $L\ M\ N$ are the direction cosines of ds $d\sigma$ & r respectively

then $$\iint \frac{ds\,d\sigma}{r^2} \begin{vmatrix} L & M & N \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}$$

$$= \iint \frac{ds\,d\sigma}{r^2}\left[\left(1-\overline{\frac{dr}{ds}}^2\right)\left(1-\overline{\frac{dr}{d\sigma}}^2\right)-\left(r\frac{d^2r}{ds\,d\sigma}\right)^2\right]^{\frac{1}{2}}$$

$$= 4\pi n$$

the integration being extended round both curves and n being the algebraical number of times that one curve embraces the other in the same direction.

R. Ricca, B. Nipoti. Gauss' linking number revisited.
Journal of Knot Theory and Its Ramifications (2011)

NeoHookean materials

Doležalová, Hencl, Malý '23, Thm. 1.1: If $\boldsymbol{u}_j$ homeomorphisms with

$$\int_\Omega |D\boldsymbol{u}_j|^2 + \sqrt{K_{\boldsymbol{u}_j}}\,\mathrm{d}\boldsymbol{x}$$

bounded, then $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ with $\boldsymbol{u}$ satisfying INV.

Doležalová, Hencl, Molchanova, arXiv:2212.06452, Thm. 5.5: If $E(\boldsymbol{u}) = \int_\Omega |D\boldsymbol{u}|^2 + H(\det D\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}$ with $H(J) \geq CJ^{-2}$, then E has a minimizer in the weak closure of

$$\mathcal{A}_{\text{hom}} := \{\boldsymbol{u} \in W^{1,2} : \boldsymbol{u} \text{ is a homeomorphism}, \det D\boldsymbol{u} > 0 \text{ a.e.}, \\ \text{Lusin's } N \text{ condition}, \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{b},\ E(\boldsymbol{u}) \leq E(\boldsymbol{b})\}.$$

Kalayanamit, arXiv:2405.12156, Thm. 2.5: The minimizer in the weak closure of $\mathcal{A}_{hom} \cap \{\boldsymbol{u} = \boldsymbol{b} \text{ in } \Omega \setminus \widetilde{\Omega}\}$ has a $W^{1,1}$ inverse.

Summary

- Direct method of the calculus of variations: compactness and lower semicontinuity.
- Sequentiall continuity of det $D\boldsymbol{u}$ with respect to weak convergence.
- Divergence structure.
- Brouwer degree. Invertibility.
- Singular minimizers.
- Harmonic dipoles. Linking numbers.
- Analysis, geometry, mechanics, and topology in the distributional determinants.