# Improved bounds for the inverse $U^{s+1}[N]$ Norm, Part I

James Leng

UCLA

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# Szeremerédi's Theorem

**Theorem (Szemerédi 1975)**

*Let $A$ be a subset of $\mathbb{N}$ with*

$$\limsup_{N\to\infty} \frac{|A \cap \{1, \ldots, N\}|}{N} > 0.$$

*Then for each positive integer $k$, $A$ has a nontrivial arithmetic progression of length $k$.*

# Linear Equations in Primes

## Theorem (Green-Tao-Ziegler 2010)

*Let $\mathcal{P}$ be the set of primes. Then*

$$|\{n, d \leq N : n, n+d, n+2d, \dots, n+(k-1)d \in \mathcal{P}\}|$$

$$= (1 + o_k(1)) I \prod_{p \text{ prime}} \beta_p$$

*where*

$$I = \int_2^N \int_2^N \frac{dxdy}{\log x \log(x+y) \cdots \log(x+(k-1)y)}$$

$$\beta_p = \begin{cases} \left(\frac{p}{p-1}\right)^{k-1} \left(1 - \frac{k-1}{p}\right) & p > k \\ \frac{1}{p} \left(\frac{p}{p-1}\right)^{k-1} & p \leq k. \end{cases}$$

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- and $r_3(N) \leq N\exp(-\log(N)^c)$ for some $c > 0$ (Kelley-Meka 2023).
- Finally, we have $r_4(N) = O(N\log(N)^{-c})$ (Green-Tao 2017).

# Quantitative Bounds for Linear Equations

Relying on breakthrough work of Manners,

**Theorem (Tao-Teräväinen 2021)**

*Let $\mathcal{P}$ be the set of primes. Then*

$$|\{n, d \leq N : n, n+d, n+2d, \ldots, n+(k-1)d \in \mathcal{P}\}|$$

$$= \left(1 + O\left(\frac{1}{\log\log(N)^{c_k}}\right)\right) I \prod_{p \text{ prime}} \beta_p$$

*where*

$$I = \int_2^N \int_2^N \frac{dxdy}{\log x \log(x+y) \cdots \log(x+(k-1)y)}$$

$$\beta_p = \begin{cases} \left(\frac{p}{p-1}\right)^{k-1}\left(1 - \frac{k-1}{p}\right) & p > k \\ \frac{1}{p}\left(\frac{p}{p-1}\right)^{k-1} & p \leq k. \end{cases}$$

# New Results for Szemerédi

**Theorem (L.-Sah-Sawhney 2024)**

*Let $r_k(N)$ be the size of the largest subset of $\{1, \ldots, N\}$ without a $k$-term arithmetic progression. Then*

$$r_k(N) = O\left(\frac{N}{\exp(\log\log(N)^{c_k})}\right)$$

*for some $c_k > 0$.*

# New Results for Linear Equations

## Theorem (L. 2024)

*Let $\mathcal{P}$ be the set of primes and let $A > 0$. Then*

$$|\{n, d \le N : n, n+d, n+2d, \ldots, n+(k-1)d \in \mathcal{P}\}|$$

$$= \left(1 + O_A\left(\frac{1}{\log(N)^A}\right)\right) I \prod_{p \text{ prime}} \beta_p$$

*where*

$$I = \int_2^N \int_2^N \frac{dxdy}{\log x \log(x+y) \cdots \log(x+(k-1)y)}$$

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# Improvements

Three sources for the improvements:

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- Improved estimates on *equidistribution of nilsequences*, or rather, *exponential sums of bracket polynomials* (L. 2023).
- Quasi-polynomial $U^{s+1}[N]$ inverse theorem (L. 2023, L.-Sah-Sawhney 2024).

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- Quasi-polynomial inverse sumset results (Croot-Sisask, Sanders 2010-2011).
- Improved estimates on *equidistribution of nilsequences*, or rather, *exponential sums of bracket polynomials* (L. 2023).
- Quasi-polynomial $U^{s+1}[N]$ inverse theorem (L. 2023, L.-Sah-Sawhney 2024).
- Most of the rest of the talk will focus on equidistribution.

# Gowers Norm

We define for $f : \mathbb{Z} \to \mathbb{C}$ finitely supported

$$\|f\|_{U^{s+1}(\mathbb{Z})}^{2^{s+1}} = \sum_{n,h_1,\dots,h_{s+1}} \prod_{\omega \in \{0,1\}^{s+1}} C^{|\omega|} f(n + h \cdot \omega)$$

where $C$ denotes conjugation. For example,

$$\|f\|_{U^2(\mathbb{Z})}^4 = \sum_{n,h_1,h_2} f(n)\overline{f(n+h_1)f(n+h_2)}f(n+h_1+h_2).$$

We define

$$[N] = \{0, 1, \dots, N-1\}$$

$$\|f\|_{U^{s+1}[N]} = \frac{\|f 1_{[N]}\|_{U^{s+1}(\mathbb{Z})}}{\|1_{[N]}\|_{U^{s+1}(\mathbb{Z})}}.$$

# Inverse Theorem

## Theorem (Green-Tao-Ziegler 2010)

*Fix* $\delta \in (0, 1/2)$. *Suppose that* $f : [N] \to \mathbb{C}$ *is* 1*-bounded and*

$$\|f\|_{U^{s+1}[N]} \geq \delta.$$

*Then there exists a nilmanifold* $G/\Gamma$ *of degree* $s$*, complexity at most* $M$*, and dimension at most* $d$ *as well as a function* $F$ *on* $G/\Gamma$ *which is at most* $K$*-Lipschitz such that*

$$|\mathbb{E}_{n\in[N]}[f(n)\overline{F(g(n)\Gamma)}]| \geq \varepsilon,$$

*where we may take*

$$d \leq O_\delta(1) \text{ and } \varepsilon^{-1}, K, M \leq O_\delta(1).$$

# Inverse Theorem Discussion

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- Associates an *algebraic object* to the *Gowers norm* which is *analytic*.

# Manners Inverse Theorem

## Theorem (Manners 2018)

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$$|\mathbb{E}_{n \in [N]}[f(n)\overline{F(g(n)\Gamma)}]| \geq \varepsilon,$$

*where we may take*

$$d \leq \delta^{-O_s(1)} \text{ and } \varepsilon^{-1}, K, M \leq \exp(\exp(\delta^{-O_s(1)})).$$

# Improved Inverse Theorem

**Theorem (L.-Sah-Sawhney 2024)**

*Fix* $\delta \in (0, 1/2)$. *Suppose that* $f : [N] \to \mathbb{C}$ *is* 1*-bounded and*

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*Then there exists a nilmanifold* $G/\Gamma$ *of degree* $s$, *complexity at most* $M$, *and dimension at most* $d$ *as well as a function* $F$ *on* $G/\Gamma$ *which is at most* $K$*-Lipschitz such that*

$$|\mathbb{E}_{n \in [N]}[f(n)\overline{F(g(n)\Gamma)}]| \geq \varepsilon,$$

*where we may take*

$$d \leq \log(1/\delta)^{O_s(1)} \text{ and } \varepsilon^{-1}, K, M \leq \exp(\log(1/\delta)^{O_s(1)}).$$

# Exponential Sums

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- Let $\mathbb{E}_{n\in[N]} f(n) = \frac{1}{N}\sum_{n=0}^{N-1} f(n)$.

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- Let $\mathbb{E}_{n \in [N]} f(n) = \frac{1}{N} \sum_{n=0}^{N-1} f(n)$.
- Let $[x]$ be the nearest integer to $x$ (rounded up) and let $\{x\} = x - [x]$.

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where $B$ is a (real) bracket polynomial. Here are examples:

- $B(n) = \alpha n[\beta n]$ with $\alpha, \beta \in \mathbb{R}$
- $B(n) = \alpha n\{\beta n\}$ with $\alpha, \beta \in \mathbb{R}$
- $B(n) = p(n)$ for some $p \in \mathbb{R}[x]$
- $B(n) = p(n)\{q(n)[r(n)]\}$ where $p, q, r \in \mathbb{R}[x]$.
- More generally, anything you can create with the "bracket" and "polynomial" operations.

# Inverse-type theorem

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If $\alpha, \beta = 0$, then $S = 1$.
Conclusion: cannot have a nontrivial uniform estimate on exponential sums of *all* bracket polynomials. However, we can hope to prove some inverse-type estimate:

**Problem**

*If $|\mathbb{E}_{n\in[N]} e(\alpha n[\beta n])| \geq \delta$ (and $\delta$ is suitably small compared to $N$), can we hope to prove something about $\alpha$ and $\beta$?*

# Previous Literature

**Theorem (Green-Tao (2007), Tao-Terävaïnen (2021))**

*If* $|\mathbb{E}_{n \in [N]} e(\alpha n[\beta n])| \geq \delta$, *and* $\delta^{-O(1)} \ll N$, *then there exists some integers* $k_1, k_2$ *with* $|k_i| \ll \delta^{-O(1)}$ *and*

$$\|k_1 \alpha + k_2 \beta\|_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O(1)}/N.$$

Here, $A \ll B$ denotes $A \leq CB$ for some constant $C$, and $\|x\|_{\mathbb{R}/\mathbb{Z}}$ is the distance from $x$ to the nearest integer.

# Many Bracket Terms

A more representative example of the situation could be

$$B(n) = \sum_{i=1}^{d} \alpha_i n[\beta_i n].$$

**Theorem (Green-Tao 2007, Tao-Teräväinen 2021)**

*If* $|\mathbb{E}_{n\in[N]} e(B(n))| \geq \delta$, *and* $\delta^{-\exp(O(d^{O(1)}))} \ll N$, *then there exists* $k_1, k_2 \in \mathbb{Z}^d$ *with* $|k_i| \ll \delta^{-\exp(O(d^{O(1)}))}$ *and*

$$\|k_1 \cdot \vec{\alpha} + k_2 \cdot \vec{\beta}\|_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-\exp(O(d^{O(1)}))}/N.$$

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- Bounds are *double exponential in dimension*.
- One also obtains bounds double exponential in dimension for exponential sums of arbitrary bracket polynomials.*
- Major obstacle is *induction on dimensions*. A parameter decrease of $\delta \mapsto \delta^2$ is unacceptable since it iterates to $\delta^{2^d}$.

# Improvement*

As before, let

$$B(n) = \sum_{i=1}^{d} \alpha_i n[\beta_i n].$$

**Theorem (L. 2023)**

*If* $|\mathbb{E}_{n\in[N]} e(B(n))| \geq \delta$, *and* $\delta^{-O(d^{O(1)})} \ll N$, *then there exists* $k_1, k_2 \in \mathbb{Z}^d$ *with* $|k_i| \ll \delta^{-O(d^{O(1)})}$ *and*

$$\|k_1 \cdot \vec{\alpha} + k_2 \cdot \vec{\beta}\|_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O(d^{O(1)})}/N.$$

*The much stronger statement that there are "enough linear relations" to reduce to a "lower degree bracket polynomial" holds and will be discussed later.

# Sketch of Proof

Start with

$$B(n) = \sum_{i=1}^{d} \alpha_i n[\beta_i n]$$

$$S = \mathbb{E}_{n \in [N]} e(B(n)).$$

Suppose $|S| \geq \delta$.

# Sketch of Proof

Start with

$$B(n) = \sum_{i=1}^{d} \alpha_i n[\beta_i n]$$

$$S = \mathbb{E}_{n \in [N]} e(B(n)).$$

Suppose $|S| \geq \delta$. Apply the van der Corput inequality to obtain that there are $\delta^{O(1)} N$ many $h \in [N]$ such that

$$|\mathbb{E}_{n \in [N]} e(B(n+h) - B(n))| \geq \delta^{O(1)}.$$

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Observe however that

$$\begin{aligned} \alpha n([\beta(n+h)] - [\beta n] - [\beta h]) &= \alpha n(\{\beta n\} + \{\beta h\} - \{\beta(n+h)\}) \\ &\equiv \{\alpha n\}(\{\beta n\} + \{\beta h\} - \{\beta(n+h)\}) \pmod 1 \end{aligned}$$

We now analyze a term of the form $e(\{\alpha n\}\{\beta n\})$.

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$$F(\{\alpha n\}, \{\beta n\}) = \sum_{|k| \leq \delta^{-O(1)}} a_{k_1,k_2} e(k_1\{\alpha n\} + k_2\{\beta n\}) + O(\delta^2)$$

with $|a_{k_1,k_2}| \leq 1$. However, $e(k_1\{\alpha n\} + k_2\{\beta n\}) = e(k_1 \alpha n + k_2 \beta n)$!

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with $|a_{k_1,k_2}| \leq 1$. However, $e(k_1\{\alpha n\} + k_2\{\beta n\}) = e(k_1 \alpha n + k_2 \beta n)$! Crucially, given $d$ many brackets, this operation loses at most $\delta^{O(d^{O(1)})}$ which is an *okay factor to lose*.

# Collecting Data

- $B(n+h) - B(n) \equiv \sum_{i=1}^{d} \alpha_i n[\beta_i h] + \alpha_i h[\beta_i n] + \alpha_i h[\beta_i h] + [\text{Lower order terms}] \pmod 1$.

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- Here, lower order terms denote terms that look like $\{\alpha n\}\{\beta n\}, \{\alpha n\}\{\beta h\}$, or $\{\alpha h\}\{\beta h\}$.
- **Slogan: Fourier expand lower order terms**.
- Since we are summing $|\mathbb{E}_n e(B(n+h) - B(n))|$, $\alpha_i h[\beta_i h]$ terms do not matter.

# More Bracket Manipulations

We may write

$$\begin{aligned}\alpha_i n[\beta_i h] + \alpha_i h[\beta_i n] &\equiv \alpha_i n(\beta_i h - \{\beta_i h\}) + \{\alpha_i h\}(\beta_i n - \{\alpha_i n\}) \pmod 1\\ &\equiv \beta_i n\{\alpha_i h\} - \alpha_i n\{\beta_i h\} + \alpha_i\beta_i nh + [\text{Lower order terms}]\\ &\equiv \zeta n \cdot \{\gamma h\} + \xi nh + [\text{Lower order terms}] \pmod 1\end{aligned}$$

where $\zeta = (\vec{\beta}, -\vec{\alpha})$ and $\gamma = (\vec{\alpha}, \vec{\beta})$,

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_d), \vec{\beta} = (\beta_1, \dots, \beta_d).$$

# Continuatation of Argument

Hence, for $\delta^{O(1)}N$ many $h \in [N]$, we have

$$|\mathbb{E}_{n\in[N]} e(\zeta n \cdot \{\gamma h\} + \xi nh + [\text{Lower order terms}])| \geq \delta^{O(1)}.$$

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By Fourier expanding the lower order terms, we obtain

$$|\mathbb{E}_{n \in [N]} \sum_{\alpha', \beta'} a_{\alpha', \beta'} e(\zeta n \cdot \{\gamma h\} + \xi nh + \alpha' n + \beta' h)| \geq \delta^{O(1)}$$

where

$$\sum_{\alpha', \beta'} |a_{\alpha', \beta'}| \leq \delta^{-O(d^{O(1)})}.$$

Applying the pigeonhole principle, we can find $\alpha'$ such that

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$$\sum_{\alpha',\beta'} |a_{\alpha',\beta'}| \leq \delta^{-O(d^{O(1)})}.$$

Applying the pigeonhole principle, we can find $\alpha'$ such that

$$|\mathbb{E}_{n\in[N]} e(\zeta n \cdot \{\gamma h\} + \xi nh + \alpha' n)| \geq \delta^{O(d^{O(1)})}.$$

So

$$\|\zeta \cdot \{\gamma h\} + \xi h + \alpha'\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{\delta^{O(d^{O(1)})}}{N}.$$

# Bracket polynomial lemma

**Lemma (Green-Tao 2007)**

*Let $N, \delta > 0$ be fixed with $0 < \delta < 1/10$ and $a, \alpha \in \mathbb{R}^d$ and $\beta \in \mathbb{R}$, with $|a| \leq 1/\delta$. Suppose there are at least $\delta N$ many $n \in [N]$ such that*

$$\|\beta + a \cdot \{\alpha n\}\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{\delta N}.$$

*Then either $\|a\|_\infty \ll \delta^{-O(d^{O(1)})}/N$ or there exists a nonzero vector $\eta \in \mathbb{Z}^d$ with $|\eta| \leq \delta^{-O(d^{O(1)})}$ such that $\|\eta \cdot \alpha\|_{\mathbb{R}/\mathbb{Z}} \ll \frac{\delta^{-O(d^{O(1)})}}{N}$.*

# Refined Bracket Polynomial Lemma

## Lemma (L. 2023)

*Let* $N, \delta, M, K > 0$ *be fixed with* $0 < \delta < 1/10$ *and* $a, \alpha \in \mathbb{R}^d$ *and* $\beta \in \mathbb{R}$, *with* $|a| \leq M$. *Suppose there are at least* $\delta N$ *many* $n \in [N]$ *such that*

$$\|\beta + a \cdot \{\alpha n\}\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{K}{N}.$$

*Then either* $N \ll (MK/\delta)^{O(d^{O(1)})}$ *or else there exists* $d \geq r \geq 0$, $w_1, \dots, w_r \in \mathbb{Z}^d$ *and* $\eta_1, \dots, \eta_{d-r} \in \mathbb{Z}^d$ *such that* $w_i, \eta_j$ *are linearly independent,* $|w_i|, |\eta_j| \leq (\delta/M)^{-O(d^{O(1)})}$, $\langle w_i, \eta_j \rangle = 0$, *and*

$$|w_i \cdot a| \leq \frac{(\delta/MK)^{-O(d^{O(1)})}}{N}$$

$$\|\eta_j \cdot \alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{(\delta/MK)^{-O(d^{O(1)})}}{N}.$$

# Aftermath

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- This is where "induction on dimensions" is beaten.
- The refined bracket polynomial lemma gives "enough linear relations" on $(\vec{\alpha}, \vec{\beta})$ so that when we simplify it, $\zeta n \cdot \{\gamma h\}$ becomes a "lower order term."
- One can bootstrap this procedure to compute equidistribution estimates for arbitrary bracket polynomials, also with good bounds.

# Proof of the refined bracket polynomial lemma

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$$\eta_1\alpha_1 + \cdots + \eta_d\alpha_d \approx 0 \pmod 1.$$

Suppose for simplicity, $\eta_1 = 1$ and $\approx$ were a genuine equality.

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$$a \cdot \{\alpha h\} = \tilde{a} \cdot \{\alpha h\} + a_1 P(h)$$

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By pigeonholing in the value of $P(h)$, we may pass to the hypothesis that

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where $\tilde{a}$ has first coordinate zero, so it is dimension $d - 1$. There are several problems with this iteration.

# Problem 1: $|\tilde{a}|$ might get too large

We naively have $|\tilde{a}| \leq 2|\eta||a|$, and combined with $|a| \leq M$, this leads to an iteration of $M \mapsto (M/\delta)^{O(d^{O(1)})}$.

# Solution 1: Use Minkowski's Theorem

**Theorem**

*If $K \subset \mathbb{R}^d$ is a convex body symmetric about the origin with vol$(K) > 2^d$, then $K$ contains a nonzero point in $\mathbb{Z}^d$.*

Apply this to a tube lying in the direction of $a$. This implies that we can choose $\eta$ to lie very close to the direction of $a$. This will in fact give $|\tilde{a}| \leq |a|$.

# Problem 2: Pigeonholing in $P$ is expensive.

$P(h)$ takes $(M/\delta)^{O(d^{O(1)})}$ many values. Pigeonholing in one of these values causes the $\delta N$ many $h$'s we work with to decrease to $(\delta/M)^{O(d^{O(1)})}N$ many $h$'s. This is quite problematic as it causes the iteration of $\delta \mapsto (\delta/M)^{O(d^{O(1)})}$.

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$$|\mathbb{E}_{n \in [N']} e(an \cdot \{\alpha h\} + \beta h)| \gg K^{-1} \gg 1$$

for $\delta N$ many $h \in [N]$ with $N' \gg N$. Then the hypothesis after accounting for $\eta$ becomes

$$|\mathbb{E}_{n \in [N']} e(\tilde{a} n \cdot \{\alpha h\} + \beta h + \{\alpha_1 n\} P(h))| \gg K^{-1}$$

for $\delta N$ many $h \in [N]$.

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for $\delta N$ many $h \in [N]$. Then by Fourier expanding the lower order terms, we no longer obtain horrible losses in $\delta$. The losses instead get shifted to $K$.

# Problem 3: Loss in $K$ is quite bad

This could be quite bad, since this incurs losses of $K \mapsto K^{O(d^{O(1)})}$, which iterates to double exponential bounds in iteration.

# Solution 3: "remember" the bracket polynomial from previous iterations

Instead of iterating

$$\|a \cdot \{\alpha h\} + \beta\|_{\mathbb{R}/\mathbb{Z}} = O(1/N)$$

we iterate

$$|\mathbb{E}_{n \in [N']} e(\tilde{a} n \cdot \{\alpha h\} + \beta h + P_j(n, h))| \gg K^{-1}.$$

With new lower order terms, we simply append to the old lower order terms. This iteration is of the shape $K_j = K^{O((dj)^{O(1)})}$ which is single exponential in dimension.

# Thank you!