A Resolution of the Arrow Impossibility Theorem

(based on "Borda's Rule and Arrow's Independence Condition," Journal of Political Economy, forthcoming)

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- when IIA modified appropriately and May's (1952) axioms for majority rule and weak consistency condition added
    - Impossibility Theorem no longer holds
    - Borda count (rank-order voting) is unique voting rule satisfying all conditions
    - because other conditions are satisfied by nearly all SWFs in literature, result shows that modified IIA uniquely distinguishes Borda count from other SWFs

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- SWF is mapping from profiles of individuals' preferences over alternatives to *social preferences*
- Arrow didn't believe that society actually has preferences
  - saw social ranking as *contingency plan*
    - if top choice not feasible, can go with second choice, etc.

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With 3 or more alternatives (candidates), there exists no SWF satisfying

- Unrestricted domain (U)
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- Non-dictatorship (ND)
- Independence of Irrelevant Alternatives (IIA)

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U: SWF defined for all possible profiles of individual preferences

P: if everyone prefers *x* to *y*, then *x* ranked above *y* socially

ND: there exists no individual who always gets her way

- if she prefers *x* to *y*, *x* is socially preferred to *y*

- U, P, ND are so weak that satisfied by practically all SWFs used or studied
  - e.g. plurality rule (*x* ranked above *y* if more individuals rank *x* first then rank *y* first)

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  social preferences between *x* and *y* should depend only on individuals' preferences between *x* and *y*

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$1/3$	$1/3$	$1/3$
x	y	z
y	z	x
z	x	y

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    - Condorcet Paradox

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- still, IIA has compelling justification:

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- majority prefer *y* to *z*
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  - Condorcet Paradox

- still, IIA has compelling justification:

  to rule out vote splitting and spoilers

	40%	**25%**	**35%**
	Trump	Rubio	Kasich
Scenario 1	Kasich	Kasich	Trump
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- Under plurality rule, Trump wins in Scenario 1 (with 40%)

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- but majority of voters (60%) prefer Kasich to Trump

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- only reason Trump wins is that Rubio *spoils* election for Kasich

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- Under plurality rule, Trump wins in Scenario 1 (with 40%)
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  - splits off some of Kasich’s first-place votes

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- Under plurality rule, Trump wins in Scenario 1 (with 40%)
- but majority of voters (60%) prefer Kasich to Trump
- only reason Trump wins is that Rubio *spoils* election for Kasich
  - splits off some of Kasich’s first-place votes
- IIA rules this out

	40%	25%	35%
Scenario 1	Trump	Rubio	Kasich
	Kasich	Kasich	Trump
	Rubio	Trump	Rubio

	40%	25%	35%
Scenario 2	Trump	Kasich	Kasich
	Kasich	Trump	Trump
	Rubio	Rubio	Rubio

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Scenario 1	Trump	Rubio	Kasich
	Kasich	Kasich	Trump
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	40%	**25%**	**35%**
Scenario 2	Trump	Kasich	Kasich
	Kasich	Trump	Trump
	Rubio	Rubio	Rubio

- in scenario 2, pretty much *any* SWF ranks Kasich above Trump

Scenario 1

40%	**25%**	**35%**
Trump	Rubio	Kasich
Kasich	Kasich	Trump
Rubio	Trump	Rubio

Scenario 2

40%	**25%**	**35%**
Trump	Kasich	Kasich
Kasich	Trump	Trump
Rubio	Rubio	Rubio

- in scenario 2, pretty much *any* SWF ranks Kasich above Trump
  - Kasich ranked first by 60% and second 40%

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Scenario 2	Trump	Kasich	Kasich
	Kasich	Trump	Trump
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- in scenario 2, pretty much *any* SWF ranks Kasich above Trump
  - Kasich ranked first by 60% and second 40%
  - Trump just reverse

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	Kasich	Trump	Trump
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- if SWF satisfies IIA, must also rank Kasich above Trump in Scenario 1

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	Kasich	Trump	Trump
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  - no one’s ranking of Trump and Kasich changes

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- hence, IIA rules out spoilers

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  - Rubio is spoiler if
    - Kasich wins when everyone ranks Rubio low (below Kasich and Trump)

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	Kasich	Trump	Trump
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- if SWF satisfies IIA, must also rank Kasich above Trump in Scenario 1
  - no one’s ranking of Trump and Kasich changes
- hence, IIA rules out spoilers
  - Rubio is spoiler if
    - Kasich wins when everyone ranks Rubio low (below Kasich and Trump)
    - Trump wins if some voters switch to rank Rubio first (above Kasich and Trump)

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  - stronger than necessary to rule out spoilers
  - makes taking account of preference intensities impossible

Scenario 3

45%	55%
x	y
z	x
y	z

Under the Borda count

x gets 3 × 45 + 2 × 55 = 245 points

y gets 3 × 55 + 1 × 45 = 210 points

z gets 2 × 45 + 1 × 55 = 145 points

so the social ranking is $\begin{array}{c} x \\ y \\ z \end{array}$

Scenario 3

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Under the Borda count

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so the social ranking is $\begin{matrix} x \\ y \\ z \end{matrix}$

Scenario 4

45%	55%
x	y
y	x
z	z

Under Borda count

social ranking = $\begin{matrix} y \\ x \\ z \end{matrix}$

Scenario 3

45%	55%
x	y
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Under the Borda count

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Scenario 4

45%	55%
x	y
y	x
z	z

Under Borda count

social ranking = $\begin{matrix}y\\x\\z\end{matrix}$

- social ranking is $\begin{matrix}x\\y\end{matrix}$ in scenario 3

Scenario 3

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y	z

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Scenario 4

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x	y
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Under Borda count

social ranking = $\begin{matrix} y \\ x \\ z \end{matrix}$

- social ranking is $\begin{matrix} x \\ y \end{matrix}$ in scenario 3
- social ranking is $\begin{matrix} y \\ x \end{matrix}$ in scenario 4

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Scenario 4

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x	y
y	x
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Under Borda count

social ranking = $\begin{matrix} y \\ x \\ z \end{matrix}$

- social ranking is $\begin{matrix} x \\ y \end{matrix}$ in scenario 3
- social ranking is $\begin{matrix} y \\ x \end{matrix}$ in scenario 4
- this violates IIA

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Scenario 4

45%	55%
x	y
y	x
z	z

Under Borda count

social ranking = $\begin{matrix} y \\ x \\ z \end{matrix}$

- But z *doesn't* split first-place votes with y in Scenario 3

Scenario 3

45%	55%
x	y
z	x
y	z

Under the Borda count

x gets 3 × 45 + 2 × 55 = 245 points

y gets 3 × 55 + 1 × 45 = 210 points

z gets 2 × 45 + 1 × 55 = 145 points

so the social ranking is $\begin{array}{c} x \\ y \\ z \end{array}$

Scenario 4

45%	55%
x	y
y	x
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Under Borda count

social ranking = $\begin{array}{c} y \\ x \\ z \end{array}$

- But z *doesn't* split first-place votes with y in Scenario 3
  - z *never* ranked high (above Kasich and Trump)

Scenario 3

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Under the Borda count

x gets 3 × 45 + 2 × 55 = 245 points

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so the social ranking is $\begin{matrix}x\\y\\z\end{matrix}$

Scenario 4

45%	55%
x	y
y	x
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Under Borda count

social ranking = $\begin{matrix}y\\x\\z\end{matrix}$

- But *z doesn't* split first-place votes with *y* in Scenario 3
  - *z never* ranked high (above Kasich and Trump)
  - so can't justify IIA on anti-spoiler grounds

Scenario 3

45%	55%
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Under the Borda count

x gets 3 × 45 + 2 × 55 = 245 points

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so the social ranking is $\begin{matrix} x \\ y \\ z \end{matrix}$

Scenario 4

45%	55%
x	y
y	x
z	z

Under Borda count

social ranking = $\begin{matrix} y \\ x \\ z \end{matrix}$

- But z *doesn't* split first-place votes with y in Scenario 3
  - z *never* ranked high (above Kasich and Trump)
  - so can't justify IIA on anti-spoiler grounds
- also, z's position in group 1's preferences provides potential info about intensity

Scenario 3

45%	55%
x	y
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Under the Borda count

x gets 3 × 45 + 2 × 55 = 245 points

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so the social ranking is $\begin{matrix} x \\ y \\ z \end{matrix}$

Scenario 4

45%	55%
x	y
y	x
z	z

Under Borda count

social ranking = $\begin{matrix} y \\ x \\ z \end{matrix}$

- But z *doesn't* split first-place votes with y in Scenario 3
  - z *never* ranked high (above Kasich and Trump)
  - so can't justify IIA on anti-spoiler grounds
- also, z's position in group 1's preferences provides potential info about intensity
  - in Scenario 3, z lies between x and y

Scenario 3

45%	55%
x	y
z	x
y	z

Under the Borda count

x gets $3 \times 45 + 2 \times 55 = 245$ points

y gets $3 \times 55 + 1 \times 45 = 210$ points

z gets $2 \times 45 + 1 \times 55 = 145$ points

so the social ranking is $\begin{matrix} x \\ y \\ z \end{matrix}$

Scenario 4

45%	55%
x	y
y	x
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Under Borda count

social ranking = $\begin{matrix} y \\ x \\ z \end{matrix}$

- But z *doesn't* split first-place votes with y in Scenario 3
  - z *never* ranked high (above Kasich and Trump)
  - so can't justify IIA on anti-spoiler grounds
- also, z's position in group 1's preferences provides potential info about intensity
  - in Scenario 3, z lies between x and y
    - implies gap between x and y substantial

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so the social ranking is $\begin{matrix}x\\y\\z\end{matrix}$

Scenario 4

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x	y
y	x
z	z

Under Borda count

social ranking = $\begin{matrix}y\\x\\z\end{matrix}$

- But *z doesn't* split first-place votes with *y* in Scenario 3
  - *z never* ranked high (above Kasich and Trump)
  - so can't justify IIA on anti-spoiler grounds
- also, *z*'s position in group 1's preferences provides potential info about intensity
  - in Scenario 3, *z* lies between *x* and *y*
    - implies gap between *x* and *y* substantial
  - in Scenario 4, *z* lies below both *x* and *y*

Scenario 3

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  - z *never* ranked high (above Kasich and Trump)
  - so can't justify IIA on anti-spoiler grounds
- also, z's position in group 1's preferences provides potential info about intensity
  - in Scenario 3, z lies between x and y
    - implies gap between x and y substantial
  - in Scenario 4, z lies below both x and y
    - implies gap between x and y smaller

Scenario 3

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Under the Borda count

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so the social ranking is $\begin{array}{c} x \\ y \\ z \end{array}$

Scenario 4

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  - so can't justify IIA on anti-spoiler grounds
- also, z's position in group 1's preferences provides potential info about intensity
  - in Scenario 3, z lies between x and y
    - implies gap between x and y substantial
  - in Scenario 4, z lies below both x and y
    - implies gap between x and y smaller
- so IIA shouldn't apply

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- but IIA prevents us from taking account of intensities at all

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- MIIA is relaxation of IIA that is necessary and sufficient for making cardinal inferences
  - necessary to relax IIA at least this much
  - sufficient to relax IIA only this much

But MIIA strong enough to rule out spoilers

Scenario 2

40%	**25%**	**35%**
Trump	Kasich	Kasich
Kasich	Trump	Trump
Rubio	Rubio	Rubio

Scenario 1

40%	**25%**	**35%**
Trump	Rubio	Kasich
Kasich	Kasich	Trump
Rubio	Trump	Rubio

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	Trump	Kasich	Kasich
Scenario 2	Kasich	Trump	Trump
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	40%	**25%**	**35%**
	Trump	Rubio	Kasich
Scenario 1	Kasich	Kasich	Trump
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- If Kasich ranked above Trump socially in Scenario 2, then Kasich ranked above Trump socially in Scenario 1

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    - makes condition applicable to plurality rule

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Ranking Consistency (RC): if each of several disjoint populations have same strict *social ranking*, then top alternative of ranking is also top social alternative for union of populations

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Proof:

- for $|X| = 2$ follows from May (1952)
- suppose $|X| = 3$
- will give proof when preferences restricted to $\left\{\begin{matrix} x & y & z \\ y & z & x \\ z & x & y \end{matrix}\right\}$

Consider profile

$1/3$	$1/3$	$1/3$
x	z	y
y	x	z
z	y	x

Condorcet Paradox profile

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violating transitivity of social preferences

- analogous conclusion if social preference is $\begin{array}{c} y \\ x \end{array}$

- analogous conclusion if social preference is $\begin{matrix} y \\ x \end{matrix}$

hence,

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    score for x is $3a+2b+1-a-b$

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- don’t need RC or continuity in the proof for domain

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- now consider full domain

$$\left\{\begin{matrix} x & y & x & z & y & z \\ z & z & y & x & x & y \\ y & x & z & y & z & x \end{matrix}\right\}$$

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  - actually, this isn't right because RC pertains to strict preferences

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(from MIIA and PR)

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- perturb other subprofiles by ε to get social ranking $\begin{array}{c} x \\ y \\ z \end{array}$
- apply RC to get $x \underset{F}{\succ} y$ for overall profile

$$(*)\qquad \begin{array}{ccc} \underline{1/3+\varepsilon} & \underline{1/3-\varepsilon} & \underline{1/3} \\ x & y & x \\ y & z & y \\ z & x & z \end{array} \xrightarrow{F} \begin{array}{c} x \\ y \\ z \end{array}$$

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Now, perturb subprofiles to get ranking $\begin{array}{c} y \\ x \\ z \end{array}$

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- now send ε and ε^{-} to zero
  - continuity says $\left\{\succ_{\cdot} \mid x \succsim_{F(\succ_{\cdot})} y\right\}$ and $\left\{\succ_{\bullet} \mid y \succsim_{F(\succ_{\bullet})} x\right\}$ closed

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  - continuity says $\left\{\succ_\bullet \mid x \succsim_{F(\succ_\bullet)} y\right\}$ and $\left\{\succ_\bullet \mid y \succsim_{F(\succ_\bullet)} x\right\}$ closed
  - so $x \sim_{F(\succ_\bullet)} y$, where $\succ_\bullet$ is overall profile

so $x \sim_{Bor} y \Rightarrow x \sim_F y$

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- Suppose $x \sim_F y$ for some $\succ_{\bullet}$
  - if $x \succ_{Bor} y$
  - then can raise *y* and lower *x* until reach profile $\succ_{\bullet}^{*}$ where $x \sim_{Bor(\succ_{\bullet}^{*})} y$

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- Thus $x \sim_F y \Leftrightarrow x \sim_{Bor} y$
- So PR then establishes result

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Corollary: *F* satisfies U, MIIA, A, N, PR, and RC $\Longleftrightarrow$

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Corollary: F satisfies U, MIIA, A, N, PR, and RC $\Leftrightarrow$

$$x \succ_F y \Leftrightarrow x \succ_{Bor} y$$