Kinetic Mean Field Games

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The Many Facets of Kinetic Theory
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Based on joint works with Alpár Mészáros and David Ambrose

Introduced in *Lasry-Lions 2006, 2007*, *Huang-Malhamé-Caines 2006*.

Mean Field Games describe **differential games** with a large number of interacting (small, indistinguishable) players.

Applications to macroeconomics, crowd motion, finance...

'Mean Field' $\rightarrow$ Population described by a **density function** m representing the state of a typical player.

'Game' $\rightarrow$ Players can **choose** their dynamics.

Some MFGs have 'kinetic' structure.

How can we use kinetic techniques to understand them?

Rules of a Mean Field Game

The typical player chooses a **control** $\alpha : [0, T] \to \mathbb{A}$ for the (S)DE describing their **state** $z(t)$

$$\mathrm{d}z = b(z; \alpha)\mathrm{d}t + \sigma \mathrm{d}W,$$

aiming to minimise a **cost**

$$J(\alpha; m) = \mathbb{E}\Big[\underbrace{\int_0^T L[z_t, \alpha_t, m_t]\mathrm{d}t}_{\text{Running cost}} + \underbrace{G[z_T, m_T]}_{\text{Terminal cost}} \Big].$$

Nash Equilibrium

Strategy α_* such that no player can gain by deviating from it alone.

$$J(\alpha; m^{\alpha_*}) \geq J(z_0, \alpha_*; m^{\alpha_*}) \quad \text{for all admissible } \alpha : [0, T] \to \mathbb{A}.$$

Mean Field Games System

Nash equilibria are described by a forward-backward PDE system:

$$\begin{cases} \partial_t m + \mathrm{div}_z\,(b(z;\alpha^*[m,Du])m) = \frac{1}{2}D^2 : (\sigma\sigma^\top m), & m|_{t=0} = m_0 \\ -\partial_t u + \widetilde{H}(z,m,Du) = \frac{1}{2}\sigma\sigma^\top : D^2 u, & u(T,z) = G[m_T](z). \end{cases}$$

The **Hamiltonian** is defined by

$$\widetilde{H}(z,m,p) := \sup_{\alpha}\{b(z,\alpha)\cdot p - L(z,\alpha,m)\}.$$

The optimal control satisfies

$$\alpha_* \in \mathrm{argmax}\{b(z,\alpha)\cdot Du - L(z,\alpha,m)\}.$$

Key Example

Many MFG works consider the “controlled velocity” setting $b(z, \alpha) = \alpha$

$$\mathrm{d}Z_t = \alpha_t \mathrm{d}t + \mathrm{d}W_t$$

corresponding to PDE systems of the form

$$\begin{cases} -\partial_t u - \frac{1}{2}\Delta u + H(z, m, Du) = 0, \\ \partial_t m - \frac{1}{2}\Delta m - \mathrm{div}(m D_p H(z, m, Du)) = 0, \\ m|_{t=0} = m_0, \quad u|_{t=T} = g(z, m_T). \end{cases}$$

However, this control system may not be appropriate for all applications.

- Aiyagari-Bewley-Huggett model for household wealth
  Achdou-Buera-Lasry-Lions-Moll 2014, Ambrose 2021
- Flocking via acceleration control
  Nourian-Caines-Malhamé 2011

Generalising the Control System

A first generalisation: linear control systems

$$\mathrm{d}Z_t = BZ_t\mathrm{d}t + \Pi\alpha_t\mathrm{d}t + \sigma\mathrm{d}W_t$$

For example, if players control their **acceleration**, $\ddot{x} = \alpha$, in phase space $z = (x, v)$, we have the linear system

$$\begin{cases} \dot{x} = v \\ \dot{v} = \alpha \end{cases}$$

Acceleration-controlled MFGs are of the form

$$\begin{cases} -\partial_t u - v \cdot D_x u + H(z, m, D_v u) = 0, \\ \underbrace{\partial_t m + v \cdot D_x m}_{\text{kinetic free transport}} - \mathrm{div}_v(m D_p H(z, m, D_v u)) = 0, \\ m|_{t=0} = m_0, \quad u|_{t=T} = g(x, v, m_T). \end{cases}$$

Well-posedness of MFG PDEs

Well-posedness: Case $\dot{x} = \alpha$

Huge literature, from both PDE and stochastic analysis perspectives

Ambrose, Cardaliaguet, Carmona, Cirant, Delarue, Goffi, Gomes, Graber, Lacker, Mészáros, Pimentel, Porretta, Sánchez-Morgado, Silva, Tonon, Voskanyan...

Key Factors

- Noise structure $\quad \mathrm{d}X_t^i = \alpha_t^i \mathrm{d}t + \sigma \mathrm{d}W_t^i + \mathrm{d}B_t$
  - Idiosyncratic - degeneracy?
  - Common
- Hamiltonian
  - Additive **separability**? $\quad \tilde{H}(m,p) = H(p) - F[m]$

    $\to$ Unrealistic for some applications,

    e.g. *congestion modelling* $H(m,p) \sim \frac{|p|^2}{1+m}$
  - m dependence: could be

    Regularising e.g. $F[m] \in C^2$ for any $m \in \mathscr{P}$

    Local $F[m](z) = f(m(z)), f : \mathbb{R} \to \mathbb{R}$

Well-posedness: Controlled Acceleration

All results for *separable* models:

Deterministic:

→ Regularising coupling ($F : \mathscr{P}_1 \to C^2$)

Achdou-Mannucci-Marchi-Tchou 2020, & state constraints 2021;
Cannarsa-Mendico 2020;
Bardi-Cardaliaguet 2021

(Degenerate) noise: $\mathrm{d}X_t = V_t \mathrm{d}t, \quad \mathrm{d}V_t = \alpha_t \mathrm{d}t + \mathrm{d}W_t^{(d)}$

→ Local couplings, (quadratic/Lipschitz Hamiltonian)

Mimikos-Stamatopoulos 2024

Two New(er) Results

Deterministic games with **local** couplings

GP-Mészáros 2022

- Uses a variational structure and approach.
- Separable Hamiltonians

Non-separable Hamiltonians with degenerate noise

Ambrose-GP-Mészáros 2024+

- Uses alignment condition between noise and control.
- Hamiltonian can be local.

Deterministic Local MFGs

A Variational Approach

Deterministic games with local coupling

Goal: **First order** kinetic MFG with **local** Hamiltonian

Idea: Some MFGs have a *variational structure*
— the MFG system is the (formal) optimality conditions for a pair of optimisation problems in duality.
c.f. Cardaliaguet-Graber 2015 $\dot{x} = \alpha, x \in \mathbb{T}^d$

Prototype model

$$\begin{cases} -\partial_t u - v \cdot D_x u + \frac{1}{r}|D_v u|^r = m^{q-1}, \\ \partial_t m + v \cdot D_x m - \mathrm{div}_v(m D_v u |D_v u|^{r-2}) = 0, \\ m|_{t=0} = m_0, \quad u|_{t=T} = m_T^{s-1}. \end{cases} \qquad r > 1, q \geq s > 1.$$

Weak solution = distributional solution (subsolution for HJB)
satisfying an energy equality.

x space can be $\mathbb{M} = \mathbb{T}^d$ or $\mathbb{R}^d$.

Theorem (Well-posedness; GP-Mészáros 2022)

For any initial condition $0 \leq m_0 \in C_b \cap L^1(\mathbb{M} \times \mathbb{R}^d)$, there **exists** a weak solution (u, m) of the MFG system.

This solution is **unique**, in that if (u', m') is also a weak solution, then $m = m'$ almost everywhere and $u = u'$ almost everywhere on the set $\{m > 0\}$.

$\rightarrow m_0$ may have vanishing regions, or be fully supported.

Strategy of Proof

Show that (c.f. Cardaliaguet-Graber)

1. The optimisation problems are indeed dual;

$$\inf_{(m,w)\in K_A} \mathcal{A}(m,w) = - \inf_{(u,\beta,\gamma)\in K_B} \mathcal{B}(u,\beta,\gamma).$$

2. Optimisers exist.

Challenges in the kinetic case:

- Unbounded domain – at least $v \in \mathbb{R}^d$
- $H \sim |D_v u|^r$: Loss of coercivity $\Rightarrow$ loss of compactness?

Optimisation Problem with Kinetic Constraint

A key step: prove existence of minimiser for

$$\mathcal{B}(u,\beta,\gamma) := \frac{1}{q'}\int_0^T \int \beta^{q'} \, \mathrm{d}x\mathrm{d}v\mathrm{d}t - \int u_0 m^0 \, \mathrm{d}x\mathrm{d}v + \frac{1}{s'}\int \gamma^{s'} \, \mathrm{d}x\mathrm{d}v$$

subject to

$$-\partial_t u - v \cdot D_x u + \frac{1}{r}|D_v u|^r \leq \beta, \quad u_T \leq \gamma \qquad (*)$$

Since $(*)$ is an inequality:

- Bounds on $\beta, \gamma \Rightarrow$ *Upper* bounds on u.
- *Lower* bounds come from bounds on $(u_0)_-$.

But $(u_0)_-$ is only controlled in regions where $m_0 > 0$.

What should we do if m^0 vanishes?

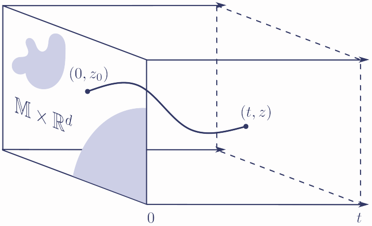
Reachable set

Consider the reachable set $\mathcal{U}_{m_0}$.

$(t, x, v) \in \mathcal{U}_{m_0}$ iff there exists a control $\alpha \in C_b^1$ such that the trajectory

$$\begin{cases} \dot{x}^\alpha = v^\alpha \\ \dot{v}^\alpha = \alpha, \end{cases} \qquad (x_t^\alpha, v_t^\alpha) = (x, v)$$

satisfies $(x_0^\alpha, v_0^\alpha) \in \{m_0 > 0\}$.



By considering test functions satisfying

$$\partial \phi + v \cdot D_x \phi + \mathrm{div}_v(\alpha \phi) = 0,$$

we obtain bounds on u on $\mathcal{U}_{m_0}$. Here

$$\mathcal{U}_{m_0} = \{0\} \times \{m_0 > 0\} \cup (0, T] \times \mathbb{M} \times \mathbb{R}^d.$$

Loss of Coercivity

Extract a limit point of a minimising sequence for

$$\mathcal{B}(u,\beta,\gamma) := \frac{1}{q'}\int_0^T\int \beta^{q'}\,\mathrm{d}x\mathrm{d}v\mathrm{d}t - \int u_0 m^0\,\mathrm{d}x\mathrm{d}v + \frac{1}{s'}\int \gamma^{s'}\,\mathrm{d}x\mathrm{d}v$$

subject to

$$-\partial_t u - v\cdot D_x u + \frac{1}{r}|D_v u|^r \le \beta, \quad u_T \le \gamma.$$

Energy estimates give uniform bounds for $D_v u \in L^r_{\mathrm{loc}}(\mathcal{U}_{m_0})$.

No estimate for $D_x u$?

Compactness is recovered by use of **averaging lemmas**.

Averaging lemmas describe a partial regularisation effect of the kinetic free transport operator.

$$\partial_t u + v \cdot D_x u = g \in L^2$$

Then **velocity averages** enjoy additional regularity

$$\rho_\phi[u](t,x) := \int_{\mathbb{R}^d} u(t,x,v)\phi(v)\mathrm{d}v \in H^{1/2-\epsilon}_{t,x} \qquad \phi \in L^\infty_c(\mathbb{R}^d)$$

(Golse-Lions-Perthame-Sentis 1988)

In practice:

- L^1 setting Golse-St Raymond 2002
  — Compactness only, extra technical conditions.
- From averages to the full value function
  — Using $D_v u \in L^r_{\mathrm{loc}}(\mathcal{U}_{m_0})$.

Non-Separable Hamiltonians

Non-separable MFGs: Well-posedness

Non-local Carmona-Delarue 2018, Cardaliaguet-Cirant-Porretta 2023; Gangbo-Mészáros-Mou-Zhang 2022, Mészáros-Mou 2024, Bansil-Mészáros-Mou 2023+

Local Ambrose 2018, 2022, Cirant-Gianni-Mannucci 2020, Ambrose-Mészáros 2023

→ Ambrose 2018, 2022: Local well-posedness for strong solutions of second-order MFGs with non-separable local Hamiltonians.

'Local' formulated through small parameters

$$\begin{cases} -\partial_t u - \frac{1}{2}\Delta u - \epsilon H(t,z,m,Du) = 0, \\ \partial_t m - \frac{1}{2}\Delta m + \epsilon \, \mathrm{div}_v \left(m D_p H(t,z,m,Du)\right) = 0, \\ u(T,\cdot) = \delta g(z,m_T), \quad m(0,\cdot) = m^0, \end{cases}$$

Kinetic-Type Non-Separable MFGs

Control system

$$\mathrm{d}Z_t = BZ_t\mathrm{d}t + \Pi\alpha_t\mathrm{d}t + \sigma\mathrm{d}W_t$$

- Alignment of controls and noise $\rightarrow |\sigma^\top\xi|^2 \geq c_0|\Pi^\top\xi|^2$
- H, g are any C^{s+2} functions, $H(t,z,0,0) = g(z,0) \equiv 0$.

'Local' well-posedness:

$$\begin{cases} -\partial_t u - (Bz)\cdot Du - \frac{1}{2}\sigma\sigma^\top : D^2 u = \epsilon H(t,z,m,\Pi^\top Du), \\ \partial_t m + \operatorname{div}(Bzm) - \frac{1}{2}D^2(\sigma\sigma^\top m) = -\epsilon \operatorname{div}\left(mD_pH(t,z,m,\Pi^\top Du)\right), \\ u(T,\cdot) = \delta g(z,m_T), \quad m(0,\cdot) = m^0, \end{cases}$$

Theorem (Ambrose-GP-Mészáros 24+)

Let s be an integer such that $s > \lceil N/2 \rceil + 2$.

For any $m^0 \in H^s_z$ and $T > 0$ the MFG system has a unique classical solution (u,m) for all sufficiently small ϵ, δ.

Sketch of Proof

Starting point → Ambrose 2018, 2022 (case $b(z, \alpha) = \alpha$).

A fixed point argument:

$$\begin{cases} -\partial_t u - v \cdot D_x u - \frac{1}{2}\Delta_v u = \epsilon H(m, D_v u), \\ \partial_t m + v \cdot D_x m - \frac{1}{2}\Delta_v m = -\epsilon \, \mathrm{div}_v \left(m D_p H(m, D_v u)\right), \\ u|_{t=T} = \delta g(m_T), \quad m|_{t=0} = m^0, \end{cases}$$

$$\text{Diffusion equation} \underset{\text{Composition}}{\overset{\text{Linear estimates}}{\rightleftarrows}} \text{Hamiltonian nonlinearity}$$

In the kinetic case:

- Choosing the right norm for the linear estimates
- Caution in the composition estimates

Estimates for the Diffusion Equation

We measure the H^s regularity of m using **time dependent** Sobolev norms.

$$\partial_t m + v \cdot D_x m - \frac{1}{2}\Delta_v m = S, \qquad m|_{t=0} = m^0.$$

Use a basis of vector fields commuting with the transport operator

$$(D_x, D_v) \to (tD_x + D_v, D_v)$$

- Inspired by hypocoercivity techniques e.g. Hérau 2007

$$\|m\|_{H^s}^2 \sim_t \sum_{|\beta| \leq s} \|\gamma^\beta(t) m\|_{H^s}^2$$

Advantages

- Compatible with forward-backward structure
- Optimises the time dependence of estimates.

Composition Estimates

Require Sobolev composition estimates in the **twisted** norms.

$$\|H(m, D_v u)\|_{H^s} \leq \mathcal{F}\left(\|m\|_{H^s}, \|D_v u\|_{H^s}\right)$$

'Alignment' — highest order terms must involve derivatives in the diffusive directions

$$-(\partial_t + v \cdot D_x - \frac{1}{2}\Delta_v)\gamma^\beta u = \epsilon \gamma^\beta H(m, D_v u)$$

- Allows local Hamiltonians
- Caution with embedding estimates
- Time dependence matters – must be L^2 type

Wrapping up

Kinetic techniques can be used to analyse Mean Field Games with generalised linear control dynamics.

Noise structure & separability/locality of the Hamiltonian remain important factors, as in the velocity-controlled case.

New challenges & insights arise from the interaction between these factors and a generalised control system:

- The role of the reachable set in the variational setting;
- Allowing degenerate noise in the non-separable case through alignment with controls.

Thank you!