Quasipolynomial bounds for U^4 norm in $\mathbb{F}_p^n$

Luka Milićević

Mathematical Institute of the Serbian Academy of Sciences and Arts

Additive Combinatorics, ICMS, Edinburgh
July 24, 2024

This research was supported by the Science Fund of the Republic of Serbia, Grant No. 11143, Approximate Algebraic Structures of Higher Order: Theory, Quantitative Aspects and Applications – A-PLUS.

Science Fund
of the Republic of Serbia

Inverse theorems in $\mathbb{F}_p^n$

Inverse theorems in $\mathbb{F}_p^n$

Obstructions.

Inverse theorems in $\mathbb{F}_p^n$

Obstructions. Extremal solutions to $\|f\|_{\mathsf{U}^k} = 1$ for $f\colon \mathbb{F}_p^n \to \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Inverse theorems in $\mathbb{F}_p^n$

Obstructions. Extremal solutions to $\|f\|_{\mathsf{U}^k} = 1$ for $f\colon \mathbb{F}_p^n \to \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Phases of non-classical polynomials.

Inverse theorems in $\mathbb{F}_p^n$

Obstructions. Extremal solutions to $\|f\|_{U^k} = 1$ for $f\colon \mathbb{F}_p^n \to \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Phases of non-classical polynomials.

If $p \geq k$, we get only polynomials.

Inverse theorems in $\mathbb{F}_p^n$

Obstructions. Extremal solutions to $\|f\|_{\mathsf{U}^k} = 1$ for $f \colon \mathbb{F}_p^n \to \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Phases of non-classical polynomials.

If $p \geq k$, we get only polynomials. *High-characteristic case.*

Inverse theorems in $\mathbb{F}_p^n$

Obstructions. Extremal solutions to $\|f\|_{\mathsf{U}^k} = 1$ for $f\colon \mathbb{F}_p^n \to \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Phases of non-classical polynomials.

If $p \geq k$, we get only polynomials. *High-characteristic case.*

If $p < k$, we have the *low-characteristic case*.

Inverse theorems in $\mathbb{F}_p^n$

Obstructions. Extremal solutions to $\|f\|_{\mathsf{U}^k} = 1$ for $f\colon \mathbb{F}_p^n \to \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Phases of non-classical polynomials.

If $p \geq k$, we get only polynomials. *High-characteristic case.*

If $p < k$, we have the *low-characteristic case*.

Theorem (Bergelson, Tao, Ziegler, 2010)

Let $f\colon \mathbb{F}_p^n \to \mathbb{D}$ be a function such that $\|f\|_{\mathsf{U}^k} \geq c$. **Assume that $p \geq k$.** Then there exists a polynomial $q\colon \mathbb{F}_p^n \to \mathbb{F}_p$ of degree at most $k-1$ such that $\left|\mathbb{E}_x f(x)\omega^{q(x)}\right| \geq \Omega_c(1)$.

Inverse theorems in $\mathbb{F}_p^n$

Obstructions. Extremal solutions to $\|f\|_{\mathsf{U}^k} = 1$ for $f\colon \mathbb{F}_p^n \to \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Phases of non-classical polynomials.

If $p \geq k$, we get only polynomials. *High-characteristic case.*

If $p < k$, we have the *low-characteristic case*.

Theorem (Bergelson, Tao, Ziegler, 2010)

Let $f\colon \mathbb{F}_p^n \to \mathbb{D}$ be a function such that $\|f\|_{\mathsf{U}^k} \geq c$. **Assume that $p \geq k$.** Then there exists a polynomial $q\colon \mathbb{F}_p^n \to \mathbb{F}_p$ of degree at most $k-1$ such that $\Big|\mathbb{E}_x f(x)\omega^{q(x)}\Big| \geq \Omega_c(1)$.

Theorem (Tao, Ziegler, 2012)

Let $f\colon \mathbb{F}_p^n \to \mathbb{D}$ be a function such that $\|f\|_{\mathsf{U}^k} \geq c$. Then there exists a (non-classical) polynomial $q\colon \mathbb{F}_p^n \to \mathbb{T}$ of degree at most $k-1$ such that $\Big|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))\Big| \geq \Omega_c(1)$.

Quantitative results

Quantitative results

In high-characteristic:

Quantitative results

In high-characteristic:

- (Gowers, M., 2017) U^4,
  $$|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp(\exp(\text{quasi-pol}(2c^{-1})))\Big)^{-1}.$$

Quantitative results

In high-characteristic:

- (Gowers, M., 2017) U^4,
  $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp(\exp(\text{quasi-pol}(2c^{-1})))\Big)^{-1}$.
- (Gowers, M., 2020) U^k, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp^{O_k(1)}(2c^{-1})\Big)^{-1}$.

Quantitative results

In high-characteristic:

- (Gowers, M., 2017) U^4, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp(\exp(\text{quasi-pol}(2c^{-1})))\Big)^{-1}$.
- (Gowers, M., 2020) U^k, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp^{O_k(1)}(2c^{-1})\Big)^{-1}$.
- (Lovett; Kim, Li, Tidor, 2022) U^4, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \exp(-\text{quasi-pol}(2c^{-1}))$.

Quantitative results

In high-characteristic:

- (Gowers, M., 2017) U^4, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp(\exp(\text{quasi-pol}(2c^{-1})))\Big)^{-1}$.
- (Gowers, M., 2020) U^k, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp^{O_k(1)}(2c^{-1})\Big)^{-1}$.
- (Lovett; Kim, Li, Tidor, 2022) U^4, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \exp(-\text{quasi-pol}(2c^{-1}))$.

Theorem (Gowers, M., 2020)

Let $f \colon \mathbb{F}_p^n \to \mathbb{D}$ be a function such that $\|f\|_{\mathrm{U}^k} \geq c$. Then there exists a multilinear form α in $k-1$ variables such that

$$\Big|\mathbb{E}_{a_1,\dots,a_{k-1},x}\, \partial_{a_1} \dots \partial_{a_{k-1}} f(x) \omega^{\alpha(a_1,\dots,a_{k-1})}\Big| \geq \Big(\exp^{O_k(1)}(c^{-1})\Big)^{-1}.$$

Quantitative results

In high-characteristic:

- (Gowers, M., 2017) U^4,
  $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp(\exp(\text{quasi-pol}(2c^{-1})))\Big)^{-1}$.
- (Gowers, M., 2020) U^k, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp^{O_k(1)}(2c^{-1})\Big)^{-1}$.
- (Lovett; Kim, Li, Tidor, 2022) U^4,
  $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \exp(-\text{quasi-pol}(2c^{-1}))$.

Theorem (Gowers, M., 2020)

Let $f\colon \mathbb{F}_p^n \to \mathbb{D}$ be a function such that $\|f\|_{\mathrm{U}^k} \geq c$. Then there exists a multilinear form α in $k-1$ variables such that

$$\Big|\mathbb{E}_{a_1,\dots,a_{k-1},x}\, \partial_{a_1} \dots \partial_{a_{k-1}} f(x) \omega^{\alpha(a_1,\dots,a_{k-1})}\Big| \geq \Big(\exp^{O_k(1)}(c^{-1})\Big)^{-1}.$$

Low-characteristic case.

Quantitative results

In high-characteristic:

- (Gowers, M., 2017) U^4,
  $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp(\exp(\text{quasi-pol}(2c^{-1})))\Big)^{-1}$.
- (Gowers, M., 2020) U^k, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp^{O_k(1)}(2c^{-1})\Big)^{-1}$.
- (Lovett; Kim, Li, Tidor, 2022) U^4,
  $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \exp(-\text{quasi-pol}(2c^{-1}))$.

Theorem (Gowers, M., 2020)

Let $f\colon \mathbb{F}_p^n \to \mathbb{D}$ be a function such that $\|f\|_{\mathsf{U}^k} \geq c$. Then there exists a multilinear form α in $k-1$ variables such that
$$\Big|\mathbb{E}_{a_1,\dots,a_{k-1},x}\,\partial_{a_1}\dots\partial_{a_{k-1}} f(x)\omega^{\alpha(a_1,\dots,a_{k-1})}\Big| \geq \Big(\exp^{O_k(1)}(c^{-1})\Big)^{-1}.$$

Low-characteristic case.

- (Tidor, 2022) U^4

Quantitative results

In high-characteristic:

- (Gowers, M., 2017) U^4,
  $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp(\exp(\text{quasi-pol}(2c^{-1})))\Big)^{-1}$.
- (Gowers, M., 2020) U^k, $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \Big(\exp^{O_k(1)}(2c^{-1})\Big)^{-1}$.
- (Lovett; Kim, Li, Tidor, 2022) U^4,
  $|\mathbb{E}_x f(x)\,\mathrm{e}(q(x))| \geq \exp(-\text{quasi-pol}(2c^{-1}))$.

Theorem (Gowers, M., 2020)

Let $f\colon \mathbb{F}_p^n \to \mathbb{D}$ be a function such that $\|f\|_{\mathrm{U}^k} \geq c$. Then there exists a multilinear form α in $k-1$ variables such that

$$\Big|\mathbb{E}_{a_1,\dots,a_{k-1},x}\,\partial_{a_1}\dots\partial_{a_{k-1}} f(x)\omega^{\alpha(a_1,\dots,a_{k-1})}\Big| \geq \Big(\exp^{O_k(1)}(c^{-1})\Big)^{-1}.$$

Low-characteristic case.

- (Tidor, 2022) U^4
- (M., 2023) U^5 and U^6 in $\mathbb{F}_2^n$.

Quantitative results

Quantitative results

Theorem (M., 2024^+)

Let $f\colon \mathbb{F}_p^n \to \mathbb{D}$ be a function such that $\|f\|_{\mathsf{U}^4} \geq c$. Then there exists a (non-classical) polynomial $q\colon \mathbb{F}_p^n \to \mathbb{T}$ of degree at most 3 such that $\Big|\mathbb{E}_x\, f(x)\, \mathrm{e}(q(x))\Big| \geq \exp(-\log^{O(1)}(2c^{-1}))$.