Recent developments in Ergodic Ramsey Theory

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ICMS – Additive Combinatorics

Theorem (Szemerédi's theorem, 1975)

If $A \subset \mathbb{N}$ *has* $\bar{d}(A) := \limsup_{N\to\infty} \frac{|A \cap [N]|}{N} > 0$*, then* $\forall k \in \mathbb{N}$*,* $\exists a, n \in \mathbb{N}$ *such that* $\{a + in : i \in [k]\} \subset A$*.*

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- Question: Does the limit exist?
- In particular, if $\bar{d}(A) > 0$, then $\exists \ell$ such that $\forall N \in \mathbb{N}$, $\exists n \in [N-\ell, N]$ and $a \in \mathbb{N}$ with $\{a + in : i \in [k]\} \subset A$.
- Question: How does ℓ grow as $\bar{d}(A) \to 0$?

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Theorem (Furstenberg-Katznelson, 1978)

If $A \subset \mathbb{Z}^d$ *has* $\bar{d}(A) := \limsup_{N\to\infty} \frac{|A \cap [N]^d|}{N^d} > 0$*, then A contains a homothetic image of any finite set.*

- The first non-ergodic proofs of this fact appeared in the mid 2000's.

- All these are generalized by the density Hales-Jewett theorem, proved by Furstenberg and Katznelson in 1991.
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Corollary (IP_r Szemerédi theorem)

For every $\delta > 0$ and $k \in \mathbb{N}$, $\exists r \in \mathbb{N}$ such that $\forall A \subset \mathbb{N}$ with $\bar{d}(A) > \delta$ and any distinct $n_1, \ldots, n_r \in \mathbb{N}$ there exist $n = \sum_{i \in I} n_i$ (for some non-empty $I \subset [r]$) and $a \in \mathbb{N}$ such that $\{a + in : i \in [k]\} \subset A$.

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- In 1996, Bergelson and Leibman obtained polynomial (and multidimensional) extensions of van der Waerden's theorem and Szemerédi's theorem, using ergodic theory.

Corollary

If $\bar{d}(A) > 0$ and $p_1, \ldots, p_k \in \mathbb{Z}[x]$ have $p_i(0) = 0$, then $\exists a, n \in \mathbb{N}$ s.t. $\left\{a + p_i(n) : i \in [k]\right\} \subset A$.

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Conjecture (Special case of density polynomial Hales-Jewett)

For every $\delta > 0$ $\exists r \in \mathbb{N}$ such that $\forall A \subset \mathbb{N}$ with $\bar{d}(A) > \delta$ and any distinct $n_1, \ldots, n_r \in \mathbb{N}$ there exist $n = \sum_{i \in I} n_i$ (for some non-empty $I \subset [r]$) and $a \in \mathbb{N}$ such that $\{a, a + n^2\} \subset A$.

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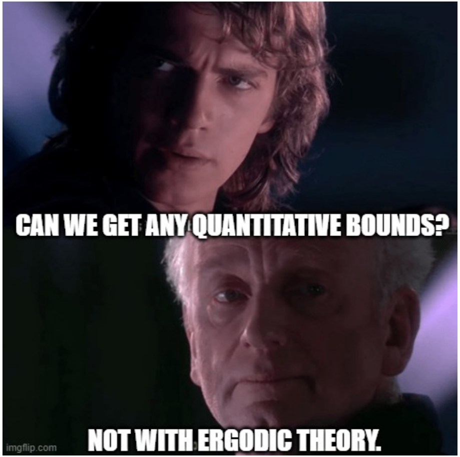

Definition

- A *topological dynamical system* is a pair (X, T) where X is a compact metric space and $T : X \to X$ is a homeomorphism.
- A *measure preserving system* is a triple (X, μ, T) where (X, T) is a t.d.s. and μ is a probability measure on X invariant under T, in the sense that $\mu(TA) = \mu(A)$ for every Borel set $A \subset X$.

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Furstenberg Correspondence Principle

- Given $A \subset \mathbb{N}$, let $a = 1_A \in \{0,1\}^{\mathbb{Z}}$ and let $T : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the left shift.

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- For $N \in \mathbb{N}$ let $\mu_N = \mathop{\mathbb{E}}_{n \leq N} \delta_{T^n a}$ and "let" $\mu = \lim \mu_N$. Then (X, μ, T) is a m.p.s.

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- Let $E := \big\{(x_n)_{n \in \mathbb{Z}} \in X : x_0 = 1\big\}$. Then $A = \{n \in \mathbb{Z} : T^n a \in E\}$ and $\mu(E) = \bar{d}(A)$.
- In fact, for any $n_1, \ldots, n_k \in \mathbb{N}$, $\bar{d}\big(A \cap (A - n_1) \cap \cdots \cap (A - n_k)\big) \geq \mu\big(E \cap T^{-n_1} E \cap \cdots \cap T^{-n_k} E\big)$.

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Theorem (Furstenberg, 1977)

Let (X, μ, T) be a m.p.s. and $E \subset X$ with $\mu(E) > 0$. Then for every $k \in \mathbb{N}$,

$$\liminf_{N-M\to\infty} \mathop{\mathbb{E}}_{M<n<N} \mu(E \cap T^{-n} E \cap \cdots \cap T^{-kn} E) > 0.$$

Question

Let (X, μ, T) be a m.p.s. and $f_1, \ldots, f_k \in L^\infty(X)$. Does the sequence

$$\frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k \tag{1}$$

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- These works introduced seminorms in m.p.s., suspiciously similar to Gowers norms.

$$\|f\|_{\mathrm{HK}^k}^{2^k} = \lim_{N\to\infty} \mathop{\mathbb{E}}_{\vec{n}\in[N]^k} \int_X \prod_{\vec{\omega}\in\{0,1\}^k} \mathcal{C}^{|\vec{\omega}|} T^{\vec{n}\cdot\vec{\omega}} f.$$

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- If $\|f_i\|_{\mathrm{HK}^k} = 0$ for some i, the limit in (1) is 0; if $\|f_i\|_{\mathrm{HK}^k} > 0$ then f_i correlates with a *nilfunction*.
- This served as an impetus for the inverse conjecture/theorem for Gowers norms.

- If $f : \mathbb{N} \to \mathbb{C}$ is bounded, realizing it as a function in $L^\infty(X, \mu, T)$ using Furstenberg's Correspondence Principle, we can define the Host-Kra seminorm of f as

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  where $\mathop{\mathbb{E}}_{x\in\mathbb{N}}$ means $\lim_{j\to\infty} \mathop{\mathbb{E}}_{x\in[M_j]}$ for some increasing sequence (M_j) for which all the limits exist.

- For comparison, $\lim_{N\to\infty} \|f\|_{U^k[N]}^{2^k} \approx \lim_{N\to\infty} \mathop{\mathbb{E}}_{\vec{n}\in[N]^k} \mathop{\mathbb{E}}_{x\in[N]} \prod_{\vec{\omega}\in\{0,1\}^k} \mathcal{C}^{|\vec{\omega}|} f(x + \vec{n}\cdot\vec{\omega})$.

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Example

Let $\lambda : \mathbb{N} \to \{-1, 1\}$ be the Liouville function ($\lambda(p) = -1$ for prime p and $\lambda(nm) = \lambda(n)\lambda(m)$).

- Green and Tao proved (around 2010) that $\lim_{N\to\infty} \|\lambda\|_{U^k[N]} = 0$ for all k.

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- A proof that $\|\lambda\|_{\mathrm{HK}^k[N]}=0$ for all k would imply (logarithmic) Chowla conjecture.

Question 1 (Erdős and Graham, \$250)

Is the equation $x^2 + y^2 = z^2$ partition regular?

In other words, if $\mathbb{N}$ is finitely colored, is there always a monochromatic pythagorean triple?

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- In 2021, Chow, Lindqvist and Prendiville established partition regularity for a large family of equations in sufficiently many variables, including $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2$.

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- Let $\mathcal{M} := \{f : \mathbb{N} \to S^1 : \forall m, n \in \mathbb{N},\ f(mn) = f(m)f(n)\} = (\widehat{\mathbb{Q}^{>0}, \times})$.

In 2013, Frantzikinakis and Host noticed a connection between Question 2 and measures on $\mathcal{M}$.

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- Let $\mathcal{M} := \{f : \mathbb{N} \to S^1 : \forall m, n \in \mathbb{N},\ f(mn) = f(m)f(n)\} = (\widehat{\mathbb{Q}^{>0}, \times})$.

In 2013, Frantzikinakis and Host noticed a connection between Question 2 and measures on $\mathcal{M}$.

- A function $f \in \mathcal{M}$ is *aperiodic* if for every $a, b \in \mathbb{N}$, $\mathop{\mathbb{E}}_{n\in[N]} f(an + b) \to 0$ as $N \to \infty$.

Theorem

- *Frantzikinakis-Host (2017): For every aperiodic $f \in \mathcal{M}$ and $\forall s \in \mathbb{N}$, $\|f\|_{U^s[N]} \to 0$ as $N \to \infty$.*

Question 1 (Erdős and Graham, \$250)

Is the equation $x^2 + y^2 = z^2$ partition regular?
In other words, if $\mathbb{N}$ is finitely colored, is there always a monochromatic pythagorean triple?

- For 2 colors this was verified by Heule, Kullmann and Marek in 2016 with the aid of a computer.
- In 2021, Chow, Lindqvist and Prendiville established partition regularity for a large family of equations in sufficiently many variables, including $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2$.

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Using this result they showed that in any finite coloring of $\mathbb{N}$ there is a solution to $9x^2 + 16y^2 = z^2$ with x and y of the same color.

Conjecture (Sarnak)

If (X, T) is a t.d.s. with 0 entropy, then for every $x \in X$ and $f \in C(X)$, $\lim_{N\to\infty} \mathbb{E}_{n\in[N]} f(T^n x)\lambda(n) = 0$.

Zero entropy means that for every $r > 0$ and $f \in C(X)$, the set $\left\{ f(T^n x)_{n=1}^N : x \in X \right\} \subset X^N$ is contained $\exp(o(N))$ balls of radius r.

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Theorem (Frantzikinakis-Host, 2018)

If (X,T) is a t.d.s. with 0 entropy and it has a unique invariant measure, then for every $x \in X$ and $f \in C(X)$, $\mathop{\mathbb{E}}^{\log}_{n\in[N]} f(T^n x)\lambda(n) \to 0$ *as* $N \to \infty$.

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Corollary

If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $h_1,\dots,h_k \in \mathbb{N}$, then $\mathop{\mathbb{E}}^{\log}_{n\le N} e^{2\pi i n\alpha}\lambda(n+h_1)\cdots\lambda(n+h_k) \to 0$ *as* $N \to \infty$.

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Question: What about rational α?

Theorem (Hindman, 1974)

For any finite coloring of $\mathbb{N}$ *there exists* $B \subset \mathbb{N}$ *infinite s.t.* $FS(B) := \left\{ \sum_{n \in F} n : F \subset B, 0 < |F| < \infty \right\}$ *is monochromatic.*

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If $A \subset \mathbb{N}$ has $d(A) > 0$, are there $B \subset \mathbb{N}$ infinite and $t \in \mathbb{N}$ such that $A - t \supset FS(B)$?

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Theorem (M.-Richter-Robertson, 2018)

If $A \subset \mathbb{N}$ *has* $\bar{d}(A) > 0$*, then there are infinite sets* $B, C \subset \mathbb{N}$ *such that* $B + C \subset A$*.*

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Theorem (Kra-M.-Richter-Robertson, 2022)

If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$, then $\exists$ $B \subset \mathbb{N}$ infinite and $t \in \{0, 1\}$ such that

$$B \oplus B := \{b + b' : b, b' \in B, b \neq b'\} \subset A - t.$$

Theorem (Kra-M.-Richter-Robertson, 2022)

If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$ then for every $k \in \mathbb{N}$ there exist infinite sets $B_1, \ldots, B_k \subset \mathbb{N}$ such that $B_1 + \cdots + B_k \subset A$.

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Conjecture ("Density version" of Hindman's theorem)

If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$ then for each $k \in \mathbb{N}$ there exist $t \in \mathbb{N}$ and infinite $B \subset \mathbb{N}$ such that

$$\left\{ \sum_{n \in F} n : F \subset B \text{ with } 0 < |F| \leq k \right\} \subset A - t.$$

- For every $m \in \mathbb{N}$, there exists a 2-coloring of $\mathbb{N}$ without a monochromatic sumset $B + mB$.

Question

Is it true that for any finite coloring of $\mathbb{N}$, there is an infinite set $B \subset \mathbb{N}$ and some $m \in \mathbb{N}$ such that $B + mB$ is monochromatic?

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- Conditionally on the Dickson-Hardy-Littlewood prime tuples conjecture,

$$\exists B \subset \mathbb{N} \text{ infinite s.t.} \qquad \mathbb{P} - 1 \supset \left\{ \sum_{n \in F} n : F \subset B \text{ with } 0 < |F| \leq k \right\}.$$

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Theorem (Tao-Ziegler, 2023)

There exist infinite sets $B, C \subset \mathbb{N}$ such that $\{b + c : b \in B, c \in C, b < c\} \subset \mathbb{P}$.