Fair Mixing and participatory democracy

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based on:

"*Fair Mixing: the case of dichotomous preferences*" (Haris Aziz and Anna Bogomolnaia, Hervé Moulin), *ACM Trans. Econ. Comput.*, 2020

“*Collective choice under dichotomous preferences*”, A Bogomolnaia, H Moulin, R Stong, J. Econ.. Theory, 2005

“*Distribution Rules Under Dichotomous Preferences: Two Out of Three Ain't Bad*”, F Brandl, F Brandt, D Peters, C Stricker, Proc. 22th ACM Economics and Computations, 2021

“*Funding public projects: A case for the Nash product rule*", F Brandl, F Brandt, M Greger, D Peters, C Stricker, W Suksompong, J. Math. Economics, 2022

“*Participatory Budgeting: Models and Approaches*”, H Aziz, N Shah, in Pathways Between Social Science and Computational Social Science, Springer Link, 2021

the problem:

the set $N = \{1, ..., n\}$ of voters need to agree on a *convex combination* $z \in \triangle A$ over the set of (mutually exclusive) outcomes $A = \{a_1, ..., a_m\}$.

z in ΔA represents

- a lottery over outcomes
- a time sharing arrangement
- the division of a budget between several projects

the basic modeling assumption:

agents have *dichotomous preferences* over outcomes: *like/dislike*

it simplifies the mathematics

it simplifies the choice of the citizens

$$[u] = [u_i^a]_{i \in N, a \in A}$$

$N \downarrow A \rightarrow$	a	b	c	d	e
1	0	0	0	1	0
2	0	0	1	1	1
3	1	1	0	0	0
4	1	0	1	0	0
5	0	1	0	1	1

we must choose a *mixture*

$z = (z_a, z_b, z_c, z_d, z_e) \geq 0$, $z_a + z_b + z_c + z_d + z_e = 1$

guided by: *the utilities*: $U_1 = z_d$, $U_2 = z_c + z_d + z_e$, etc..

instance/problem: $M = (N, A, u)$

a central concern: <u>Fairness</u> as *protection of minorities*

a modest principle: everyone is entitled to some benefit from the public decision

we also want

<u>Efficiency</u> (Pareto optimality)

<u>Incentive Compatibility</u> (e.g. strategy-proofness)

A **rule** assigns to each problem $M = (N, A, u)$

a *set* of mixtures $f(M) = \{z \in \Delta(A)\}$
with a *single valued* feasible utility profile $U = F(M) = [u] \cdot z$

Hard wired in the definition of a rule:

$\longrightarrow$ it ignores null or full rows (indifferent agents)

$\longrightarrow$ null columns (useless outcomes)

$\longrightarrow$ clone outcomes (welfarist viewpoint)

More hard wired properties of the rules

Anonymous (ANO): f and F are symmetric functions of $(u_1, \cdots, u_n)$
they commute with permutations of N

Neutral (NEUT): f and F commute with permutations of A

easily compatible unlike in deterministic voting: *no need to break ties*

the next two properties have real bite
but they may conflict with the previous ones or with other fairness principles

Efficiency (EFF): $F(M) \leq U \Longrightarrow F(M) = U$, for all M, U feasible at M

in the example $z \in \Delta(A)$ is inefficient *iff* $z_e > 0$ and/or $z_b \cdot z_c > 0$

checking efficiency is a linear program

Strategy-proofness (SP): $u_i \cdot f(u) \geq \max_{z' \in f(u|^i u'_i)} u_i \cdot z'$ for all i and u'_i

a famous impossibility result in the generalisation of our model to vNM preferences (Gibbard, 1977, Zhou 1990):

whether voters report only their ordinal preferences, or their full vNM utilities
Efficiency + Strategy-proofness (SP) + Anonymity $= \varnothing$

This incompatibility disappears in the dichotomous domain

The *Utilitarian rule* (UTIL) (aka *Approval voting*) averages all *deterministic utility profiles* with largest approval

$$F^{ut}(M) = avg\{u^a | a \in \operatorname{arg\,max}_{b \in A} \sum_{i \in N} u_i^b\}$$

Simple fact**:** *UTIL is Efficient, Strategy-proof, Anonymous, and Neutral* but UTIL has a fatal flaw

it is uncompromising, ignores minorities entirely

a minimal individual guarantee

Individual Fair Share (IFS): $U_i \geq \frac{1}{n}$ for all i

bad news: if $n \geq 5$ and $|A| \geq 5$

$$\text{EFF} + \text{Strategy-proofness} + \text{Positive Fair Share} = \varnothing$$

first a weaker version of the result Bogomolnaia & Moulin 2005

sharp version Brandl et al. 2021 *with a computer-aided proof!*

a first attempt around the impossibility result

Egalitarian rule (EGAL) equalizes utilities in the leximin sense

$$F^{eg}(M) = \text{arg} \max_{U \in [u] \cdot \Delta(A)} \succ_{leximin}$$

in the example EGAL picks $\frac{1}{2}a + \frac{1}{2}d$, while UTIL picks d

EGAL meets EFF + IFS

Assume the public outcomes are non rival but excludable: (club meeting, cable TV broadcast..)

Excludable Strategy-proofness (EXSP):

$$u_i \cdot f(u) \geq \max_{z' \in f(u|^i u'_i)} (u_i \wedge u'_i) \cdot z' \text{ for all } i \text{ and } u'_i$$

Proposition

The Egalitarian rule is Excludable Strategy-proof, EFF, and IFS but EGAL has a fatal flaw

Clone Invariance:

adding any number of clones of i has no effect

numbers do not matter!

one individual preference matter as much as one thousand identical preferences remedies to Clone Invariance:

– No Show (almost) always hurts the absentee

Strict Participation (PART)

$$U_i(N) \;\geq\; \max_{z \in f(N \diagdown i)} u_i \cdot z \text{ ; } U_i(N) > \min_{z \in f(N \diagdown i)} u_i \cdot z \text{ if } \min_{z \in f(N \diagdown i)} u_i \cdot z < 1$$

– Individual welfare guarantees add up *among clones*

Unanimous Fair Share (UFS)

$$\text{for all coalition } S \;:\; u_i = u_j \text{ for } i, j \in S \Longrightarrow U_i \geq \frac{|S|}{n} \text{ for all } i \in S$$

the three rules we (finally) propose meet PART and UFS.

Two are variants of the familiar "Random Dictator":
$\sigma \in per(N)$ is an ordering of N, the σ-*Priority rule* F^{σ}: ensures $u_{\sigma(1)} = 1$; $u_{\sigma(2)} = 1$ as well if 1 and 2 like a common outcome; $u_{\sigma(3)} = 1$ if $\sigma(3)$ likes an outcome common with all happy agents before her; and so on ..

Random Priority rule (RP) $F^{rp}(M) = \frac{1}{n!} \sum_{\sigma \in per(N)} U^{\sigma}$

Conditional Utilitarian rule(CUT): (recall we drop indifferent agents)
let $\tau(u_i) = \{a \in A | u_{ia} = 1\}$

$$F^{cut}(M) = \frac{1}{n} \sum_{i \in N} avg\{u^a | a \in \arg\max_{b \in \tau(u_i)} \sum_{i \in N} u_i^b\}$$

each active agent spreads her $\frac{1}{n}$- share of decision power equally between the outcomes with maximal support among those she likes (Duddy (2015))

Proposition:

i) *Both CUT and RP are Strategy-proof; they meet* PART and UFS. They are not Efficient for $|A| > 3$ and $|N| > 4$

ii) *CUT is polynomial, RP is #P-complete to compute (Aziz, Brandt, Brill (2013))*

iii) *Total utility for CUT is never below that of RP*

iv) *for all* M, $F^{cut}(M)$ *is efficient if* $F^{rp}(M)$ *is efficient; the converse is not true*

v) *CUT is at most (and can be)* $O(n^{-\frac{1}{3}})$*-inefficient, while RP can be* $O(\frac{\ln(n)}{n})$*-inefficient*

and the winner is CUT!

Open questions

– what is the worst case inefficiency of RP?

– in the impartial culture context, what is the probability that RP or CUT is efficient? what about some expected measure of their inefficiency?

numerical experiments show that CUT is more than 90% efficient under the impartial culture

Two group guarantees with much more bite than UFS:

Average Fair Share (AFS)

$$\{\exists a \in A : u_{ia} = 1 \text{ for all } i \in S\} \implies \frac{1}{|S|} U_S \geq \frac{|S|}{n}.$$

in the canonical example: $\frac{1}{3}(U_1 + U_2 + U_5) \geq \frac{3}{5}$ and $\frac{1}{2}(U_3 + U_4) \geq \frac{2}{5}$

Core Fair Share (CFS)

$$\nexists z \in \mathbf{\Delta}(A) \text{ s. t. } \forall i \in S : U_i \leq \frac{|S|}{n}(u_i \cdot z) \text{ and } \exists i \in S : U_i < \frac{|S|}{n}(u_i \cdot z).$$

each coalition can cumulate its individual shares of decision power and form core objections

– AFS and CFS are not logically related

– RP and CUT violate both AFS and CFS

Nash Max Product rule (NMP)

$$f^{nmp}(M) = \arg\max_{z \in \Delta(A)} \sum_{i \in N} \ln(u_i \cdot z)$$

strictly convex program $\Longrightarrow$ well defined.

Proposition:

i) *The Nash rule is Efficient; it meets Strict Participation, Average Fair Share and Core Fair Share*

ii) *it is not SP, and not even Excludable Strategy-proof*

iii) *its exact computational complexity is unknown, but it is easily approximated by C-plex methods*

Open questions

– for what sizes of N and A is the Nash rule EXSP ?

– is there a rule meeting EFF, EXSP and Strict Participation?

– is there a rule meeting EFF, EXSP and Unanimous Fair Share?

Thank You