Graph and Mean-Field Limits for Interacting Particle Systems on Weighted Deterministic and Random Graphs

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The Many Facets of Kinetic Theory
25 September 2024

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INTRODUCTION

Collective dynamics models

Social dynamics model

$$\frac{d}{dt}x_i(t) = \frac{1}{N}\sum_{j=1}^{N} a_{ij}\,(x_j(t) - x_i(t)),$$

where:

- $x_i \in \mathbb{R}^d$ is the state variable (opinion, position)
- $a_{ij} \in \mathbb{R}$ is the **interaction coefficient**.

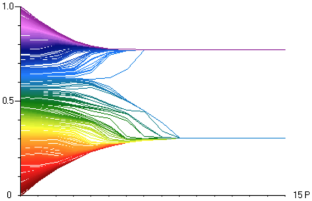
Hegselmann-Krause dynamics

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N} a(\|x_i - x_j\|)(x_j - x_i), \quad x_i \in \mathbb{R}^d, \quad i \in \{1, \dots, N\} \qquad \text{(HK)}$$

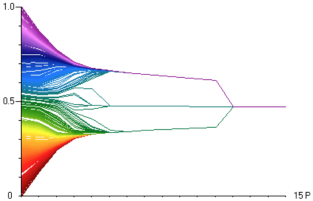
with $a_{ij} = a(\|x_i - x_j\|)$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is the *influence function*.

Two types of questions

- **Self-organization**: emergence of well organized group patterns.



Clustering



Consensus

[Hegselmann and Krause, '02]

- **Large Population Limit**: N the number of agents goes to infinity.

The classical approach : The mean-field limit

- **No longer** follow each agent's **individual trajectory**,
- the population is represented by its **probability density**,
- the **limit measure** $\mu_t(x)$ represents the density of agents with opinion x at time t.

HK model: macroscopic

$$\partial_t \mu_t + \nabla \cdot (V[\mu_t]\mu_t) = 0 \qquad V[\mu_t](x) = \int_{\mathbb{R}^d} a(\|x-y\|)(y-x)d\mu_t(y).$$

- **Limitation**: *Indistinguishability* of the particles $\Rightarrow$ reduces the span of models that can be studied.

The new approach : The graph limit

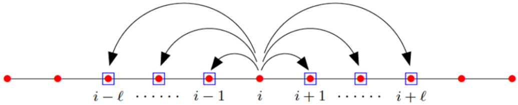
The θ-nearest-neighbor interactions model

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=i-\ell}^{i+\ell}(x_j - x_i) \qquad \text{with } \ell = \lfloor \theta N \rfloor, \theta \in [0,1] \qquad (\theta\text{-nearest})$$

- (θ-nearest) : system of ODE on **graph** $G_N =< V(G_N), E(G_N) >$ with

$$V(G_N) = \{1, 2, \ldots, N\} \qquad E(G_N) = \{(i,j) \in \{1,2,\ldots,N\}^2 \mid 0 < dist(i,j) \leq \ell\}$$

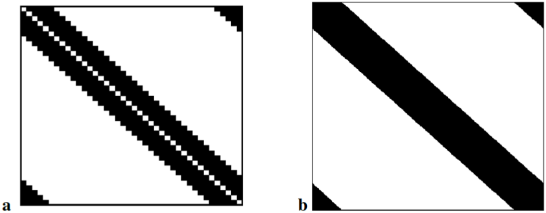
where $dist(i,j) = \min\{|i-j|, N - |i-j|\}$.



Scheme of the θ-nearest-neighbor interactions [Biccari, Ko, Zuazua, '19]

- Let $w^{G_N} : [0,1]^2 \to \{0,1\}$

$$w^{G_N}(\xi,\zeta) = 1 \qquad \text{if } (i,j) \in E(G_N) \text{ and } (\xi,\zeta) \in \left[\frac{i-1}{N}, \frac{i}{N}\right) \times \left[\frac{j-1}{N}, \frac{j}{N}\right).$$



Plot of the support of the function w^{G_N} representing the adjacency matrix of the ℓ-nearest-neighbor graph (a) and that of its limit W (b) [Medvedev, '13].

- $\{w^{G_N}\}$ converges to the $\{0,1\}$-valued function $w(\xi,\zeta) = \chi_{[0,\theta]}(|\xi - \zeta|)$.

The graph limit *(or the continuum limit)*

Let $I = [0,1]$, $I_1^N := [0, \frac{1}{N})$ and $\forall i \in \{1, \ldots, N\}$, $I_i^N := [\frac{i-1}{N}, \frac{i}{N})$. Let $w : I^2 \to \mathbb{R}$ a *graphon* on I^2.

Define a sequence of weighted graphs $G_N =< \{1, \ldots, N\}, \{1, \ldots, N\}^2, \bar{w}^N >$ with:

$$\bar{w}_{ij}^N = N^2 \iint_{I_i^N \times I_j^N} w(\xi, \zeta) d\xi \, d\zeta.$$

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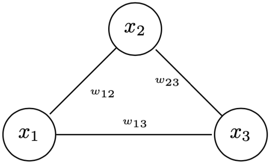
The nonlinear heat equation on G_N

$$\frac{d}{dt} x_i = \frac{1}{N} \sum_{j=1}^N (\bar{w}^N)_{ij} \phi(x_j - x_i), \quad x_i \in \mathbb{R}^d, \quad i \in \{1,\dots,N\}$$



with $w_{ij} = (\bar{w}^N)_{ij}$.

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Theorem [Medvedev, '13]: Graph Limit

If $w \in L^\infty(I)$, it holds

$$\|x - x_N\|_{C([0,T];L^2(I))} \xrightarrow[N \to +\infty]{} 0$$

where x is the solution to the integro-differential equation

$$\partial_t x(t, \xi) = \int_I w(\xi, \zeta) \, \phi(x(t, \zeta) - x(t, \xi)) d\zeta.$$

The mean-field limit

◇ **The exchangeable particle system**

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N}\phi(x_j - x_i)$$

The exchangeable mean-field limit

$$\partial_t \mu_t(x) + \nabla_x \cdot \left(\left(\int_{\mathbb{R}^d} \phi(y-x)\mu_t(dy)\right)\mu_t(x)\right) = 0$$

The mean-field limit

◇ **The non-exchangeable particle system**

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N} w_{ij}\phi(x_j - x_i)$$

The mean-field limit

◇ **The non-exchangeable particle system**

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N} w_{ij}\phi(x_j - x_i)$$

The non-exchangeable mean-field limit

$$\partial_t \mu_t^{\xi}(x) + \nabla_x \cdot \left(\left(\int_I \int_{\mathbb{R}^d} w(\xi,\zeta)\phi(y-x)\mu_t^{\zeta}(dy)d\zeta \right) \mu_t^{\xi}(x) \right) = 0$$

- Kaliuzhnyi-Verbovetskyi, Medvedev, '18
- Chiba, Medvedev, '19
- Gkogkas, Kuehn, 20
- Kuehn, Xu, 21
- Jabin, Poyato, Soler, '22
- Bet, Copini, Nardi, '23

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◇ More details and **links between the two approaches**
$\Rightarrow$ **Review paper** (A., Pouradier Duteil, '24)

The different systems/equations

- The **microscopic dynamics:**

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N} w_{ij}\phi(x_j - x_i)$$

- The **graph limit** equation:

$$\partial_t x(t,\xi) = \int_I w(\xi,\zeta)\,\phi(x(t,\zeta) - x(t,\xi))d\zeta.$$

- The **non-exchangeable mean-field limit** equation:

$$\partial_t \mu_t^{\xi}(x) + \nabla_x \cdot \left(\left(\int_I \int_{\mathbb{R}^d} w(\xi,\zeta)\phi(y-x)\mu_t^{\zeta}(dy)d\zeta \right) \mu_t^{\xi}(x) \right) = 0$$

From one system/equation to another

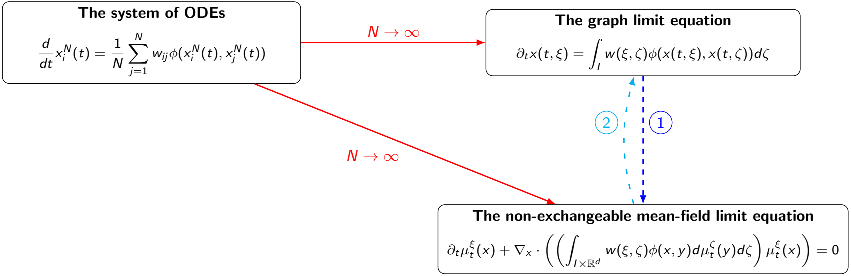


Figure: Links between the different equations.

• The red arrows corresponds to **large population limits**, respectively **graph limit** and **non-exchangeable mean-field limit**.

From graph limit to non-exchangeable limit (A., Pouradier Duteil, '24)

- Let $x(t,\xi)$ denote the **solution** to the **graph limit equation**. Let $\overline{\mu}_t$ denote a **"continuous" empirical measure** defined by

$$\overline{\mu}_t(\xi,x) = \int_I \delta_{x(t,\zeta)}(x)\delta_\zeta(\xi)d\zeta.$$

- For all test functions $f \in C^\infty(I \times \mathbb{R}^d)$,

$$\begin{aligned} & \frac{d}{dt}\int_{I\times\mathbb{R}^d} f(\xi,x)d\overline{\mu}_t(\xi,x)d\xi = \frac{d}{dt}\int_I f(\xi,x(t,\xi))d\xi \\ = & \int_I \nabla_x f(\xi,x(t,\xi)) \cdot \left(\int_I w(\xi,\zeta)\phi(x(t,\xi),x(t,\zeta))d\zeta\right) d\xi \\ = & \int_{I\times\mathbb{R}^d} \nabla_x f(\xi,x) \cdot \left(\int_{I\times\mathbb{R}^d} w(\xi,\zeta)\phi(x,y)d\overline{\mu}_t(\zeta,y)d\zeta\right) d\overline{\mu}_t(\xi,x)d\xi, \end{aligned}$$

$\Longrightarrow \overline{\mu}_t(\xi,x)$ **solution** of the **Vlasov equation**

$$\partial_t\mu_t^\xi(x) + \nabla_x \cdot \left(\left(\int_{I\times\mathbb{R}^d} w(\xi,\zeta)\phi(x,y)d\mu_t^\zeta(y)d\zeta\right)\mu_t^\xi(x)\right) = 0$$

From the non-exchangeable mean-field limit to the graph limit (d=1)

We denote

$$\bar{x}(t,\xi) := \int_{\mathbb{R}} x \, d\mu_t^{\xi}(x).$$

Then,

$$\begin{aligned} \partial_t \bar{x}(t,\xi) =& \partial_t \int_{\mathbb{R}} x \, d\mu_t^{\xi}(x) = \int_{\mathbb{R}} \partial_x(x) \left(\int_{I\times\mathbb{R}} w(\xi,\zeta)\phi(x,y) d\mu_t^{\zeta}(y) d\zeta \right) d\mu_t^{\xi}(x) \\ =& \int_{\mathbb{R}} \left(\int_{I\times\mathbb{R}} w(\xi,\zeta)\phi(x,y) d\mu_t^{\zeta}(y) d\zeta \right) d\mu_t^{\xi}(x). \end{aligned}$$

Hypothesis

We suppose that

$$\phi(x,y) = (\lambda_1 x + \lambda_2 y),$$

with $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.

Example: the original **Hegselmann-Krause** for which the interation corresponds to $(y-x)$.

We obtain

$$
\begin{aligned}
\partial_t \bar{x}(t,\xi) &= \int_{\mathbb{R}} \left(\int_{I\times\mathbb{R}} w(\xi,\zeta)(\lambda_1 x + \lambda_2 y) d\mu_t^{\zeta}(y) d\zeta \right) d\mu_t^{\xi}(x) \\
&= \int_I w(\xi,\zeta) \left(\lambda_1 \int_{\mathbb{R}} x d\mu_t^{\xi}(x) + \lambda_2 \int_{\mathbb{R}} y d\mu_t^{\zeta}(y) \right) d\zeta \\
&= \int_I w(\xi,\zeta) \left(\lambda_1 \bar{x}(t,\xi) + \lambda_2 \bar{x}(t,\zeta) \right) d\zeta \\
&= \int_I w(\xi,\zeta) \phi(\bar{x}(t,\xi), \bar{x}(t,\zeta)) d\zeta.
\end{aligned}
$$

• **Obtaining a closed equation** in the general **(nonlinear)** case: **still open** (for further comments, see Paul, Trélat, '22).

Purpose of the talk

Discussion around **three variants** of the previous model:

- **adaptive dynamical** networks,
- **random weighted** graphs,
- **higher-order** interactions.

References:

- *Mean-field and graph limits for collective dynamics models with time-varying weights*, A., Pouradier Duteil, '21,
- *Graph limit for interacting particle systems on weighted random graphs*, A., Pouradier Duteil, '23,
- *Large-population limits of non-exchangeable particle systems*, A., Pouradier Duteil, '24,
- *Mean-field limit of non-exchangeable multi-agent system over hypergraphs with unbounded rank*, A., Pouradier Duteil, Poyato, '24.

ADAPTIVE DYNAMICAL NETWORK

Adaptive dynamical network

• **Real-life interactions**: not only are **relationships influence our opinions**, but our opinions also exert a **reciprocal effect**, inducing **alterations in the network structure** of our relationships.

$\Longrightarrow$ the **connectivity of the network evolves over time** and this evolution can **depend on the states** of the system itself.

Definition

We will say that a network is **adaptive** if the **evolution of the edge** (i, j) explicitly **depends on the states of the nodes** i and j.

General form:

$$\begin{cases} \dfrac{d}{dt} x_i(t) = f_i(x_i(t), t) + \displaystyle\sum_{j=1}^{N} w_{ij}(t)\phi(x_i(t), x_j(t), t) & \text{for all } i \in \{1, \cdots, N\}, \\ \dfrac{d}{dt} w_{ij}(t) = h_{ij}(w^N(t), x^N(t), t), \end{cases}$$

where $x^N = (x_i)_{1 \leq i \leq N}$ and $w^N = (w_{ij})_{1 \leq i,j \leq N}$

Weight-varying opinion dynamics (A. Pouradier Duteil, '21)

Opinion dynamics with time-varying influence

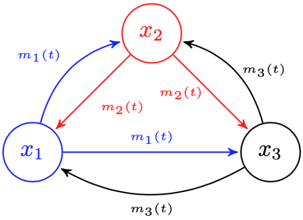
$$\begin{cases} \frac{d}{dt} x_i(t) = \frac{1}{N} \sum_{j=1}^{N} m_j(t) \phi(x_j(t) - x_i(t)) \\ \frac{d}{dt} m_i(t) = \psi_i(m(t), x(t)) \end{cases} \qquad (\mathrm{D}_N)$$

where:

- $x_i \in \mathbb{R}^d$ is the state variable (opinion, position)
- $m_i \in \mathbb{R}^+$ is the agent's weight
- $N = \sum_{i=1}^N m_i(0)$ is the (initial) total weight of the system
- ϕ is the interaction function (often, $\phi(x_j - x_i) = a(\|x_i(t) - x_j(t)\|)(x_j(t) - x_i(t)))$
- ψ_i dictate the weight dynamics. We suppose $\sum_i \psi_i \equiv 0$.

The model viewed on a graph

$$\begin{cases} \dfrac{d}{dt}x_i(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} m_j(t)\phi(x_j(t) - x_i(t)) \\ \dfrac{d}{dt}m_i(t) = \psi_i(m(t), x(t)) \end{cases}$$



- The edge weights depend on time $m_i(t)$.
- Their evolution is coupled with the evolution of the nodes $x_i(t)$.

Convergence of the microscopic system to the Graph limit equation

Theorem [A., Pouradier Duteil, '21]

Under suitable **regularity assumptions** on ϕ and ψ and a **sublinear growth bound** for ψ, then for x_N, m_N defined as, for $\xi \in [0,1]$ and $t \in [0,T]$,

$$\begin{cases} x_N(\xi,t) = P_c^N(x^N(t)) := \sum_{i=1}^N x_i^N(t)\mathbf{1}_{[\frac{i-1}{N},\frac{i}{N})}(\xi) \\ m_N(\xi,t) = P_c^N(m^N(t)) := \sum_{i=1}^N m_i^N(t)\mathbf{1}_{[\frac{i-1}{N},\frac{i}{N})}(\xi). \end{cases}$$

where (x^N, m^N) solution to the microscopic dynamics (D_N) with appropriate initial conditions, there exists $(x,m) \in \mathcal{C}([0,T]; L^\infty(I,\mathbb{R}^d)) \times \mathcal{C}([0,T]; L^\infty(I,\mathbb{R}))$ such that

$$\|x - x_N\|_{\mathcal{C}([0,T];L^2(I,\mathbb{R}^d))} \xrightarrow[N\to+\infty]{} 0 \quad \text{and} \quad \|m - m_N\|_{\mathcal{C}([0,T];L^2(I,\mathbb{R}))} \xrightarrow[N\to+\infty]{} 0.$$

Moreover, the limit functions x and m are solutions to **the graph limit equation**

$$\begin{cases} \partial_t x(\xi,t) = \int_I m(\zeta,t)\phi(x(\xi,t)-x(\zeta,t))d\zeta; & x(\cdot,0) = x_0 \\ \partial_t m(\xi,t) = \psi(\xi, x(\cdot,t), m(\cdot,t)); & m(\cdot,0) = m_0. \end{cases} \tag{GL}$$

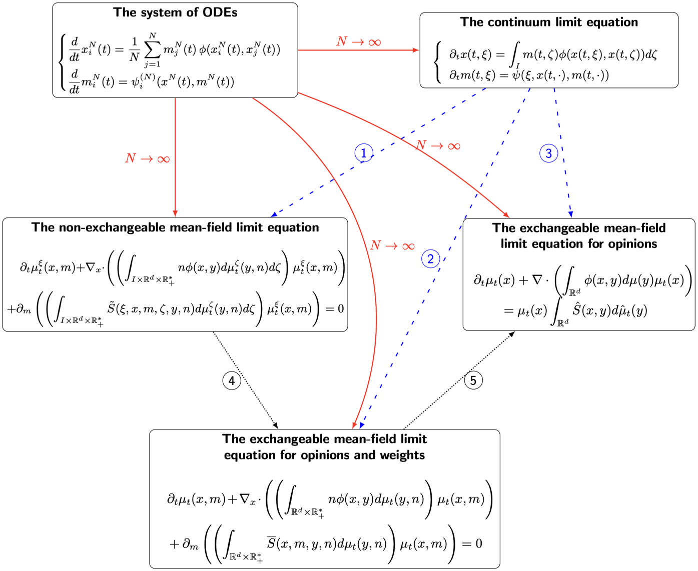


Figure: Links between the different equations (A., Pouradier-Duteil, '24)

Other results

- The setting of **Kuramoto-type model** (Gkogkas, Kuehn, Xu, '23)

$$\begin{cases} \dfrac{d}{dt}x_i = \omega_i(x_i, t) + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} w_{ij}\phi\left(x_i, x_j\right) \qquad \text{for all } i \in \{1, \cdots, N\} \\ \dfrac{d}{dt}w_{ij} = -\varepsilon\left(w_{ij} + H(x_i, x_j)\right) \end{cases}$$

- Generalization of the **evolving-weight dynamics** (Throm, '23)

$$\begin{cases} \dfrac{d}{dt}x_i = \omega_i(x, t) + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} w_{ij}\phi\left(x_i, x_j\right) \qquad \text{for all } i \in \{1, \cdots, N\} \\ \dfrac{d}{dt}w_{ij} = \psi_{ij}^{(N)}(x(t), w(t)) \end{cases} \tag{1}$$

WEIGHTED RANDOM GRAPHS

About random graphs

- **Random graph:** a graph which is generated by a random process.

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- **Example 1:** **Erdos-Rényi graph**: the edge between a pair of distinct nodes is inserted with probability p.

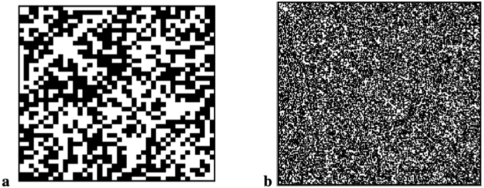

Figure: Pixel pictures of the Erdos-Rényi graph with $N = 40$ and $p = 0.5$ (left), $N = 600$ and $p = 0.5$ (right) [Medvedev, 2014]

About random graphs

- **Random graph:** a graph which is generated by a random process.

- **Example 2 : Small world graph**: replacing a random set of the local connections by randomly chosen long-range ones.

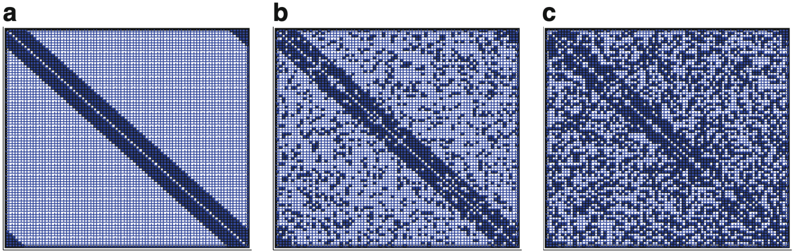

Figure: Pixel pictures of the Small world graph, p starts at 0 and increases from left to right [Medvedev, 2014]

Dynamical systems on W-random graph

• Let $\overline{\xi} = (\xi_1, \xi_2, \xi_3, \dots)$ and $\overline{\xi}^N = (\xi_1, \xi_2, \dots, \xi_N)$ where $\xi_i, i \in \mathbb{N}$ are i.i.d. random variables with $\mathcal{L}(\xi_1) = \mathcal{U}(I)$.

Definition [Medvedev, '14]

A **W-random graph** on N nodes generated by the random sequence $\overline{\xi}$, denoted $G_N = \mathbb{G}(\overline{\xi}_N, W)$ is such that the edges of G_N are **selected at random** and

$$\mathbb{P}((i,j) \in E(G_N)) = W(\xi_i, \xi_j) \text{ for each } (i,j) \in \{1, \dots, N\}^2 \text{ for } i \neq j.$$

The decision wether to include a pair $(i,j) \in \{1, \dots, N\}^2$ is made **independently** as for the decisions of other pairs.

Dynamical systems on W-random graph

$$\frac{d}{dt} x_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \sigma_{ij} \phi(x_j^N(t) - x_i^N(t))$$

with $\mathcal{L}(\sigma_{ij} | \overline{\xi}) = \mathcal{B}(W(\xi_i, \xi_j))$.

Random graph limit

Dynamical systems on W-random graph

$$\frac{d}{dt}x_i^N(t) = \frac{1}{N}\sum_{j=1}^{N}\sigma_{ij}\phi(x_j^N(t) - x_i^N(t)) \qquad (\tilde{S}_N^{\mathrm{r-r}})$$

with $\mathcal{L}(\sigma_{ij}|\overline{\xi}) = \mathcal{B}(W(\xi_i, \xi_j))$.

Random graph limit

Dynamical systems on W-random graph

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with $\mathcal{L}(\sigma_{ij}|\bar{\xi}) = \mathcal{B}(W(\xi_i, \xi_j))$.

Medvedev obtains the **convergence** to

The random graph limit equation

$$\partial_t x(\xi, t) = \int_I W(\xi, \zeta)\phi(x(\zeta, t) - x(\xi, t))d\zeta. \qquad (C)$$

Random graph limit

Theorem [Medvedev, '14]: Random Graph Limit

Suppose $W \in \mathcal{W}_0$, a class of symmetric measurable function on I^2 with values on I. ϕ is a **Lipschitz continuous function** on $\mathbb{R}$ and $g \in L^\infty(I)$. Let $T > 0$ and suppose that the solution of (C) $x(\xi, \zeta)$ satisfies the following inequality

$$\min_{t\in[0,T]} \int_I \left\{ \int_I W(\xi,\zeta)\phi(x(\zeta,t) - x(\xi,t))^2 d\zeta - \left(\int_I W(\xi,\zeta)\phi(x(\zeta,t) - x(\xi,t) d\zeta \right)^2 \right\} \geq c_1$$

for some positive constant c_1. Then, the solution of $(\tilde{S}_N^{\mathrm{r-r}})$ and (C) satisfy the following relation

$$\lim_{N\to+\infty} \mathbb{P}\{N^{1/2} \sup_{t\in[0,T]} \|x^N(t) - \mathbf{P}_{\bar{\xi}^N} x(\xi,t)\|_{2,N} \leq C\} = 1$$

for some constant $C > 0$ with $\mathbf{P}_{\bar{\xi}^N} x(\xi,t) = (x(\xi_1^N,t), x(\xi_2^N,t), \ldots, x(\xi_N^N,t))$ and

$$(x,y)_N := \frac{1}{N}\sum_{i=1}^N x_i y_i, \text{ and the corresponding norm } \|x\|_{2,N} := \sqrt{(x,x)_N}.$$

Weighted random graph

Example [Garlaschelli, '09]

A **weighted random graph** model in which the **probability of drawing an edge** of discrete weight $w \in \mathbb{N}$ between vertices i and j is given by

$$\mathbb{P}(\sigma_{ij}^N = w) = q_{ij}(w) = p^w(1-p).$$

Lack of a general framework !

Definition [A., Pouradier Duteil, '23]

A **q-weighted random graph** on N nodes generated by the random sequence $\bar{\xi}$, denoted G_N, is such that the weight of an edge of G_N is randomly attributed. More precisely, the **law for the weight of the edge** (i,j) is $q(\xi_i, \xi_j, .)$ where

$$\begin{array}{rccc} q: & I \times I & \rightarrow & \mathcal{P}(\mathbb{R}_+) \\ & (\xi, \zeta) & \mapsto & q(\xi, \zeta; .). \end{array}$$

The decision of the attribution of the weight of a pair $(i,j) \in \{1, \ldots, N\}^2$ is made **independently** from the decision for other pairs.

Examples

- **W-random graph** (Medvedev, '14): **Generate** between any two nodes (ξ, ζ) an edge (of weight 1) **with probability** $\mathbf{W}(\xi, \zeta)$.

$$q(\xi, \zeta; \cdot) = (1 - W(\xi, \zeta))\delta_0 + W(\xi, \zeta)\delta_1, \qquad \text{for all } \xi, \zeta \in \mathbb{R}.$$

- **Erdös-Rényi weighted random graph** (Garlaschelli, 09): **Generate** between any two nodes **an edge with weight w** $\in \mathbb{N}$, with probability $p^w(1 - p)$.

$$q(\xi, \zeta; \cdot) = (1 - p) \sum_{i=0}^{+\infty} p^i \delta_i, \qquad \text{for all } \xi, \zeta \in \mathbb{R}.$$

Weighted random graph limit

- Let $\overline{\xi} = (\xi_1, \xi_2, \xi_3, \dots)$ and $\overline{\xi}^N = (\xi_1, \xi_2, \dots, \xi_N)$ where $\xi_i, i \in \mathbb{N}$ are i.i.d. random variables with $\mathcal{L}(\xi_1) = \mathcal{U}(I)$.

Dynamical systems on q-weighted random graph

$$\begin{cases} \frac{d}{dt} x_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} \sigma_{ij} \phi(x_j^N(t) - x_i^N(t)), \\ x_i^N(0) = g(\xi_i^N), \qquad i \in \{1, \dots, N\} \end{cases} \tag{$S_N^{\mathrm{r-r}}$}$$

with $\mathcal{L}(\sigma_{ij} | \overline{\xi}) = q(\xi_i, \xi_j; \cdot)$.

We prove the convergence towards the continuum limit

The weighted random graph limit equation

$$\begin{cases} \partial_t x(\xi, t) = \int_I \left(\int_{\mathbb{R}_+} w q(\xi, \zeta; dw) \right) \phi(x(\zeta, t) - x(\xi, t)) d\zeta \\ x(\xi, 0) = g(\xi), \quad \xi \in I, \end{cases} \tag{C_2}$$

Our result

Hypothesis 1

Let $\phi \in L^\infty(\mathbb{R})$ be bounded and Lipschitz continuous, with $\|\phi\|_{\mathrm{Lip}} := L$ and $\|\phi\|_{L^\infty(\mathbb{R})} := K$.

Hypothesis 2

There exists $M > 0$ such that for all $(\xi, \zeta) \in I^2$, for all $k \in \{1, \cdots, 4\}$,

$$\left(\int_{\mathbb{R}_+} w^k q(\xi, \zeta; dw) \right)^{1/k} \leq M,$$

i.e. the first four moments of the probability measure $q(\xi, \zeta; \cdot)$ are bounded uniformly in ξ and ζ.

Our result

Theorem [A., Pouradier Duteil, 2023]: Weighted Random Graph Limit

Let ϕ satisfy Hypothesis 1, let $g \in L^\infty(I)$ and let q be a weighted random graph law satisfying Hypothesis 2. Then, as N goes to infinity, **solution x^N to the discrete system** $(S_N^{\mathrm{r-r}})$ **converges** to the **solution x of the continuous model** (C_2). More precisely,

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|x^N(t) - \mathbf{P}_{\overline{\xi}^N} x(\cdot,t)\|_{2,N} \geq \frac{C_1(T)}{\sqrt{N}}\right] \leq \frac{\tilde{C}_1}{N}$$

where the constants $C_1(T)$ and $\tilde{C}_1$ are respectively defined by
$C_1(T) := \sqrt{T}\sqrt{1+M^2K^2}e^{(\frac{1}{2}+4ML)T}$ and $\tilde{C}_1 := 3M^4K^4 + 6$.

Numerical Illustration: the weighted Erdös-Rényi random graph

- **Erdös-Rényi weighted random graph** (Garlaschelli, 09): **Generate** between any two nodes **an edge with weight w** $\in \mathbb{N}$, with probability $p^w(1-p)$.

$$q(\xi,\zeta;\cdot) = (1-p)\sum_{i=0}^{+\infty} p^i \delta_i, \qquad \text{for all } \xi,\zeta \in \mathbb{R}.$$

- **First moment** given by:

$$\bar{w}(\xi,\zeta) = \int_{\mathbb{R}^+} wq(\xi,\zeta;dw) = (1-p)\sum_{i=1}^{+\infty} ip^i = \frac{p}{1-p}$$

- **Limit equation**:

$$\begin{cases} \partial_t x(\xi,t) = \frac{p}{1-p}\int_I \phi(u(\zeta,t)-u(\xi,t))d\zeta \\ x(\xi,0) = g(\xi), \quad \xi \in I. \end{cases}$$

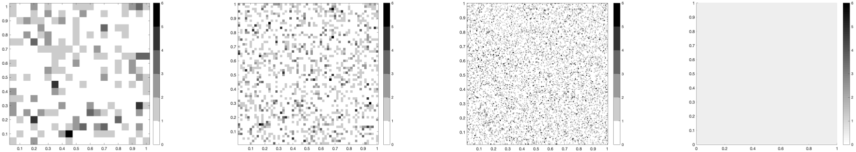

Figure: Left and Centers: Random interaction matrices generated by deterministic sequences for $N = 20$, $N = 60$ and $N = 150$, for the random weighted graphon (30), Right: Corresponding graphon.

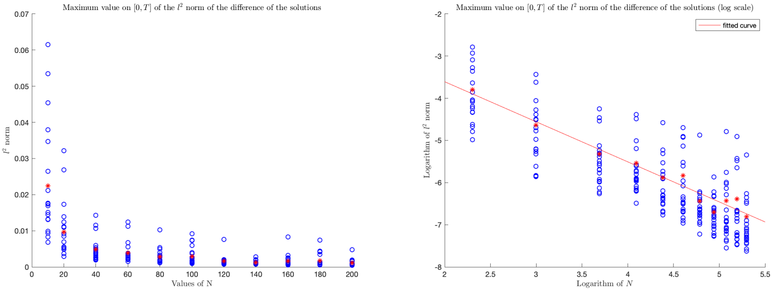

Figure: Convergence of $\sup_{t\in[0,T]} \|x^N(t) - \mathbf{P}_{\overline{\xi}^N} x(\cdot,t)\|_{2,N}$ for different values of N, with 20 runs for each value of N.

Numerical Illustration: Weighted “Small World” network

- **Model for a “small-world” network** (Watts, Strogatz, ’98): **Connect each node** with its k **closest neighbors** to form a ring lattice. Then, **rewire each edge at random** with probability p.
- **Refined model for a weighted “small-world” network**: **Connect two nodes** with an edge of **weight 1** if they are among each other’s closest k neighbors, i.e. if $|\xi_i - \xi_j| \leq r$, where $r := \frac{k}{2N}$. Then, with probability $p = \frac{|\xi_i - \xi_j|}{r}$, **rewire each edge** at random, giving the new edge a **weight drawn uniformly** in the interval $[0, 1]$.

$$q(\xi, \zeta; dw) = \begin{cases} \frac{\rho(\xi,\zeta)}{r} d\lambda_{[0,1]} + (1 - \frac{\rho(\xi,\zeta)}{r})\delta_1 & \text{if } \rho(\xi - \zeta) \leq r \\ d\lambda_{[0,1]} & \text{otherwise} \end{cases} \quad (2)$$

where $\rho(\xi, \zeta) = \min\{|\xi - \zeta|, |\xi - \zeta - 1|, |\zeta - \xi - 1|\}$.

- **First moment**:

$$\bar{w}(\xi, \zeta) = \int_{\mathbb{R}^+} wq(\xi - \zeta; dw) = \begin{cases} (1 - \frac{\rho(\xi-\zeta)}{2r}) & \text{if } \rho(\xi - \zeta) \leq r \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

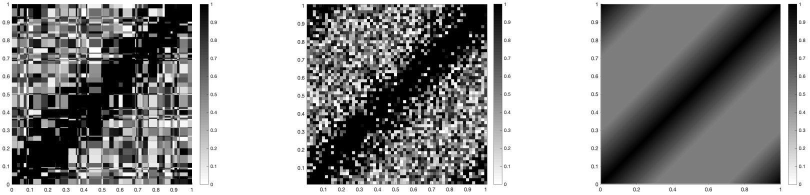

Figure: Values of the random interaction matrices generated from a random sequence (left) and a deterministic sequence (right) according to the random weighted graph law (2) for $N = 60$. Right: Corresponding continuous graphon $(\xi, \zeta) \mapsto \bar{w}(\xi, \zeta)$.

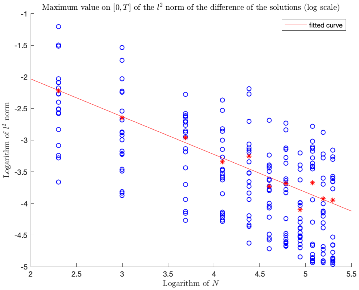

Figure: Convergence of $\sup_{t\in[0,T]} \|x^N(t) - \mathbf{P}_{\overline{\xi}^N} x(\cdot,t)\|_{2,N}$ for different values of N, with 20 runs for each value of N. Case of the random weighted graph law (30).

HIGHER ORDER INTERACTIONS

Hypergraphs

• Many **existing models** focus on **binary interactions** $\neq$ **real-life dynamics** often involve **interactions** within groups containing **more than just two individuals** (virtual group chats, physical meetings . . .)

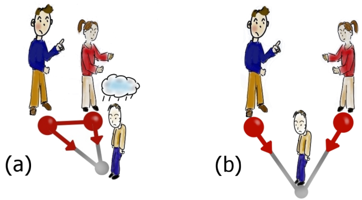

Figure: Higher-order group interactions in social context [Neuhauser et al, 2022]

Hypergraphs

- Hypergraph $H = (V, E)$ where V are the vertices, E the hyperedges.

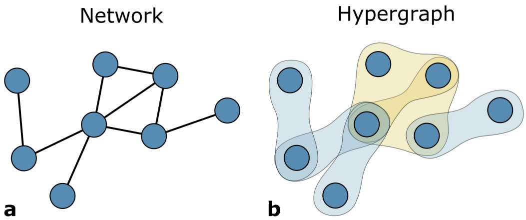

Figure: Pairwise and higher-order interactions [Battiston et al, 2021]

Models of multi-agent dynamics on hypergraphs

- Extension of the **Kuramoto-Saraguchi** model on hypergraphs (Skardal, Arenas, '20)

$$\frac{d}{dt}x_i = \sum_{j_1=1}^{N} w_{ij_1}^{N,1}\sin(x_{j_1} - x_i) + \sum_{j_1=1}^{N}\sum_{j_2=1}^{N} w_{ij_1j_2}^{N,2}\sin(2x_{j_1} - x_{j_2} - x_i)$$
$$+ \sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\sum_{j_3=1}^{N} w_{ij_1j_2j_3}^{N,3}\sin(x_{j_1} + x_{j_2} - x_{j_3} - x_i)$$

- **Higher-order opinion dynamics** on a uniform hypergraph of rank 2 (Neuhauser, Lambiotte, Schaub '22)

$$\frac{d}{dt}x_i = \sum_{j_1=1}^{N}\sum_{j_2=1}^{N} w_{ij_1j_2}^{N,2} e^{\lambda|x_{j_1} - x_{j_2}|}\left(\frac{x_{j_1} + x_{j_2}}{2} - x_i\right).$$

About Graph Theory

- **Graphons** are **natural limit objects** associated to a sequence of (dense) graphs.

Graphon space

Given $W > 0$,

$$\mathcal{G}_W := \{w \in L^\infty_+([0,1]^2) : \|w\|_{L^\infty} \leq W, \text{ and } w \text{ is symmetric}\}.$$

Cut-distance

For any two graphons $w, \bar{w} \in \mathcal{G}_W$,

labelled cut-distance: $$d_\square(w, \bar{w}) := \sup_{S,T \subset [0,1]} \left| \iint_{S \times T} (w(\xi,\zeta) - \bar{w}(\xi,\zeta))\, d\xi\, d\zeta \right|,$$

(unlabelled) cut-distance: $$\delta_\square(w, \bar{w}) = \inf_\Phi d_\square(w, \bar{w}^\Phi),$$

with $\Phi : [0,1] \longrightarrow [0,1]$ bijective, measure-preserving and $\bar{w}^\Phi(\xi,\zeta) = \bar{w}(\Phi(\xi), \Phi(\zeta))$.

- **Comparison** with the **L^1-norm**: for all $w, \bar{w} \in \mathcal{G}_W$,

$$d_\square(w, \bar{w}) \leq \|w - \bar{w}\|_{L^1}.$$

Hypergraph theory

Hypergraphons of unbounded rank (UR-hypergraphons)

Given $W > 0$, $\mathcal{H}_W := \left\{ w = (w_\ell)_{\ell \in \mathbb{N}} : \begin{array}{l} w_\ell \in L^\infty_+([0,1]^{\ell+1}),\ \|w_\ell\|_{L^\infty} \le W, \\ \text{and } w_\ell \text{ is symmetric for all } \ell \in \mathbb{N} \end{array} \right\}.$

Cut-distance

For any $w, \bar{w} \in \mathcal{H}_W$, $\forall \ell \in \mathbb{N}$, **the ℓt-th order labeled cut distance** is

$$d_{\square,\ell}(w_\ell, \bar{w}_\ell) := \sup_{S, S_1, \dots, S_\ell \subset [0,1]} \left| \int_{S \times S_1 \times \dots \times S_\ell} (w_\ell - \bar{w}_\ell)\, d\xi\, d\xi_1 \dots d\xi_\ell \right|, .$$

For any strictly positive summable sequence $(\alpha_\ell)_{\ell \in \mathbb{N}}$, we define

the labeled cut distance: $d_\square(w, \bar{w}; (\alpha_\ell)_{\ell \in \mathbb{N}}) := \sum_{\ell=1}^{\infty} \alpha_\ell\, d_{\square,\ell}(w_\ell, \bar{w}_\ell)$,

the unlabeled cut distance: $\delta_\square(w, \bar{w}; (\alpha_\ell)_{\ell \in \mathbb{N}}) = \inf_\Phi d_\square(w, \bar{w}^\Phi; (\alpha_\ell)_{\ell \in \mathbb{N}})$,

where Φ bijective, measure-preserving maps $\Phi : [0,1] \longrightarrow [0,1]$, and

$$\bar{w}_\ell^\Phi(\xi, \xi_1, \dots, \xi_\ell) = \bar{w}_\ell(\Phi(\xi), \Phi(\xi_1), \dots, \Phi(\xi_\ell)).$$

Convergence of a sequence of hypergraphs

Construction of piecewise-constant function associated to a sequence of hypergraphs

For any sequence of hypergraphs $(H_N)_{N\in\mathbb{N}}$, for all $\ell \in \mathbb{N}$,

- if $\ell \leq N-1$, for all $(\xi, \xi_1, \cdots, \xi_\ell) \in [0,1]^{\ell+1}$,

$$w_\ell^{H_N}(\xi, \xi_1, \cdots, \xi_\ell) := \sum_{i, j_1, \cdots j_\ell = 1}^{N} w_{ij_1\cdots j_\ell}^{\ell, N} N^\ell \mathbb{1}_{I_i^N \times I_{j_1}^N \times \cdots I_{j_\ell}^N}(\xi, \xi_1, \cdots, \xi_\ell),$$

- if $\ell \geq N$, for all $(\xi, \xi_1, \cdots, \xi_\ell) \in [0,1]^{\ell+1}$, $w_\ell^{H_N}(\xi, \xi_1, \cdots, \xi_\ell) = 0$.

Convergence

The sequence of hypergraphs $(H_N)_{N\in\mathbb{N}}$ is said to **converge to a UR-hypergraphon** w when

$$\lim_{N\to\infty} \delta_\square(w, w^{H_N}; (\alpha_\ell)_{\ell\in\mathbb{N}}) = 0$$

for some positive and summable sequence $(\alpha_\ell)_{\ell\in\mathbb{N}}$.

The θ-nearest neighbor example

The hypergraph: for $\theta \in (0,1]$, for each $\ell \in \{1, \cdots, N-1\}$,

$$w^{\ell,N}_{ij_1\cdots j_\ell} = \begin{cases} 1 & \text{if } \max\limits_{k_1,k_2 \in \{i,j_1\ldots,j_\ell\}} |k_1 - k_2| \le \theta N, \\ 0 & \text{otherwise.} \end{cases}$$

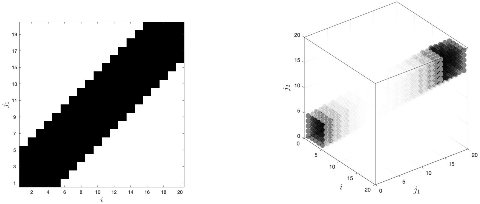

Figure: Pixel representation for $\ell = 1, 2$ with $\theta = 0.3$ and $N = 20$.

The **unbounded rank hypergraphon**: for all $\ell \in \mathbb{N}$,

$$w_\ell(\xi_0, \xi_1, \cdots, \xi_\ell) = \begin{cases} 1 & \text{if } \max\limits_{i,j \in \{0,\cdots,\ell\}} |\xi_i - \xi_j| \le \theta \\ 0 & \text{otherwise.} \end{cases}$$

Non-exchangeable mean-field limit for higher order case

The microscopic dynamics

$$\begin{cases} \dfrac{dX_i^N(t)}{dt} = \displaystyle\sum_{\ell=1}^{N-1} \sum_{j_1,\dots,j_\ell=1}^{N} w_{ij_1\cdots j_\ell}^{\ell,N}\, K_\ell(X_i^N(t), X_{j_1}^N(t), \dots, X_{j_\ell}^N(t)), \\ X_i^N(0) = X_{i,0}^N, \qquad i \in \{1, \cdots, N\}. \end{cases}$$

The non-exchangeable mean-field limit

$$\begin{cases} \partial_t \mu_t^\xi + \mathrm{div}_x(F_w[\mu_t](\cdot,\xi)\,\mu_t^\xi) = 0, \quad t \geq 0,\, x \in \mathbb{R}^d,\, \xi \in [0,1], \\ \mu_{t=0}^\xi = \mu_0^\xi. \end{cases}$$

where

$$F_w[\mu_t](x,\xi) := \sum_{\ell=1}^{\infty} \int_{[0,1]^\ell} w_\ell(\xi,\xi_1,\dots,\xi_\ell) \times \left(\int_{\mathbb{R}^{d\ell}} K_\ell(x,x_1,\dots,x_\ell)\, d\mu_t^{\xi_1}(x_1) \cdots d\mu_t^{\xi_\ell}(x_\ell) \right) d\xi_1, \dots\, d\xi_\ell.$$

Our main result

Theorem [A., Pouradier Duteil, Poyato, '24]

Assume that **the kernels K_ℓ** satisfy some **regularity assumptions** and the **coupling weights** some **suitable scaling**. Suppose additionnally that both of them satisfy **some symmetries**.

For any $(X_{1,0}^N, \ldots, X_{N,0}^N)$ with *i.i.d.* $X_{i,0}^N$ (but N dependent law) such that there exists $p \in [1,2]$ for which $X_{i,0}^N$ satisfies

$$\sup_{N\in\mathbb{N}} \max_{1\leq i\leq N} \mathbb{E}|X_{i,0}^N|^p < \infty,$$

consider the unique solutions $(X_1^N, \ldots, X_N^N)$ to the microscopic dynamics. Then, there is a subsequence $N_k \to \infty$ such that **the mean-field limit of the multi-agent system** is characterized in a suitable sense by a **solution to the Vlasov-type equation** for some $(\mu_t^\xi)_{\xi\in[0,1]} \subset \mathcal{P}(\mathbb{R}^d)$ and some $w = (w_\ell)_{\ell\in\mathbb{N}}$ such that $\sup_{\ell\in\mathbb{N}} \|w_\ell\|_{L^\infty} \leq W$.

Strategy of the proof

Intermediate Particle Systems

$$\begin{cases} \dfrac{d\bar{X}_i^N}{dt} = \displaystyle\sum_{\ell=1}^{N-1} \sum_{j_1,\dots,j_\ell=1}^{N} w_{ij_1\cdots j_\ell}^{\ell,N} \, \mathbb{E}_i^N K_\ell(\bar{X}_i^N, \bar{X}_{j_1}^N, \dots, \bar{X}_{j_\ell}^N), \\ \bar{X}_i^N(0) = X_{i,0}^N, \end{cases}$$

where $\mathbb{E}_i^N = \mathbb{E}[\,\cdot\,|\bar{\mathcal{F}}_i^N]$ denotes the expectation conditioned to the natural filtration

$$\bar{\mathcal{F}}_i^N(t) = \sigma(\{\bar{X}_i^N(s) : 0 \le s \le t\}).$$

Error estimate

$$\left(\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}|X_i^N(t) - \bar{X}_i^N(t)|^p\right)^{1/p} \le e^{(\tilde{C}_\infty^N + C_p^N)t} \varepsilon_p^N,$$

with

$$\tilde{C}_\infty^N \le W \sum_{\ell=1}^{\infty} L_\ell, \quad C_p^N \le W \sum_{\ell=1}^{\infty} \ell L_\ell, \quad \varepsilon_p^N \le 2W \sum_{\ell=1}^{\infty} \frac{\sqrt{\ell!}\, B_\ell}{N^{\ell/2}}.$$

Associated PDE system

We denote their associated laws

$$\bar{\lambda}_t^{N,i} := \mathrm{Law}(\bar{X}_i^N(t)), \quad t \geq 0,\, 1 \leq i \leq N.$$

Solution of the PDE system

Then, $(\bar{\lambda}^{N,i})_{1 \leq i \leq N}$ is a **solution in distributional sense** to the following coupled PDE system

$$\begin{cases} \partial_t \bar{\lambda}_t^{N,i} + \mathrm{div}_x (F_i^N[\bar{\lambda}_t^{N,1}, \cdots, \bar{\lambda}_t^{N,N}]\, \bar{\lambda}_t^{N,i}) = 0, \quad t \geq 0,\, x \in \mathbb{R}^d,\, 1 \leq i \leq N, \\ \bar{\lambda}_0^{N,i} = \mathrm{Law}(X_{i,0}^N), \end{cases}$$

where

$$F_i^N[\bar{\lambda}_t^{N,1}, \cdots, \bar{\lambda}_t^{N,N}](x) = \sum_{\ell=1}^{N-1} \sum_{j_1, \cdots, j_\ell = 1}^{N} w_{ij_1 \cdots j_\ell}^{\ell,N} \int_{\mathbb{R}^{d\ell}} K_\ell(x, x_1, \ldots, x_\ell)\, d\bar{\lambda}_t^{N,1}(x_1) \cdots d\bar{\lambda}_t^{N,N}(x_N).$$

Graphon reformulation

For every $N \in \mathbb{N}$, and $t \in \mathbb{R}_+$ we define

$$\mu_t^{N,\xi} := \sum_{i=1}^N \mathbb{1}_{I_i^N}(\xi)\delta_{X_i^N(t)}, \quad \bar{\mu}_t^{N,\xi} := \sum_{i=1}^N \mathbb{1}_{I_i^N}(\xi)\,\bar{\lambda}_t^{N,i}, \quad \xi \in [0,1],$$

$$w_\ell^N(\xi,\xi_1,\dots,\xi_\ell) := \sum_{i,j_1,\dots,j_\ell=1}^N \mathbb{1}_{I_i^N \times I_{j_1}^N \times \dots \times I_{j_\ell}^N}(\xi,\xi_1,\dots,\xi_\ell)\, N^\ell\, w_{ij_1\cdots j_\ell}^{\ell,N}, \quad \xi,\xi_1,\cdots,\xi_\ell \in [0,1],$$

for all $1 \leq \ell \leq N-1$, $w_\ell^N \equiv 0$ for all $\ell \geq N$.

Lemma

Under the previous assumptions, consider the **unique solution** $(\bar{X}_1^N,\dots,\bar{X}_N^N)$ to the **intermediate particle system**, their **associated laws** $(\bar{\lambda}^{N,i})_{1\leq i\leq N}$ and the **graphon reformulation** $(\bar{\mu}^N, w^N)$. Then, $\bar{\mu}^N$ is a **distributional solution to the Vlasov equation with hypergraphon** $w^N = (w_\ell^N)_{\ell\in\mathbb{N}}$ and initial datum $\bar{\mu}_{t=0}^{N,\xi} = \sum_{i=1}^N \mathbb{1}_{I_i^N}(\xi)\,\mathrm{Law}(X_{i,0})$.

Functional setting

Fibered probability measures

Consider any $\nu \in \mathcal{P}([0,1])$. We define the **space of fibered probability measures** by

$$\mathcal{P}_\nu(\mathbb{R}^d \times [0,1]) := \{\mu \in \mathcal{P}(\mathbb{R}^d \times [0,1]) : \pi_{\xi\#}\mu = \nu\},$$

where $\pi_\xi(x,\xi) = \xi$ projection on the second component, and then$\pi_{\xi\#}\mu$ stands for the marginal of μ in the second component.

Consider any $\nu \in \mathcal{P}([0,1])$ and any $p \in [1,\infty]$, we define

$$\mathcal{P}_{p,\nu}(\mathbb{R}^d \times [0,1]) := \left\{\mu \in \mathcal{P}_\nu(\mathbb{R}^d \times [0,1]) : \int_0^1 d_{\mathrm{BL}}^p(\mu^\xi, \delta_0)\, d\nu(\xi) < \infty\right\},$$

$$d_{p,\nu}(\mu_1,\mu_2) := \left(\int_0^1 d_{\mathrm{BL}}^p(\mu_1^\xi, \mu_2^\xi)\, d\nu(\xi)\right)^{1/p}, \quad \mu_1,\mu_2 \in \mathcal{P}_{p,\nu}(\mathbb{R}^d \times [0,1]).$$

Stability estimates for the Vlasov equation

Theorem

For any initial data $\mu_0, \bar{\mu}_0 \in \mathcal{P}_{p,\nu}(\mathbb{R}^d \times [0,1])$ with $p \in [1, \infty)$, let $\mu, \bar{\mu} \in C(\mathbb{R}_+, \mathcal{P}_{p,\nu}(\mathbb{R}^d \times [0,1]))$ be the **unique global-in-time distributional solutions** issued at μ_0 with given w (respectively, $\bar{\mu}_0$ and $\bar{w}$). Then, we have

$$d_{p,\nu}(\mu_t, \bar{\mu}_t) \leq e^{(C_p + L_F)t} \left(d_{p,\nu}(\mu_0, \bar{\mu}_0) + \frac{D_\infty^{1/q}}{L_F} \delta_\square(w, \bar{w}; (4^\ell \|\hat{K}_\ell\|_{L^1})_{\ell \in \mathbb{N}})^{1/p} \right),$$

for every $t \geq 0$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Figure: Social graph (http://inicia.org.ar/blog/7-claves-para-hacer-networking/)

Thank you for your attention !