École Polytechnique Fédérale de Lausanne

“Uniformity norms and Hindman’s conjecture”

By Florian K. Richter

Additive Combinatorics Conference
ICMS, Bayes Centre, Edinburgh

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Hindman’s Conjecture

Formulation

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. In [Hin79] Hindman posed the following conjecture.

Hindman's Conjecture

For any finite coloring of $\mathbb{N}$ there exists a monochromatic set of the form $\{x, y, x+y, xy\}$ for some $x, y \in \mathbb{N}$.

[Hin79] Neil Hindman, *Partitions and sums and products of integers*, Transactions of the American Mathematical Society **247** (1979), 227–245.

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Hindman's conjecture postdates a stronger conjecture by Erdős [Erd77], which was disproved by Hindman. However, the following is still standing.

[Erd77] Paul Erdős. *Problems and results on combinatorial number theory III*, Number theory day (Proc. Conf., Rockefeller Univ., New York, 1976), Lecture Notes in Math., **Vol. 626**, Springer, Berlin, 1977.

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Conjecture (Graham-Rothschild-Spencer)

For any $N \in \mathbb{N}$ and any finite coloring of $\mathbb{N}$ there exist $x_1 < \dots < x_N \in \mathbb{N}$ such that all finite sums and products formed from the x_i are monochromatic.

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Progress in $\mathbb{N}$

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Any 2-coloring of $\mathbb{N}$ contains many monochromatic sets of the form $\{x, y, x+y, xy\}$.

[Bow22] Matt Bowen, *Monochromatic sums and products in 2-colorings of* $\mathbb{N}$, arXiv: 2205.12921 (2022).

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Difficulty: The affine semigroup $(\mathbb{N}, +, \cdot)$ is not amenable, obstructing analytical approaches (e.g. from Fourier analysis or ergodic theory).

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Theorem (Green-Sanders, 2016)

For any $r \in \mathbb{N}$ there is a cofinite set of primes p such that for any r-coloring of $\mathbb{F}_p$ there exists a monochromatic set of the form $\{x, y, x + y, xy\}$.

[GS16] Ben Green and Tom Sanders, *Monochromatic sums and products*, Discrete Analysis **613** (2016).

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Any finite coloring of $\mathbb{Q}$ admits a monochromatic pattern $\{x, y, x + y, xy\}$.

[BS22] Matt Bowen and Marcin Sabok, *Monochromatic products and sums in the rationals*, arXiv:**2210.12290v1**.

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Theorem (Alweiss, 2023)

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[Alw23] Ryan Alweiss, *Monochromatic Sums and Products over $\mathbb{Q}$*, arXiv:**2307.08901** (2023).

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For any $\delta > 0$ there is a cofinite set of primes p such that any set $A \subset \mathbb{F}_p$ of relative density $\geqslant \delta$ contains $\{x, x+y, xy\}$.

[Shk10] I. D. Shkredov. *On monochromatic solutions of some nonlinear equations in $\mathbb{Z}/p\mathbb{Z}$*, Mat. Zametki, 88(**4**):625–634, 2010.

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Theorem (Bergelson-Moreira, 2015)

Let $(K, +, \cdot)$ be any infinite countable field. Any set $A \subset K$ of positive density (with respect to a double-Følner sequence) contains $\{x+y, xy\}$.

[BM15] V. Bergelson and J. Moreira. *Ergodic theorem involving additive and multiplicative groups of a field and $\{x+y, xy\}$ patterns*, ETDS, 37(**3**), 2015.

[Shk10] I. D. Shkredov. *On monochromatic solutions of some nonlinear equations in $\mathbb{Z}/p\mathbb{Z}$*, Mat. Zametki, 88(**4**):625–634, 2010.

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Any set with positive natural density contains $\{x+y, xy+1\}$.

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Theorem

There exists a (multiplicatively invariant) density on $\mathbb{N}$ such that any set with positive measure under this density contains $\{x+y, xy\}$.

Densities on $\mathbb{N}$

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A **density on $\mathbb{N}$** is a function $D\colon 2^{\mathbb{N}} \to [0, 1]$ such that

- **Unit Range:** $D(\emptyset) = 0$ and $D(\mathbb{N}) = 1$
- **Monotonicity:** if $A \subset B$ then $D(A) \leqslant D(B)$
- **Subadditivity:** for all $A, B \subset \mathbb{N}$ one has $D(A \cup B) \leqslant D(A) + D(B)$

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We say that D is **multiplicatively invariant** if $D(A/m) = D(A)$, where

$$A/m = \{n \in \mathbb{N} : nm \in A\}.$$

Densities on $\mathbb{N}$

Examples I

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The **Cesàro average** and the **logarithmic average** of $f\colon \mathbb{N} \to \mathbb{C}$ over a finite set $A \subset \mathbb{N}$ are defined respectively as

$$\mathbb{E}_{n\in A}\, f(n) = \frac{1}{|A|}\sum_{n\in A} f(n) \qquad \text{and} \qquad \mathbb{E}^{\log}_{n\in A}\, f(n) = \frac{\sum_{n\in A}\frac{f(n)}{n}}{\sum_{n\in A}\frac{1}{n}}.$$

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- The **upper logarithmic density** of a set $A \subset \mathbb{N}$ is defined as

$$\overline{\delta}(A) = \limsup_{N\to\infty} \frac{\sum_{n\in A} \frac{1}{n}}{\sum_{n\leqslant N} \frac{1}{n}} = \limsup_{N\to\infty} \mathbb{E}^{\log}_{n\leqslant N} 1_A(n).$$

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A **multiplicative Følner sequence** on $\mathbb{N}$ is a sequence of finite sets $\Psi = (\Psi_M)_{M \in \mathbb{N}}$ such that

$$\frac{|\Psi_M \cap \Psi_M/m|}{|\Psi_M|} \xrightarrow{M \to \infty} 1, \qquad \forall m \in \mathbb{N}.$$

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REMARK: $(\mathbb{N}, +, \cdot)$ is not amenable and there exists no Følner sequence on $\mathbb{N}$ that is both additively and multiplicatively invariant.

Main Results

for patterns of the form $\{x + y, xy + 1\}$

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Example: The set $A = \bigcup_{n\geqslant 4}[2^{2^n}, 2^{2^n+1})$ satisfies $\overline{d}(A) > 0$, but does not contain patterns of the from $\{x+y, xy+1\}$.

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$$\delta^{\times}(A) = \limsup_{M\to\infty} \mathbb{E}_{m\in\Psi_M}\Bigg(\underbrace{\lim_{k\to\infty} \mathbb{E}^{\log}_{n\leqslant N_k} 1_A(mn)}_{\text{logarithmic density of } A/m} \Bigg),$$

where $N_1 < N_2 < \ldots \in \mathbb{N}$ is any sequence such that all limits above exist.

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Observation

$\delta^{\times}(A) > 0 \iff$ *The logarithmic density of A/m is bigger than ε for a set of m that has positive multiplicative density.*

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Theorem 2 (R. 2024+)

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Main Theorems

Outline of proof

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Local Almost Periodicity

Definitions

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EXAMPLE: In [MRT15] it is shown that the classical Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ is locally aperiodic (with Cesàro averages).

[MRT15] K. Matomäki, M. Radziwiłł, and T. Tao. *An averaged form of Chowla's conjecture*. Algebra Number Theory, 9(9):2167–2196, 2015.

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[HK09] B. Host and B. Kra. *Uniformity seminorms on* $\ell^\infty(\mathbb{N})$ *and applications*. J. Anal. Math. 108 (2009).

Structure Theorem II

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Any $f \in \ell^\infty(\mathbb{N})$ can be split into $f = f_{nil} + f_{uni}$, where

- f_{nil} is “locally k-step nil”,
- and $\|f_{uni}\|_{U^{k+1}(\mathbb{N})} = 0$.

Generalized von Neumann Theorems

Classical cases

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Generalized von Neumann Theorem (Gowers norms)

If $\lim_{N\to\infty} \|f\|_{U^k[N]} = 0$ then

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Question: What about a generalized von Neumann theorem for $f(n+m)f(nm)$?

Sums, Products, and Uniformity

controlling $\{x + y, xy\}$ via uniformity seminorms

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Proposition 2

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van der Corput's inequality

Old and new

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$$\left(\mathbb{E}_{h\in[H]}\left|\mathbb{E}^{\log}_{n\leqslant N} f(n)g(n+h)\right|\right)^2 \leqslant 2\,\mathbb{E}_{h\in[H]}\left|\mathbb{E}^{\log}_{n\leqslant N} f(n)\overline{f(n+h)}\right| + \frac{1}{H} + o_{N\to\infty}(1).$$

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Orthogonality Criterion (Daboussi-Delange-Kátai-Bourgain-Sarnak-Ziegler)

Let $P \subset \mathbb{P}$ be a finite set of primes and $g\colon \mathbb{N} \to S^1$ completely multiplicative. Then

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$$\approx \left(\mathbb{E}^{\log}_{p\in P} u_p\, \mathbb{E}^{\log}_{n\leqslant N} f\left(\tfrac{n}{p}\right) g(n)\, p\, 1_{p|n}\right)^2$$

Property of log averages

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Property of log averages

Rearrange and Cauchy-Schwarz

New Orthogonality Criterion

Proof

New Orthogonality Criterion

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Let $P \subset \mathbb{P}$ be a finite set of primes. Then

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Proposition 1

If f is locally aperiodic then

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$$\left|\mathbb{E}^{\log}_{m\leqslant M}\mathbb{E}^{\log}_{n\leqslant N} f(n+m)f(nm)\right|^2 \lessapprox \mathbb{E}^{\log}_{p,q\in P}\left|\mathbb{E}^{\log}_{m\leqslant M}\mathbb{E}^{\log}_{n\leqslant N} f(qn+pm)\overline{f(pn+qm)}\right|.$$

$\square$

Thank you

New Orthogonality Criterion

Proof

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If $\overline{\delta}(A) > 0$ then A contains $\{x + y, xy + 1\}$.

New Orthogonality Criterion

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New Orthogonality Criterion

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Theorem 3

Suppose $A \subset \mathbb{N}$. For all $\varepsilon > 0$ there exist $W, k \in \mathbb{N}$ such that

$$\mathbb{E}^{\log}_{p_1 \in \mathbb{P}_W} \cdots \mathbb{E}^{\log}_{p_k \in \mathbb{P}_W} \mathbb{E}^{\log}_{n \in \mathbb{N}} 1_A(n + p_1 \cdots p_k) 1_A(np_1 \cdots p_k + 1) \geqslant \delta(A)^2 - \varepsilon.$$

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The main idea is that:

$$\begin{aligned}
&\mathbb{E}^{\log}_{p_1 \in \mathbb{P}_W} \cdots \mathbb{E}^{\log}_{p_k \in \mathbb{P}_W} \mathbb{E}^{\log}_{n \in \mathbb{N}} 1_A(n + p_1 \cdots p_k) 1_A(np_1 \cdots p_k + 1) \\
&\qquad \approx_\varepsilon \mathbb{E}^{\log}_{p_1 \in \mathbb{P}_W} \cdots \mathbb{E}^{\log}_{p_k \in \mathbb{P}_W} \mathbb{E}^{\log}_{n \in \mathbb{N}} 1_A(n + 1) 1_A(np_1 \cdots p_k + 1) \\
&\qquad = \mathbb{E}^{\log}_{p_1 \in \mathbb{P}_W} \cdots \mathbb{E}^{\log}_{p_k \in \mathbb{P}_W} \mathbb{E}^{\log}_{n \in \mathbb{N}} 1_{A-1}(n) 1_{A-1}(np_1 \cdots p_k).
\end{aligned}$$