Some recent progress on sum-product problem

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Erdős-Szemerédi 1983, quote:

Let $1 < a_1 < \ldots < a_n$ be a sequence of integers. Consider the integers of the form

$$a_i + a_j,\ a_i a_j : \qquad 1 \leq i \leq j \leq n. \tag{1}$$

It is tempting to conjecture that for every $\epsilon > 0$ there is an n_0, so that for every $n \geq n_0$, there are more than $n^{2-\varepsilon}$ distinct integers of the form (1).

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Moreover, we conjecture that for every k and $n \geq n_0(k)$ there are more than $n^{k-\varepsilon}$ distinct integers of the form

$$a_{i_1} + \ldots + a_{i_k}, \qquad \prod_{j=1}^{k} a_{i_j}. \tag{2}$$

Perhaps our conjectures remain true if the $a's$ are real or complex numbers.

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- Over $\mathbb{C}$, $\epsilon > \frac{2}{3}$ (Konyagin-R, 2014).
- Over $\mathbb{F}_p$, $|A| < p^{1/2}$, $\epsilon > \frac{1}{4}$ (Mohammadi-Stevens, 2022).

Asymptotic cases: FSMP and FPMS

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$$|AA| \leq M|A| \Rightarrow |A+A| \geq M^{-\frac{2}{\epsilon}}|A|^{2-4\epsilon}.$$

for $\mathbb{Z}$ Zhelezov-Palvölgyi 2020 after Bourgain-Chang 2004, and now, stronger, $\mathbb{Z}$ from weak PFR of Gowers, Green, Manners, Tao 2023:

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- Over $\mathbb{F}_p$ partial results only.

Energy: second moment of convolution

- (Additive) energy:

$$E(A) = \sum_{s \in A+A} r_{A+A}^2(s) = \sum_{d \in A-A} r_{A-A}^2(d)$$
$$= |\{a_1 - a_2 = a_3 - a_4 : a_i \in A\}| .$$

  Multiplicative energy $E^\times(A)$.

- Trivially $|A|^2 \le E(A) \le |A|^3$. Cauchy-Schwarz:

$$|A \pm A| \ge \frac{|A|^4}{E(A)} .$$

- Bookkeeping:

$$f(t) := \sum_{a \in A} e(at) , \qquad E(A) = \|f\|_{L^4(\mathbb{T})}^4 = \|f\|_4^4 .$$

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PFR + Subspace theorem, now for $\mathbb{C}$

$$\exists A' \subseteq A, \quad |A'| \geq |A|/2 : \quad |\{a_1 - a_2 = c : \; a_{1,2} \in A'\}| \leq M^C |A|.$$

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- Is it true that

$$\exists A' \subseteq A: \qquad |AA| + \frac{|A'|^4}{E(A')} \geq |A|^{2-o(1)} \ ?$$

Balog-Wooley example, 2017

NO. The best one can expect:

$$\exists A' \subseteq A: \qquad |AA| + \frac{|A'|^4}{E(A')} \geq |A|^{\frac{5}{3}-o(1)}.$$

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Given n, take $p > n^2$ and

$$A = \bigcup_{j=1}^{n} p^j [n^2] .$$

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The same for any positive density subset $A' \subseteq A$.

Matchnig the BW bound

Theorem (Hanson, R, Shkredov, Zhelezov, 2023): Let $A \subset \mathbb{Z}$. Then

$$\exists A' \subseteq A : \quad |A'| \gg |A|^{1-o_r(1)} : \quad |AA| + \frac{|A'|^4}{E(A')} \geq |A|^{\frac{5}{3}-o_r(1)} \,,$$

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Corollary: For A as above and $M = \frac{|AA|}{|A|}$

$$|AA + AA| \gg \min\left(M, \frac{|A|}{M}\right) |A|^{2-o_r(1)}.$$

Ingredients

Proof ingredients:

- Structure theorem
- "*A martingale that occurs in harmonic analysis*" (Gundy and Varopoulus, 1976)
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Theorem: Let $p_1, \ldots, p_r$ be primes and Γ a subgroup of $\mathbb{Q}^*$, generated by them. For fixed $a_1, \ldots, a_l \in \mathbb{C}^*$,

$$\left|\left\{g_1, \ldots, g_l \in \Gamma \ : \ \sum_{i=1}^{l} a_i g_i = 1, \ \text{no 0 subsum}\right\}\right| \leq (8l)^{4l^2+lr+1} .$$

Structure theorem

Let r_0 = max # of prime factors, $r \leq r_0$. $\exists\, A' \subseteq A$: $|A'| \geq |A|^{1-o_r(1)}$, a set of r primes, generating group Γ, and coprime therewith B:

- 50% pairs (b, b') coprime ,
- $\forall\ a \in A',\ a = (p_1^{v_1} \cdot \ldots \cdot p_r^{v_r})b \ := \ gb$, i.e. $A' = \bigsqcup_{b \in B} b\Gamma_b$,
- $\exists L$: each $|\Gamma_b| \approx L$, and $L = \prod_{j=1}^{r} L_j$,
- roughly L_j, $j = 1, \ldots, r$ of p_j valuations in each Γ_b.

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- So $L|B| = |A|^{1-o_r(1)}$, and we lose L on products:

$$|AA| \gg \frac{|A|^2}{L}.$$

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Lemma: (Chang, 2003) Let $A \subset \mathbb{Z}$, symmetric, p prime and $A_v := \{a \in A : \nu_p(a) = v\}$. Then

$$E(A) \ll \sum_v \{ E(A_v, A) = |\{a_1 - a_2 = a_3 - a_4 : a_1, a_2 \in A_v\}| \}$$
$$\leq \sqrt{E(A)} \sum_v \sqrt{E(A_v)} .$$

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Thus

$$E(A') \ll L^2 E(B) \leq L^2 |B|^3 , \qquad |AA| + \frac{|A'|^4}{E(A')} \geq |A|^{3/2 - o_r(1)} .$$

From 3/2 to 5/3

- **Main Proposition:** (Martingales) Let Γ be generated by $p_1, \ldots, p_r$, let $A = \bigcup_{b \in B} b\Gamma_b$, symmetric. Then

$$E(A) \ll_r |\{a_1 - a_2 = a_3 - a_4 : \; v_\Gamma(a_1) = v_\Gamma(a_2),\; v_\Gamma(a_3) = v_\Gamma(a_4)\}|$$

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- **Lemma:** (Subspace) For $0 < \epsilon < 1/6$, $r \leq \log^{1-6\epsilon} |B|$,

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Together, using $|A|^{1-o_r(1)} = L|B|$, we win L^2 on energy:

$$\frac{|A'|^4}{E(A')} \ge \frac{|A|^{4-o_r(1)}}{|A|^2 + L|B|^{3+o(1)}} = \frac{|A|^{5-o_r(1)}}{|A|^3 + |AA|^2} \,. \qquad \square$$

One common prime: $\mathrm{rank}(\Gamma) = 1$

Input:

$$A = \bigcup_{v} (A_v = p^v B_v) = \bigcup_{b \in B} \{b\} p^{V(b)} .$$

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No Subspace theorem needed for $r = 1$.

$$|AA| + \frac{|A'|^4}{E(A')} \ge |A|^{\frac{5}{3}-o(1)}\,. \quad \square$$

Proof of Main Prop: Square function, rank 1

Main Prop, rank 1: One has $\|f\|_4 \ll \|S_p[f]\|_4$, with

$$f(t) = \sum_{v \geq 0} \sum_{\nu_p(a)=v} \hat{f}(a) e(at),$$

$$S_p[f](t) = \left(\sum_{v \geq 0} \left| \sum_{\nu_p(a)=v} \hat{f}(a) e(at) \right|^2 \right)^{1/2} = \left(\sum_{v \geq 0} |f_v(t)|^2 \right)^{1/2}$$

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Theorem: (Burkholder-Gundy, 1970) For $1 < q < \infty$

$$\|S_p[f]\|_{L^q(\mathbb{T})} \ll_q \|f\|_{L^q(\mathbb{T})}\,, \qquad \|M[f]\|_{L^q(\mathbb{T})} \ll_q \|S_p[f]\|_{L^q(\mathbb{T})}\,,$$

with

$$f_{\geq v}(t) := \sum_{\nu_p(a)\geq v} \hat{f}(a)e(at)\,, \qquad M[f](t) := \sup_{v\geq 0} f_{\geq v}(t)\,.$$

Square function, one prime, cont.

On the proof of BG-Theorem:

$$f_{\geq v}(t) = \sum_{v' \geq v} \left(\sum_{\nu_p(a)=v'} \hat{f}(a) e(at) \right) = p^{-v} \sum_{j=0}^{p^v-1} f\left(t + \frac{j}{p^v}\right) .$$

So,

$f_{\geq N}, f_{\geq N-1}, .., f_{\geq 1}, f$ is a martingale.

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Main Prop rank 1 is a direct consequence of Chang's lemma and Doob's martingale maximal inequality:

$$\|f\|_4^4 = \int_{\mathbb{T}} \sum_{v \geq 0} |f_v(t)|^2 |f_{\geq v}(t)|^2 dt \leq \|M[f]\|_4^2 \|S[f]\|_4^2 \ll \|f\|_4^2 \|S[f]\|_4^2.$$

Rank > 1: Khintchine, randomisation, inverse

Recall:

$$f(t) = \sum_{v \geq 0} f_v = \sum_{v \geq 0} \left(\sum_{\nu_p(a)=v} \hat{f}(a) e(at) \right) .$$

Let random $\xi(v) \in \{\pm 1\},$, independently, $f_\xi = \sum_{v \geq 0} \xi(v) f_v.$

$S_p[f] = S_p[f_\xi]$.

Furthermore, point-wise in t

$$\mathbb{E}_\xi |f_\xi(t)|^4 = \sum_{v_1, v_2, v_3, v_4} f_{v_1}(t) f_{v_2}(t) \overline{f_{v_3}(t) f_{v_4}(t)} \mathbb{E}(\xi(v_1)\xi(v_2)\xi(v_3)\xi(v_4))$$

$$\ll \left(\sum_v |f_v(t)|^2 \right)^2 = S_p[f](t)^4 .$$

Fetch another common prime q:

Let

$$f_v = \sum_{w \geq 0} f_{v,w} = \sum_{w \geq 0} \left(\sum_{\nu_p(a)=v,\, \nu_q(a)=w} \hat{f}(a) e(at) \right) , \; f = \sum_{v,w \geq 0} f_{v,w}$$

Take a random $\eta(s) \in \{\pm 1\}$, similarly (Khintchine)

$$\mathbb{E}_{\xi,\eta} \|f_{\xi,\eta}\|_4^4 \ll \left| \sum_{v,w} |f_{v,w}|^2 \right|_2^2 := \|S_{p,q}[f]\|_4^4 \, .$$

Take a realisation (ξ, η),where this is the case. Now use BG-theorem: inverse/direct/inverse/direct:

$$\|f_{\xi,\eta}\|_4 \gg \|S_p[f_{\xi,\eta}]\|_4 = \|S_p[f_\eta]\|_4 \gg \|f_\eta\|_4$$
$$\gg \|S_q[f_\eta]\|_4 = \|S_q[f]]\| \gg \|f\|_4 \, ,$$

iterate $\square$

Thank you