On quantitative Gowers uniformity and applications

Joni Teräväinen

University of Turku

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Quantitative inverse theorem

For $f\colon \mathbb{Z} \to \mathbb{C}$ with finite support, define

$$\|f\|_{U^k(\mathbb{Z})} = \Big(\sum_{n,h_1,\ldots,h_k \in \mathbb{Z}} \prod_{\omega \in \{0,1\}^k} \mathcal{C}^{|\omega|} f(n + \omega \cdot \mathbf{h}) \Big)^{1/2^k}$$

and for $f\colon [N] \to \mathbb{C}$ define

$$\|f\|_{U^k[N]} = \frac{\|f 1_{[N]}\|_{U^k(\mathbb{Z})}}{\|1_{[N]}\|_{U^k(\mathbb{Z})}}.$$

Improving on Green–Tao–Ziegler (2012), Manners (2018), we have

Theorem (Leng–Sah–Sawhney, 2024)

Let $f\colon [N] \to \mathbb{C}$ be 1-bounded, $\delta \in (0, 1/2)$, and

$$\|f\|_{U^k[N]} \geq \delta.$$

Then there is a nilmanifold G/Γ of step $k-1$, of complexity $\leq \exp((\log(1/\delta)^{C_k}))$ and of dimension $\leq (\log(1/\delta))^{C_k}$, such that for some 1-Lipschitz $F\colon G/\Gamma \to \mathbb{C}$ and polynomial sequence $g\colon \mathbb{Z} \to G$ we have

$$\Big| \frac{1}{N} \sum_{n \leq N} f(n) F(g(n)\Gamma) \Big| \geq \exp(-(\log(1/\delta)^{C_k})).$$

Consequences of quasipolynomial inverse theorem

The quasipolynomial inverse theorem leads to several new avenues and applications:

1. Simpler approach to proving Gowers uniformity.
2. A quantitative theory of higher order Fourier analysis.
3. New qualitative applications in ergodic theory.

Gowers uniformity of the primes

The cornerstone of the study of patterns in the primes is the Gowers uniformity of the von Mangoldt function.

Improving on Green–Tao–Ziegler (2012), Tao–T. (2021), we have

Theorem (Leng, 2024)

Let

$$\Lambda_w(n) = \prod_{p \le w} \frac{p}{p-1} 1_{(n,p)=1}$$

be the Cramér model with $w = \exp((\log N)^{1/10})$. Then

$$\|\Lambda - \Lambda_w\|_{U^k[N]} \ll_{k,A} (\log N)^{-A}.$$

This implies an asymptotic for linear equations in the primes with any power of logarithm savings in the error term (Leng, 2024).

Proof of Gowers uniformity of the primes

Suppose $\|\Lambda - \Lambda_w\|_{U^k[N]} \geq \delta$. Green–Tao approach:

1. Show that the inverse theorem continues to hold for unbounded functions bounded by a pseudorandom measure.
2. Show that the primes are bounded by a pseudorandom measure.
3. Deduce that $\Lambda - \Lambda_w$ correlates with a nilsequence.

Leng's result allows bypassing steps 1 and 2 by looking at the bounded function $\frac{\Lambda(n)-\Lambda_w(n)}{\log N}$.

$$\|\Lambda - \Lambda_w\|_{U^k[N]} \geq (\log N)^{-A} \implies \left\| \frac{\Lambda - \Lambda_w}{\log N} \right\|_{U^k[N]} \geq (\log N)^{-A-1}$$

$$\implies \left| \frac{1}{N} \sum_{n \leq N} (\Lambda(n) - \Lambda_w(n)) F(g(n)\Gamma) \right| \geq \exp(-(\log \log N)^{C_k}).$$

Proof of Gowers uniformity of the primes

Recall

$$\left|\frac{1}{N}\sum_{n\le N}(\Lambda(n)-\Lambda_w(n))F(g(n)\Gamma)\right| \ge \exp(-(\log\log N)^{C_k}).$$

By Vaughan's identity, $\Lambda(n)$ is a sum of $O((\log N)^{O(1)})$ type I sums $\sum_{d|n, d\le N^{1/3}} \alpha_d$ and type II sums $\sum_{n=\ell m, \ell\in[N^{1/3}, N^{2/3}]} \alpha_\ell \beta_m$. Also Λ_w is essentially a type I sum.

Type I and II sum estimates for nilsequences
$\implies$ $F(g(n)\Gamma)$ essentially periodic $\implies$

$$\left|\frac{1}{N}\sum_{\substack{n\le N \\ n\equiv b \pmod r}}(\Lambda(n)-\Lambda_w(n))\right| \gg \exp(-O((\log\log N)^{C_k}))$$

for some $1\le b\le r$.

If we assume GRH, we can conclude.

Otherwise, the Siegel–Walfisz theorem is a bottleneck.

Siegel model for the primes

Solution: use a refined model. If β is the Siegel zero and $\chi \pmod{q}$ the corresponding real character ($L(\beta,\chi)=0$), define

$$\Lambda_w^{\mathrm{Siegel}}(n) = \Lambda_w(n)(1-\chi(n)n^{\beta-1}).$$

(if no Siegel zero exists, put $\beta = 1$, $\chi = 0$).

Then the Landau–Page theorem says

$$\Big|\frac{1}{N}\sum_{\substack{n\le N\\ n\equiv b \pmod{r}}} (\Lambda(n)-\Lambda_w^{\mathrm{Siegel}}(n))\Big| \ll \exp(-(\log N)^c).$$

Removing the Siegel model

$\Lambda_w^{\text{Siegel}}$ is essentially a type I sum, so the previous argument gives

$$\|\Lambda - \Lambda_w^{\text{Siegel}}\|_{U^k[N]} \ll \exp(-(\log w)^{c_k}) \ll_A (\log N)^{-A}.$$

By the triangle inequality for Gowers norms, suffices to show

$$\|\Lambda_w - \Lambda_w^{\text{Siegel}}\|_{U^k[N]} \ll_A (\log N)^{-A}.$$

Since $\Lambda_w(n) - \Lambda_w^{\text{Siegel}}(n) = \Lambda_w(n)\chi(n)n^{\beta-1}$, we are left with

estimating

$$\|\chi\|_{U^k[N]} \asymp_k \left(\frac{1}{N^{k+1}} \sum_{n,h_1,\ldots,h_k \in [N]} \prod_{\omega \in \{0,1\}^k} \chi(n + \omega \cdot \mathbf{h}) \right)^{1/2^k}.$$

By the Weil bound, $|\sum_{n \leq N} \chi(P(n))| \ll_{\deg P} Nq^{-1/2}$ unless $P(y)$ is a square modulo q. Using this, we get

$$\|\chi\|_{U^k[N]} \ll q^{-c_k} \ll_A (\log N)^{-A}$$

since $q \gg_A (\log N)^A$ by Siegel's bound.

A general recipe

The same recipe can be used to give good quantitative bounds for the Gowers norms of any "nice" arithmetic function f, e.g.
$f(n) = d(n)$ (divisor function),
$f(n) = r(n)$ (representation function of sums of two squares),
$f(n) = \mu(n)$ (Möbius function).

1. Find a model function f_{model} that mimics the distribution of f in APs ($f - f_{\mathrm{model}}$ has negligible mean in arithmetic progressions).
2. In practice, $f(n), f_{\mathrm{model}}(n) \ll d(n)^C$ for some C. Tail bounds for $d(n) \implies f(n), f_{\mathrm{model}}(n) \ll (\log N)^A$ for some A outside a negligible set of n.
3. Apply the quasipolynomial inverse theorem to $(f(n) - f_{\mathrm{model}}(n))/(\log N)^A$.
4. In practice, f, f_{model} are linear combinations of type I and II sums, so can use the equidistribution theory of nilsequences to conclude.

Quantitative higher order Fourier analysis

Let's then look at how the quantitative inverse theorem can be used to prove quantitative results on patterns.

Some results in higher order Fourier analysis of the primes:

1. Linear patterns in primes (Green–Tao–Ziegler).
2. Polynomial patterns in primes (Tao–Ziegler).
3. Positive density sets contain APs with difference $p-1$ (Frantzikinakis–Host–Kra, Wooley–Ziegler).

(1) was quantified by Leng (2024), (3) was quantified by Tao–T. (2021). What about (2)?

Polynomial progressions in the primes

Improving on Tao–Ziegler (2018), we have

Theorem (Matthiesen–T.–Wang, 2024)

Let $P_1, \ldots, P_k$ be polynomials of degree $\leq d$, and suppose $\deg(P_i - P_j) = d$ for $i < j$. Then

$$\frac{1}{N^{d+1}} \sum_{n \leq N^d} \sum_{m \leq N} \Lambda(n + P_1(m)) \cdots \Lambda(n + P_k(m)) = \prod_p \beta_p + O((\log N)^{-A}),$$

where β_p are suitable local factors.

A similar result for the Möbius function, under a weaker hypothesis on the polynomials (for some ℓ we have $\deg(P_\ell - P_i) = d, i \neq \ell$).

For Möbius function, a qualitative result was known for any polynomials with $P_i - P_j$ nonconstant (Matomäki–Radziwiłł–Tao–T.–Ziegler, 2020).

Proof sketch

Consider a model case

$$\frac{1}{N^3}\sum_{n\leq N^2}\sum_{m\leq N}\Lambda(n)\Lambda(n+m^2)\Lambda(n+2m^2).$$

For $j\in\{1,2,3\}$, decompose $\Lambda=\Lambda_{w_j}^{\mathrm{Siegel}}+E_j$. Get a Siegel main term

$$\frac{1}{N^3}\sum_{n\leq N^2}\sum_{m\leq N}\Lambda_{w_1}^{\mathrm{Siegel}}(n)\Lambda_{w_2}^{\mathrm{Siegel}}(n+m^2)\Lambda_{w_3}^{\mathrm{Siegel}}(n+2m^2).$$

and 7 error terms such as

$$\frac{1}{N^3}\sum_{n\leq N^2}\sum_{m\leq N}\Lambda_{w_1}^{\mathrm{Siegel}}(n)E_2(n+m^2)\Lambda_{w_3}^{\mathrm{Siegel}}(n+2m^2).$$

Proof sketch

We have

$$\Lambda_{w_3}^{\text{Siegel}}(n) = \Lambda_{w_3}(n)(1-\chi(n)n^{\beta-1}) \approx \prod_{p\leq w}\frac{p}{p-1}\sum_{\substack{P^+(d)\leq w_3\\ d\leq w_3^{(\log\log N)^2}}}\mu(d)1_{d|n}(1-\chi(n)n^{\beta-1}).$$

Hence, to handle the error term it suffices to show that

$$S_d := \frac{1}{N^3}\sum_{n\leq N^2}\sum_{m\leq N}\Lambda_{w_1}^{\text{Siegel}}(n)E_2(n+m^2)1_{d|n+2m^2} \ll_A (\log N)^{-A}/d.$$

By a generalised von Neumann theorem of Peluse,

$$\left|\frac{1}{N^{d+1}}\sum_{n\leq N^d, m\leq N} f_1(n+P_1(m))\cdots f_k(n+P_k(m))\right| \ll_{P_1,\ldots,P_k} \|f_k\|_{U^s[N^d]}^{c_k}.$$

Hence,

$$S_d \ll d^{O_k(1)}\|E_2\|_{U^s[N]} \ll w_3^{O_k((\log\log N)^2)}\exp(-(\log w_2)^{c_k})$$

Done if $w_2 = w_3^{(\log\log N)^{B_k}}$, $w_3 = \exp((\log\log N)^{B_k})$.

Similarly set $w_1 = w_2^{(\log\log N)^{B_k}}$ to ensure the other error terms are small.

Proof sketch

We are left with the Siegel term

$$\frac{1}{N^3}\sum_{n\le N^2}\sum_{m\le N}\Lambda_{w_1}^{\mathrm{Siegel}}(n)\Lambda_{w_2}^{\mathrm{Siegel}}(n+m^2)\Lambda_{w_3}^{\mathrm{Siegel}}(n+2m^2).$$

Decompose $\Lambda_{w_j}^{\mathrm{Siegel}}(n)=\Lambda_{w_j}(n)-\Lambda_{w_j}\chi(n)n^{\beta-1}$. Get a main term

$$M=\frac{1}{N^3}\sum_{n\le N^2}\sum_{m\le N}\Lambda_{w_1}(n)\Lambda_{w_2}(n+m^2)\Lambda_{w_3}(n+2m^2).$$

and error terms such as

$$\frac{1}{N^3}\sum_{n\le N^2}\sum_{m\le N}\Lambda_{w_1}(n)\chi(n)n^{\beta-1}\Lambda_{w_2}(n+m^2)\Lambda_{w_3}(n+2m^2).$$

These can be evaluated using basic sieve theory, the main term M gives $(1+O((\log N)^{-A}))\prod_p \beta_p$ and the error term gives something that is smaller by a factor of $(\log N)^{-A}$ due to cancellation in character sums.

Multiple ergodic averages

Switching gears, let's look at a (qualitatve) ergodic theory problem that boils down to strong quantitative estimates for Gowers norms.

Throughout, let (X, T, ν) be a measure-preserving system and $P_i \in \mathbb{Z}[y]$.

Conjecture (Furstenberg–Bergelson–Leibman)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f_1(T^{P_1(n)}x) \cdots f_k(T^{P_k(n)}x)$$

exists for ν-almost every $x \in X$.

- Known in the cases where $k = 2$ and either P_1 or P_2 has degree ≤ 1 (Bourgain, Krause–Mirek–Tao).
- L^2-convergence is known (Host–Kra, Leibman)

Möbius ergodic averages

What about multiple ergodic averages with an arithmetic weight?

Theorem (T., 2024)

Let $k \in \mathbb{N}$ and $P_1, \ldots, P_k \in \mathbb{Z}[y]$. Let (X, T, ν) be a measure-preserving system, and let $f_i \in L^\infty(X)$. Then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f_1(T^{P_1(n)}x) \cdots f_k(T^{P_k(n)}x) = 0$$

for ν-almost every $x \in X$.

- This gives an analogue of the Furstenberg–Bergelson–Leibman conjecture for μ.
- Taking $k = 2$, $P_1(y) = y$, $P_2(y) = 2y$ solves a problem of Frantzikinakis.
- Should be possible to also handle commuting transformations.
- Get a polylogarithmic decay rate to 0.

Prime ergodic averages

What about prime weights?

Theorem (Krause–Mousavi–Tao–T., 2024+)

Let $P \in \mathbb{Z}[y]$ with $\deg P > 1$. Let (X, T, ν) be a measure-preserving system, and let $f_1, f_2 \in L^\infty(X)$. Then

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\le N} f_1(T^p x) f_2(T^{P(p)} x)$$

exists for ν-almost every $x \in X$.

Handling the prime weight requires in particular handling the constant weight 1, so we need to use additional tools (machinery of Krause–Mirek–Tao).

Lacunary subsequence trick

We sketch the proof of the Möbius result.

Let $f_1, \ldots, f_k \in L^\infty(X)$ and

$$A(N,x) := \frac{1}{N} \sum_{n \leq N} \mu(n) f_1(T^{P_1(n)}x) \cdots f_k(T^{P_k(n)}x).$$

- We have the slow variation property

$$A((1+\varepsilon)N, x) = A(N,x) + O(\varepsilon),$$

so it suffices to show that, for all $\varepsilon > 0$, $\lim_{j\to\infty} A((1+\varepsilon)^j, x)$ exists for almost every x. As a model case, consider showing that

$$\lim_{j\to\infty} A(2^j, x) \quad \text{exists almost everywhere.}$$

Applying Borel–Cantelli

Let

$$A(N,x) := \frac{1}{N}\sum_{n\le N}\mu(n) f_1(T^{P_1(n)}x)\cdots f_k(T^{P_k(n)}x).$$

As a model case, consider showing that

$$\lim_{j\to\infty} A(2^j,x) \quad \text{exists almost everywhere.}$$

- By Chebyshev's inequality,

$$\nu(\{x\in X\colon |A(2^j,x)|\ge\varepsilon\}) \le \varepsilon^{-2}\int_X |A(2^j,x)|^2\,\mathrm{d}\nu(x).$$

- If we show that

$$\int_X |A(N,x)|^2\,\mathrm{d}\nu(x) \ll (\log N)^{-2},$$

then the existence of $\lim_{j\to\infty} A(2^j,x)$ follows from the Borel–Cantelli lemma since $\sum_{j\ge 1} 1/(\log 2^j)^2 < \infty$.

$\implies$ Need strong quantitative bounds for the L^2-averages.

Reduction to additive combinatorics

Want to show that

$$\int_X |A(N,x)|^2 \, d\nu(x) \ll (\log N)^{-2},$$

where $A(N,x) = \frac{1}{N}\sum_{n\le N} \mu(n) f_1(T^{P_1(n)}x)\cdots f_k(T^{P_k(n)}x)$.

- Equivalent to showing that for all ϕ with $\|\phi\|_{L^\infty(X)} = 1$,

$$\int_X \phi(x) A(N,x) \, d\nu(x) \ll (\log N)^{-2}.$$

- Since ν is measure-preserving, reduce to showing that

$$\int_X \frac{1}{N^{d+1}} \sum_{m\le N^d} \sum_{n\le N} \mu(n)\phi(T^m x) f_1(T^{m+P_1(n)}x)\cdots f_k(T^{m+P_k(n)}x) \, d\nu(x)$$
$$\ll (\log N)^{-2}.$$

- It now suffices to give a good upper bound for

$$\frac{1}{N^{d+1}} \sum_{m\le N^d} \sum_{n\le N} \mu(n) g_0(m) g_1(m+P_1(n))\cdots g_k(m+P_k(n))$$

for any 1-bounded $g_0, g_1, \ldots, g_k : \mathbb{Z} \to \mathbb{C}$.

By van der Corput's inequality and PET induction, can show that for some $s \in \mathbb{N}$, $c_s > 0$, we have

$$\left|\frac{1}{N^{d+1}} \sum_{m \leq N^d} \sum_{n \leq N} \mu(n) g_0(m) g_1(m + P_1(n)) \cdots g_k(m + P_k(n))\right| \ll \|\mu\|_{U^s[N]}^{c_s}.$$

Using the bound $\|\mu\|_{U^s[N]} \ll_A (\log N)^{-A}$ (Leng, 2024), the proof can now be concluded.

A second approach

We can use a weaker norm to avoid these deep Gowers norms bounds in the case where the P_i have distinct degrees.
This turns out to be helpful for some applications.• For $f\colon [N] \to \mathbb{C}$, let

$$\|f\|_{u^s[N]} := \sup_{\substack{P \in \mathbb{R}[y] \\ \deg P \leq s-1}} \Big| \frac{1}{N} \sum_{n \leq N} f(n) e(P(n)) \Big|.$$

• The $u^s[N]$ norm is weaker than the $U^s[N]$ norm for $s > 2$; for $s = 2$ they are equivalent up to polynomial losses.
• $u^s[N]$ norm much easier to bound than $U^s[N]$ norm: work of Vinogradov (1930s) $\implies \|\mu\|_{u^s[N]} \ll_A (\log N)^{-A}$.

Proposition (T., 2024)

Let $P_1, \ldots, P_k \in \mathbb{Z}[y]$ have distinct degrees. Let $\theta, g_0, \ldots, g_k\colon \mathbb{Z} \to \mathbb{C}$ be 1-bounded. Then

$$\Big| \frac{1}{N^{d+1}} \sum_{m \leq N^d} \sum_{n \leq N} \theta(n) g_0(m) g_1(m + P_1(n)) \cdots g_k(m + P_k(n)) \Big| \ll \|\theta\|_{u^{k+1}[N]}^{c_k}.$$

A generalised von Neumann theorem

Proposition (T., 2024)

Let $P_1, \ldots, P_k \in \mathbb{Z}[y]$ have distinct degrees. Let $\theta, g_0, \ldots, g_k : \mathbb{Z} \to \mathbb{C}$ be 1-bounded. Then

$$\left| \frac{1}{N^{d+1}} \sum_{m \leq N^d} \sum_{n \leq N} \theta(n) g_0(m) g_1(m + P_1(n)) \cdots g_k(m + P_k(n)) \right| \ll \|\theta\|_{u^{k+1}[N]}^{c_k}.$$

Peluse's inverse theorem: when $\theta = 1$, the LHS can be bounded in terms of the average of g_1 over short intervals (and arithmetic progressions).

Proof sketch:

- Induction on k. For $k = 1$, use classical Fourier analysis.
- Use van der Corput in m to eliminate θ.
- Apply Peluse's theorem to conclude that $\Delta_h g_1$ is locally constant for many h, so g_1 is locally linear (in short intervals, g_1 is a linear phase).
- Apply van der Corput again to conclude that g_1 must be a major arc linear phase in short intervals.
- There are not too many major arc phases, so by pigeonholing can assume that g_1 is equal to a global major arc phase $e(\alpha n)$. Now apply induction.

The prime case

For proving the almost everywhere convergence of

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\leq N} f_1(T^p x) f_2(T^{P(p)} x),$$

there are three main ingredients:

- A version of Peluse's inverse theorem over the primes,
- An L^p-improving estimate for primes evaluated at polynomials,
- A major/minor arc analysis (in the spirit of Krause–Mirek–Tao).

Thank you!