# Ranking Rankings

## Forming preferences on others based on others' preferences on us & applications in social choice

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# Ranking Rankings

Psychological reciprocity suggests that people like to be liked.

- Markets: Firms indicate preferences over countries by deciding their location, while countries adjust their tax laws—supporting those that support you increases payoffs and decreases risks.
- Voting: Voters rank candidates by ideology, influencing candidates to target their supporters—aligning policies with voter types maximizes endorsement and minimizes opposition.
- Matching: Universities hire researchers after interviewing them and evaluating a good fit—hiring a colleague who wants to join your department promotes integration and collaboration.

# An Example on Matching

An open position is offered by the Universities of Amsterdam (A), Edinburgh (E), and St-Etienne (S). There are three researchers: 1, 2, 3.

| 1 | 2 | 3 | $A$ | $E$ | $S$ |
|---|---|---|---|---|---|
| $E$ | $E$ | $E$ | 1 | 1 | 1 |
| $S$ | $S$ | $S$ | 2 | 2 | 2 |
| $A$ | $A$ | $A$ | 3 | 3 | 3 |
| $E$ | $S$ | $A$ | | | |

# An Example on Matching

An open position is offered by the Universities of Amsterdam (A), Edinburgh (E), and St-Etienne (S). There are three researchers: 1, 2, 3.

| 1 | 2 | 3 | A | E | S |
|---|---|---|---|---|---|
| E | E | E | 1 | 1 | 1 |
| S | S | S | 2 | 2 | 2 |
| A | A | A | 3 | 3 | 3 |
| E | S | A | | | |

*vs.*

| 1 | 2 | 3 | A | E | S |
|---|---|---|---|---|---|
| E | E | S | 1 | 1 | 1 |
| S | S | E | 2 | 2 | 3 |
| A | A | A | 3 | 3 | 2 |
| E | A | S | | | |

Now researcher 3 reports that she prefers S, followed by E, and A.

S may prioritize researcher 3, since she seems more motivated.

E will take researcher 1, and researcher 2 will be left with A.

→ If researcher 2 untruthfully reported that she preferred S to E, then she could do better too!

# Outline

Model

Convergence to Equilibria

Axiomatic Analysis

# Model

# Matching with Reciprocal Preferences

$$P = (R, U, \succ_R, f)$$

- Two finite non-empty and disjoint sets of cardinality $n$: researchers $R = \{1, 2, 3 \ldots\}$ and universities $U = \{x, y, z, \ldots\}$.
- A pref. profile $\succ_R = (\succ_i)_{i \in R}$. Each $\succ_i \in \mathcal{L}_U$ is a linear order over $U$.
- The position of $x$ in the ranking $\succ_i$ is $p_x(\succ_i) = |\{y : y \succ_i x\}| + 1$.
- A reciprocity function $f : (\mathcal{L}_U)^n \to (\mathcal{L}_R)^n$ indicates how universities form pref. according to researchers' pref.

- Given $P$, a matching is a one-to-one function $\mu : R \to U$.

Note that when $f$ is constant, we have the classical matching setting.

# The Deferred Acceptance Algorithm

- Step 1. Each researcher proposes to her most preferred university according to $\succ_R$. Each university tentatively accepts the most preferred researcher among those researchers that have proposed, and rejects all others according to $f(\succ_R)$.
- ...
- Step $k$. Each rejected researcher proposes to her next most preferred university. Each university tentatively accepts the most preferred researcher among the proposers and the one that was tentatively accepted before, and rejects all others.

The DA algorithm stops when there are no new proposals.

Gale and Shapley. “College admissions and the stability of marriage.” The American Mathematical Monthly (1962).

# The Strategic Game

- Each researcher $i \in R$ has a strategy $\succ'_i \in \mathcal{L}_U$.
- The outcome function $\mu^{DA}$ associates the DA matching to each strategy profile $\succ'_R \in (\mathcal{L}_U)^n$ and $f(\succ'_R) \in (\mathcal{L}_R)^n$;
- Each researcher $i \in R$ evaluates the matching $\mu^{DA}(\succ'_R, f(\succ'_R))$ according to her (extended) sincere preference $\succ_i$.

A Nash Equilibrium (NE) is a strategy profile where no researcher can obtain a strictly better outcome by unilaterally deviating.

When $f$ is constant, the sincere profile of the researchers is a NE. This is also true for any $f$ when $n = 2$, but not in general.

Lester and Freedman. "Machiavelli and the Gale-Shapley algorithm." The American Mathematical Monthly (1981).

# Convergence to Equilibria

# Iterative Strategic Game

- A path in $(\mathcal{L}_U)^n$ is a (finite or infinite) sequence $(\succ_R^t)_{t\geqslant 0}$ of strategy profiles, where $\succ_R^t$ and $\succ_R^{t+1}$ differ in exactly one $\succ_i$.
- In a strict best-response path each deviating researcher moves to a strict best response according to her sincere preference $\succ_i$.
- A strict best-response path terminates after a finite number of steps $\ell \in \mathbb{N}$ if no researcher can play a strict best response to $\succ_R^\ell$.

If a strict best-response path terminates after a finite number of steps $\ell$, then $\succ_R^\ell$ is a NE and we have convergence.

# Positional Reciprocity Functions

- An order over positions is $pos = (pos_x)_{x \in U}$, where each $pos_x : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a one-to-one function.
- An order over $R$ for a position $p \in \{1, \ldots, n\}$ is a one-to-one function $s_x(p) : R \to \{1, \ldots, n\}$.
- An ordering scheme is $s = (s_p(x))_{x \in U, p \in \{1, \ldots, n\}}$.

The positional reciprocity function $f^{s,p}$ associated with $s$ and $pos$ is such that $x$ prefers $i$ over $j$ if:

- $pos_x(p_x(\succ_i)) < pos_x(p_x(\succ_j))$, or
- $pos_x(p_x(\succ_i)) = pos_x(p_x(\succ_j)) = p$ and $s_x(p)(i) < s_x(p)(j)$.

In the numerical reciprocity function, $pos_x(p) = p$ for all $p, x$.

Intuitively, $x$ prefers $i$ to $j$ if $x$ is higher in $\succ_i$ than in $\succ_j$, or if the two positions are equal but $i$ has a smaller index than $j$ in $s_x(p)$.

# Possibility of Cycles

The reciprocity function $f$ is identical to the numerical reciprocity function $f^{s,<}$ with the natural order $<$ over $R = \{1, 2, 3\}$, except that:

$$2 f_z(\succ_R) 1 \quad \text{when} \begin{cases} z \succ_1 x \succ_1 y \text{ and } z \succ_2 y \succ_2 x; & \text{or} \\ z \succ_1 y \succ_1 x \text{ and } z \succ_2 x \succ_2 y \end{cases}$$

| 1 | 2 | 3 | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|
| $z$ | $z$ | $x$ | 3 | 3 | 1 |
| $x$ | $x$ | $y$ | 1 | 1 | 2 |
| $y$ | $y$ | $z$ | 2 | 2 | 3 |
| $z$ | $y$ | $x$ | | | |

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|---|---|---|---|---|---|
| $z$ | $z$ | $x$ | 3 | 3 | 1 |
| $x$ | $x$ | $y$ | 1 | 1 | 2 |
| $y$ | $y$ | $z$ | 2 | 2 | 3 |
| $z$ | $y$ | $x$ | | | |

| 1 | 2 | 3 | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|
| $z$ | $z$ | $x$ | 3 | 3 | 2 |
| $x$ | $y$ | $y$ | 1 | 1 | 1 |
| $y$ | $x$ | $z$ | 2 | 2 | 3 |
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| 1 | 2 | 3 | x | y | z |
|---|---|---|---|---|---|
| z | z | x | 3 | 3 | 1 |
| x | x | y | 1 | 1 | 2 |
| y | y | z | 2 | 2 | 3 |
| z | y | x | | | |

| 1 | 2 | 3 | x | y | z |
|---|---|---|---|---|---|
| z | z | x | 3 | 3 | 2 |
| x | y | y | 1 | 1 | 1 |
| y | x | z | 2 | 2 | 3 |
| y | z | x | | | |

| 1 | 2 | 3 | x | y | z |
|---|---|---|---|---|---|
| z | z | x | 3 | 3 | 1 |
| y | y | y | 1 | 1 | 2 |
| x | x | z | 2 | 2 | 3 |
| z | y | x | | | |

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| 1 | 2 | 3 | x | y | z |
|---|---|---|---|---|---|
| z | z | x | 3 | 3 | 1 |
| x | x | y | 1 | 1 | 2 |
| y | y | z | 2 | 2 | 3 |
| z | y | x | | | |

| 1 | 2 | 3 | x | y | z |
|---|---|---|---|---|---|
| z | z | x | 3 | 3 | 2 |
| x | y | y | 1 | 1 | 1 |
| y | x | z | 2 | 2 | 3 |
| y | z | x | | | |

| 1 | 2 | 3 | x | y | z |
|---|---|---|---|---|---|
| z | z | x | 3 | 3 | 1 |
| y | y | y | 1 | 1 | 2 |
| x | x | z | 2 | 2 | 3 |
| z | y | x | | | |

| 1 | 2 | 3 | x | y | z |
|---|---|---|---|---|---|
| z | z | x | 3 | 3 | 2 |
| y | x | y | 1 | 1 | 1 |
| x | y | z | 2 | 2 | 3 |
| y | z | x | | | |

# Convergence Result

For any numerical reciprocity function, any iterative strategic game starting from a sincere preference profile terminates in $O(n^3)$ steps.

*Proof sketch.* Show that $\varphi_i(t) \gtrsim_i^0 \varphi_i(t+1)$ for all $i \in R$, where $\varphi_i(t)$ is the best outcome $i$ could achieve by moving in round $t$.

Induction in the number of rounds $t$. Consider $t = 0$.

- For contradiction, suppose that $x = \varphi_i(1) \succ_i^0 \varphi_i(0) = y$.
- Since $y$ is a best outcome, $i$ cannot get $x$ in round 0.
- Some $j$ ranks $x$ first in round 0, and gets it: $s_x(1)(j) < s_x(1)(i)$.
- $j$ gets her truthfully preferred match and won't move in round 0.
- If $i$ is the one who moves in round 0, the statement is obvious.
- Suppose $j'$ moves in round 0 and gets matched with $z$. If $z \neq x$, then $j$ still beats $i$ over $x$; otherwise, $j'$ ranks $x$ first and beats $j$ in round 1: $s_x(1)(j') < s_x(1)(j) < s_x(1)(i)$, done!

# Axiomatic Analysis

# Anonymity of Other Universities

A reciprocity function $f$ satisfies (AOU) if for each permutation $\pi_x : U \rightarrow U$ such that $\pi_x(x) = x$ for some $x \in U$, each researcher $i \in R$, and each preference profile $\succ_R \in (\mathcal{L}_U)^n$, the following holds:

$$f_x(\succ_1, \ldots, \succ_i, \ldots, \succ_n) = f_x(\succ_1, \ldots, \succ_i^{\pi_x}, \ldots, \succ_n)$$

That is, $x$ only cares about its position in the researchers' rankings, and not about the way other universities are ranked.

# Strong Anonymity of the Researchers

A reciprocity function $f$ satisfies (SAR) if for each inversion $\sigma : R \to R$ of some researchers $i, j \in R$, each university $x \in U$, and each researcher profile $\succ_R \in (\mathcal{L}_U)^n$, the following holds:

$$f_x(\sigma(\succ_R)) = \sigma(f_x(\succ_R))$$

No reciprocity function $f$ satisfies (SAR):
If $\succ_i = \succ_j$, universities should still strictly rank $i$ and $j$.

$$\begin{array}{cc} i & j \\ \hline x & x \\ y & y \end{array} = \begin{array}{cc} j & i \\ \hline x & x \\ y & y \end{array}$$

# Anonymity of the Researchers

A reciprocity function $f$ satisfies (AR) if for each inversion $\sigma : R \rightarrow R$ of some researchers $i, j \in R$, each university $x \in U$, and each preference profile $\succ_R \in (\mathcal{L}_U)^n$ such that $\succ_i \neq \succ_j$, the following holds:

$$f_x(\sigma(\succ_R)) = \sigma(f_x(\succ_R))$$

No reciprocity function $f$ satisfies both (AR) and (AOU):
If $p_x(\succ_i) = p_x(\succ_j)$, university $x$ should still strictly rank $i$ and $j$.

# Weak Anonymity of the Researchers

A reciprocity function $f$ satisfies (WAR) if for each inversion $\sigma : R \to R$ of some $i, j \in R$, each university $x \in U$, and each preference profile $\succ_R \in (\mathcal{L}_U)^n$ s.t. $p_x(\succ_i) \neq p_x(\succ_j)$, the following holds:

$$f_x(\sigma(\succ_R)) = \sigma(f_x(\succ_R))$$

Positional reciprocity functions satisfy (WAR).
Constant reciprocity functions violate (WAR).

# Monotonicity

A reciprocity function $f$ satisfies (M) if for each university $x \in U$ and profiles $\succ_R, \succ'_R$ that only differ on $i$ and for all $y, z \in U \setminus \{x\}$ the relative ranking between all $y, z \neq x$ is the same, the following holds:

$$\{y \in U : x \succ_i y\} \subset \{y \in U : x \succ'_i y\}$$
$$\Longrightarrow \{j \in R : if_x(\succ_R)j\} \subseteq \{j \in R : if_x(\succ'_R)j\}$$

If a reciprocity function satisfies (AOU), (WAR), and (M), then

$$p_x(\succ_i) > p_x(\succ_j) \Longrightarrow if_x(\succ_R)j$$

For contradiction, suppose that $j \succ_x i$ in first profile below:

| $i$ | $j$ |
|---|---|
| $x$ | $y$ |
| $y$ | $x$ |

| $i$ | $j$ |
|---|---|
| $x$ | $x$ |
| $y$ | $y$ |

| $i$ | $j$ |
|---|---|
| $y$ | $x$ |
| $x$ | $y$ |

From (M), $j \succ_x i$ in last profile above, contradicting (WAR).

# Independence

The reciprocity function $f$ satisfies (I) if for all preference profiles $\succ_R = (\succ_1, \ldots, \succ_i, \ldots, \succ_j, \ldots, \succ_n)$ and $\succ'_R = (\succ'_1, \ldots, \succ_i, \ldots, \succ_j, \ldots, \succ'_n)$, and each university $x \in U$, the following holds:

$$i f_x(\succ_R) j \iff i f_x(\succ'_R) j$$

The constant reciprocity function satisfies (AOU), (M), and (I).

# Characterizing Numerical Reciprocity

A reciprocity function satisfies (WAR), (AOU), (M), and (I) if and only if it is a numerical reciprocity function.

*Proof sketch.*

- The numerical reciprocity function clearly satisfies the axioms.
- From AOU, WAR, and M, we know how $x$ evaluates researchers when it is ranked in different positions.
- When researchers rank $x$ in the same position, $(I)$ forces a strict order over them.

# Back to Convergence

Recall: convergence guaranteed for numerical reciprocity functions.

No strict subset of $\{(WAR), (AOU), (M), (I)\}$ suffices for convergence.

?

- We saw an example without (AOU).
- We have (less trivial) examples without (M) and (I).
- We conjecture an example without (WAR).

# Conclusions

So far, we have:

- proposed a model for ranking rankings via reciprocity functions;
- discussed applications in social choice theory;
- shown the impact on strategic behaviour in matching;
- obtained a characterisation for natural functions;
- linked axioms and convergence.

Next, we may look into:

- technical questions: convergence time, all NE, simultaneous moves, non-myopic agents, other matching mechanisms, exact characterisations, partial info;
- generalised reciprocity: fixed points, complex strategies;
- other frameworks: voting, network formation.