# Global stability for McKean–Vlasov equations on large networks

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# Motivation

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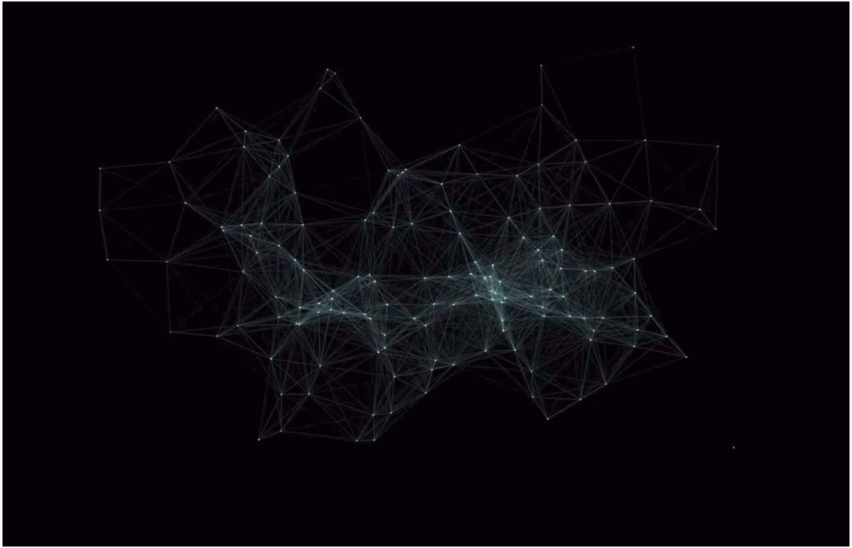
System of $N$ interacting particles described by

**stochastic McKean differential equations**

$$\mathrm{d}X_t^i = -\frac{\kappa}{N}\sum_{j\neq i}^{N} \nabla D(X_t^i - X_t^j)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \quad i = 1,\dots,N.$$

- $X_t^i$ is the $i$th particle at time $t \geq 0$,
- $X_t^i \in U := [-\frac{L}{2}, \frac{L}{2}]^d$ with periodic boundary, i.e. flat $d$-**dim torus**,
- $D : U \to \mathbb{R}$ (periodic) interaction potential, $\kappa > 0$ interaction strength,
- $B_t^i$ independent Brownian motions
- **Applications:** *Kuramoto model* of synchronization $D(x) = -\cos(\frac{2\pi x}{L})$, Hegselmann-Krause model for opinion formation, biomechanics etc.

# Motivation



System of $N$ interacting particles on network described by

**stochastic McKean differential equations**

$$dX_t^i = -\frac{\kappa}{N}\sum_{j\neq i}^{N} A_{ij}\nabla D(X_t^i - X_t^j)\,dt + \sqrt{2}\,dB_t^i, \quad i = 1,\dots,N.$$

- $X_t^i$ is the $i$th graph node at time $t \geq 0$,
- $X_t^i \in U := [-\frac{L}{2}, \frac{L}{2}]^d$ with periodic boundary, i.e. flat $d$-**dim torus**,
- $D : U \to \mathbb{R}$ (periodic) interaction potential, $\kappa > 0$ interaction strength,
- $B_t^i$ independent Brownian motions
- $A_{ij} \in \{0,1\}$ describes graph edges.
- **Applications:** E.g. opinion formation on social networks.

## Overview

**Goal:**
Show exponential decay for *McKean–Vlasov equations* which describe interacting particles on large network/graph structures in the mean-field limit.

**Method:**
Incorporate graph operators as graph limits in the Vlasov interaction term and apply the entropy method to show decay.

# Outline

## Part 1: Graph limit theory

- Dense graphs and graphons
- The problem of sparse graphs
- Graphops

## Part 2: McKean–Vlasov equations on large networks

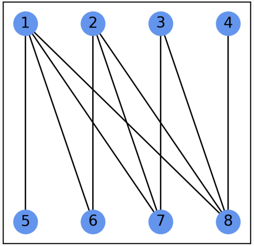
- Existence
- All-to-all coupling
- Global stability via entropy methods

# Part 1: Graph limit theory

# Graph limits

Graph $\mathcal{G}^N = (\mathcal{V}^N, \mathcal{E}^N)$, Adjacency matrix $A^N := (A_{ij})_{i,j=1\dots,N}$

**Question:** How to formulate $\lim_{N\to\infty} A^N$?

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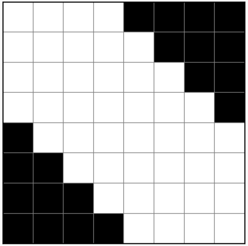
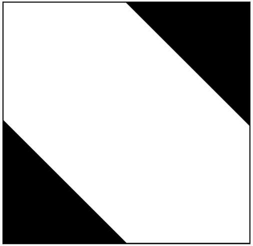




# Graph limits

$\mathcal{G}^N = (\mathcal{V}^N, \mathcal{E}^N)$, Adjacency matrix $A^N := (A^{ij})_{i,j=1\dots,N}$.

**Question:** How to formulate $\lim_{N\to\infty} A^N$?

- **Dense graphs** $|\mathcal{E}^N| \approx |\mathcal{V}^N|^2$:
  Graphons, symm. and measurable $W(\cdot,\cdot) : [0,1]^2 \to [0,1]$
  [Borgs, Chayes et al. 2008, 2012] introduce cut-metric
- **Intermediate graphs** $|\mathcal{V}^N| \ll |\mathcal{E}^N| \ll |\mathcal{V}^N|^2$:
  $L^p$-graphons (e.g. power-law graphs) as rescaled graphon convergence
  [Borgs, Chayes, et al. 2019]
- **Sparse graphs** $|\mathcal{E}^N| \leq |\mathcal{V}^N|$:
  - Bounded degree: measure-based *graphings* [Benjamini-Schramm, 2001]
  - Bounded average degree
  - ...

# Graph limits

**Shortcomings of mentioned limit theories:**

- Graphons, $L^p$-graphons, graphings require separate convergence theories.
- Many sparse and intermediate graphs not included

**Solution:** **Graphops** and *action convergence* – Unifying graph limit theory. [Backhausz, Szegedy 2018]

# Graphops

**Question:** How can we compare two graphs $\mathcal{G}_1^{N_1}$ and $\mathcal{G}_2^{N_2}$ if $N_1 \neq N_2$?

**Solution:** Describe each graph $\mathcal{G}^N$ via "actions" of adjacency matrix $A^N$:

- take an arbitrary column vector $v \in \mathbb{R}^N$, compute $A^N v$.
- consider the matrix $M_v := [v, A^N v] \in \mathbb{R}^{N \times 2}$.
- for fixed $v$, sample rows of $M_v$ uniformly $\implies$ generates prob. measure $\rho_v^N$ on $\mathbb{R}^2$ (random matrix theory).
- Graph $\mathcal{G}^N$ represented as family of measures $\{\rho_v^N : v \in \mathbb{R}^N\}$ on $\mathbb{R}^2$.

**For $N \to \infty$: Limit via convergence of probability measures.** $\implies$ Limiting measure family represents operator.

Graphs as operators $\implies$ rather general graph limit theory based on Borel probability space $(\Omega, \mathcal{A}, \mu)$.

**Definition**

An operator $A : L^\infty(\Omega) \to L^1(\Omega)$ is called a **graphop** if it is linear, bounded, self-adjoint (w.r.t. $(\cdot,\cdot)_{L^2(\Omega)}$) and positivity preserving.

**Example**

Let $\Omega = [0,1]$ and $W : [0,1]^2 \to \mathbb{R}$, $L^p$-graphon, $p \in [1,\infty]$.

$(A_W\rho)(\xi) := \int_{[0,1]} W(\xi,\tilde{\xi})\rho(\tilde{\xi})\mathrm{d}\tilde{\xi}$ satisfies $\|A_W\|_{\infty\to 1} \leq \|W\|_{L^p([0,1]^2)}$

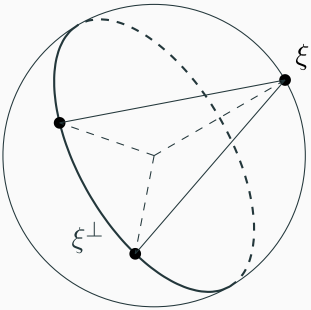
**Example**

Finite graph $\mathcal{G}^N \to$ adjacency matrix $A^N \to$ step function graphon $\to$ integral operator.

**Example**

The **spherical graphop**:

$$A_{\mathbb{S}} : L^2(\mathbb{S}^2, \mu) \to L^2(\mathbb{S}^2, \mu),$$
$$(A_{\mathbb{S}}\rho)(\xi) := \int_{\xi^\perp} \rho \, \mathrm{d}\nu_\xi(\tilde{\xi}),$$




- $\mu$ is the uniform probablity measure on $\mathbb{S}^2$
- $\xi^\perp := \{\tilde{\xi} \in \mathbb{S}^2 \mid \xi^T\tilde{\xi} = 0\}$.
- $\nu_\xi$ is the uniform probability measure on the (1-dim) submanifold $\xi^\perp$.

Not a graphing (degree not bounded),
not yet a graphon (not dense enough)!

Analytical language of graphons and graphops well suited to be incorporated into dynamical framework.

**Graphons:**

- “The mean field analysis for the **Kuramoto model** on graphs” [Chiba, Medvedev 2019]
- “**Consensus Formation** in First-Order Graphon Models with Time-Varying Topologies” [Bonnet, Duteil, Sigalotti]

**Graphops:**

- “Network dynamics on graphops” [Kuehn 2020]
- “Graphop mean-field limits for Kuramoto-type models” [Gkogkas, Kuehn 2022]
- “Vlasov equations on digraph measures” [Kuehn, Xu 2022]

# Part 2: Graphop McKean–Vlasov equations

$$\mathrm{d}X_t^i = -\frac{\kappa}{N}\sum_{j\neq i}^{N} A_{ij}\nabla_x D(X_t^i - X_t^j)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t^i, \quad i = 1,\dots,N.$$

Mean-field formulation $N \to \infty$ of (SDE) formally leads to

**graphop McKean–Vlasov equation**

$$\partial_t \rho = \kappa\,\mathrm{div}_x(\rho V[A](\rho)) + \Delta_x \rho, \tag{MVE}$$

$$V[A](\rho)(t,x,\xi) := \int_U \nabla_x D(x-\tilde{x})(A\rho)(t,\tilde{x},\xi)\mathrm{d}\tilde{x}.$$

- $\rho(t,x,\xi)$ depends on time $t$, $x \in U := [-\frac{L}{2},\frac{L}{2}]^d$ flat torus, $\xi \in \Omega$ with prob. space $(\Omega,\mathcal{A},\mu)$.
- $A$ is a graphop acting on graph variable function.
- $D \in \mathcal{W}^{2,\infty}(U)$, e.g. $D(x) = -\cos((\frac{2\pi}{L})x)$.
- $\kappa > 0$ relative interaction strength.

# Existence of classical solutions

$$\begin{cases} \partial_t \rho = \kappa \operatorname{div}_x(\rho V[A](\rho)) + \Delta_x \rho, & t \geq 0, \\ \rho(0) = \rho_0. \end{cases}$$

**Def.**: $\rho_0$ is *admissible* if, for a.e. $\xi \in \Omega$, $\rho_0(\cdot, \xi) \in H^{3+d}(U) \cap \mathcal{P}_{ac}(U)$ and for each $x \in U$, $\rho_0(x, \cdot) \in L^\infty(\Omega, \mu)$.

**Proposition**

*Let $\rho_0$ be admissible and $A$ a graphop. Then $\rho(\cdot, \cdot, \xi)$ is a unique classical solution for a.e. $\xi \in \Omega$ and $\rho(t, \cdot, \cdot) \in \mathcal{P}_{ac}(U \times \Omega)$. If $(A1_\Omega)(\xi) \leq C$, then $\rho(t, x, \cdot) \in L^\infty(\Omega)$.*

$$\|A\|_{p \to q} := \sup_{v \in L^\infty(\Omega)} \frac{\|Av\|_{L^q(\Omega)}}{\|v\|_{L^p(\Omega)}}, \quad p, q \in [1, \infty].$$

- $\xi$-regularity expected to be improvable $\to$ Open Problem.
  E.g. $\|A\|_{2\to 2} < \infty \implies \rho(t, x, \cdot) \in L^2(\Omega), t > 0$.
- Difficulty: No direct regularization effect in $\xi$ variable.

# Entropy method

**All-to-all coupling:**

$A\rho = \rho$, $\rho(t,x,\xi) \simeq \rho(t,x)$ and $V[A](\rho) = \nabla_x D \star \rho$.

**Relative entropy functional** with steady state $\rho_\infty := \frac{1}{L^d}$

$$H(\rho|\rho_\infty) := \int_U \rho \log\left(\frac{\rho}{\rho_\infty}\right) \, \mathrm{d}x$$

- $H(\rho_\infty|\rho_\infty) = 0$
- *Csiszár-Kullback-Pinsker* inequality

$$\|\rho - \rho_\infty\|^2_{L^1(U)} \leq 2H(\rho|\rho_\infty). \tag{CKP}$$

- *log-Sobolev inequality*

$$H(\rho|\rho_\infty) \leq \frac{L^2}{4\pi^2} \int_U |\nabla_x \log(\rho)|^2 \rho \, \mathrm{d}x. \tag{Sob}$$

**Idea:** $\partial_t H(\rho(t)|\rho_\infty) \leq -\alpha H(\rho(t)|\rho_\infty)$ for solutions of (MVE).

# All-to-all coupling

**Proposition (Carrillo, Gvalani et al. 2020)**

*Let* $\rho_0 \in H^{3+d}(U) \cap \mathcal{P}_{ac}(U)$ *and* $H(\rho_0|\rho_\infty) < \infty$. *If*

$$0 < \kappa < \frac{2\pi^2}{L^2 \|\Delta_x D\|_{L^\infty(U)}},$$

*then the solution* $\rho$ *is exponentially converging to* $\rho_\infty := \frac{1}{L^d}$ *with decay estimate*

$$H(\rho(t)|\rho_\infty) \le \mathrm{e}^{-\alpha t} H(\rho_0|\rho_\infty), \quad t \ge 0,$$

*where*

$$\alpha := \frac{4\pi^2}{L^2} - 2\kappa \|\Delta_x D\|_{L^\infty(U)} > 0.$$

# All-to-all coupling

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\rho(t)|\rho_\infty) = \int_U \Big(\Delta_x\rho + \kappa\nabla_x(\rho\nabla_x(D\star\rho))\Big)\log(\frac{\rho}{\rho_\infty})\,\mathrm{d}x$$
$$+ \underbrace{\int_U \rho\frac{\rho_\infty}{\rho}\frac{1}{\rho_\infty}\Big(\Delta_x\rho + \kappa\nabla_x(\rho\nabla_x(D\star\rho))\Big)\,\mathrm{d}x}_{=0}$$
$$= -\int_U |\nabla_x\rho|^2\frac{1}{\rho}\,\mathrm{d}x - \kappa\int_U \rho\nabla_x(D\star\rho)(\frac{\rho_\infty}{\rho})\nabla_x(\frac{\rho}{\rho_\infty})\,\mathrm{d}x$$
$$= -\int_U |\nabla_x\log(\rho)|^2\rho dx + \kappa\int_U \rho(\Delta_x D\star\rho)\,\mathrm{d}x.$$

**First term:** log-Sobolev inequality.

**Second term:** Replace both $\rho$ by $\rho-\rho_\infty$ then Hölder and (CKP)

$$\frac{d}{dt}H(\rho(t)|\rho_\infty) \leq (-\frac{4\pi^2}{L^2} + 2\kappa\|\Delta_x D\|_\infty)H(\rho(t)|\rho_\infty).$$

□

# Entropy method for graphop case

**Definition**

For the probability space $(\Omega, \mathcal{A}, \mu)$ the **relative entropy** for heterogeneous couplings is chosen as

$$\hat{H}(\rho|\rho_\infty) := \int_\Omega \int_U \rho \log\left(\frac{\rho}{\rho_\infty}\right) \mathrm{d}x \, \mathrm{d}\mu(\xi).$$

**Definition**

The **numerical radius** of a graphop $\|A\|_{2\to 2} < \infty$ is given as

$$n(A) := \sup\{(Af, f)_{L^2} \mid f \in L^2(\Omega), \|f\|_{L^2(\Omega)} = 1\}.$$

It holds that $n(A) \leq \|A\|_{2\to 2} \leq 2n(A)$.

**Theorem (Kuehn, W. 2023)**

*Let* $\rho_0$ *be admissible and satisfy* $\hat{H}(\rho_0|\rho_\infty) < \infty$. *Let the graphop* $A$ *satisfy* $n(A) < \infty$. *If*

$$\kappa < \frac{2\pi^2}{L^2\|\Delta_x D\|_{L^\infty(U)} n(A)},$$

*then, the solution to (MVE) is exponentially converging to* $\rho_\infty := \frac{1}{L^d}$ *with estimate*

$$\hat{H}(\rho(t)|\rho_\infty) \leq \mathrm{e}^{-\hat{\alpha}(A)t}\hat{H}(\rho_0|\rho_\infty), \quad t \geq 0,$$

*where*

$$\hat{\alpha}(A) := \frac{4\pi^2}{L^2} - 2\kappa\|\Delta_x D\|_{L^\infty(U)} n(A) > 0.$$

**Remark:** If $A$ has graphon kernel $W$ with $\|W\|_2 \leq \infty$. Then we can replace $n(A) = \|W\|_2$.

**Proof.**
As in all-to-all case:

$$\frac{d}{dt}\hat{H}(\rho|\rho_\infty) \leq -\frac{4\pi^2}{L^2}\hat{H}(\rho|\rho_\infty) + \kappa \int_\Omega \int_U \rho[\Delta_x D \star (A\rho)] \, \mathrm{d}x \, \mathrm{d}\mu(\xi).$$

**Second term:** $A$ is a linear bounded operator acting solely on the network variable $\xi \implies \Delta_x D \star (A\rho) = A\,(\Delta_x D \star \rho)$.

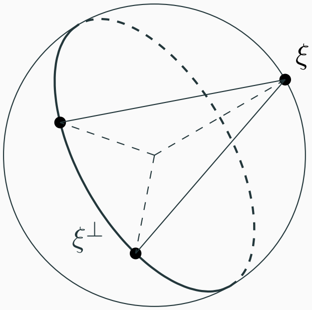
$$\begin{aligned}
&\kappa \int_\Omega \int_U \rho(\Delta_x D \star (A\rho)) \, \mathrm{d}x \, \mathrm{d}\mu(\xi) \\
&\leq \kappa \|\Delta_x D\|_{L^\infty(U)} \int_\Omega (A\|\rho - \rho_\infty\|_{L^1(U)} \|\rho - \rho_\infty\|_{L^1(U)} \, \mathrm{d}\mu(\xi) \\
&\leq \kappa \|\Delta_x D\|_{L^\infty(U)} n(A) \int_\Omega \|\rho - \rho_\infty\|^2_{L^1(U)} \, \mathrm{d}\mu(\xi) \\
&\overset{(\mathrm{CKP})}{\leq} 2\kappa \|\Delta_x D\|_{L^\infty(U)} n(A) \hat{H}(\rho|\rho_\infty)
\end{aligned}$$

$$\implies \frac{d}{dt}\hat{H}(\rho(t)|\rho_\infty) \leq -\hat{\alpha}(A)\hat{H}(\rho(t)|\rho_\infty), \quad t \geq 0.$$

**Remark:** More general graphops $\|A\|_{p \to p^*} < \infty$ do not work with our method. □
Ineq. cannot be closed!

**Example**

Let $A_{\mathbb{S}}$ be the **spherical graphop**. One can show that $\|A_{\mathbb{S}}\|_{2\to 2} = 1$ and hence Theorem 7 yields decay of solutions of (MVE) with explicit rate




$$\hat{\alpha}(A_{\mathbb{S}}) := \frac{4\pi^2}{L^2} - 2\kappa\|\Delta_x D\|_{L^\infty(U)} > 0, \quad \text{provided} \quad \kappa < \frac{2\pi^2}{L^2\|\Delta_x D\|_{L^\infty(U)}}.$$

## Sakaguchi–Kuramoto model with frequency distribution

$$\partial_t \rho = \partial_x(-\omega\rho + \kappa\rho V[A, g](\rho)) + \partial_{xx}\rho, \quad t \geq 0,$$

$$\rho(0) = \rho_0, \qquad V[A, g](\rho) := \int_{\mathbb{R}} (\sin * A\rho)\, g d\omega.$$

- $\rho(t, x, \xi, \omega)$.
- Phase $x \in \mathbb{T}^1 \simeq [-\pi, \pi]$, intrinsic frequency $\omega \in \mathbb{R}$.
- $g$ is frequency density function determined via $\rho_0$.
- Vlasov term also dependent on $g$.

Result of Theorem 7 extends to this equation for arbitrary $\|g\|_{L^1(dw)} = 1$ using

$$\overline{H}(\rho|\rho_\infty) := \int_{\mathbb{R}} \int_U \rho \log(\frac{\rho}{\rho_\infty}) dx g d\omega.$$

Decay rate independent of $g$ (but not sharp).

# Conclusion

- Introduced **graph limit theory of graphops and action convergence** which is able to deal with dense, intermediate and sparse graph structures in an analytical framework.
- **Incorporated graphops into McKean–Vlasov equations, including the Sakaguchi–Kuramoto model,** to express coupling for a wide range of graph structures in the mean-field limit.
- **Extended the entropy method to show global stability of chaotic steady state** for $L^2$ graphops under weak coupling.

# Thank you for your attention.

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