Entrywise preservers beyond Schoenberg

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Two matrix products

Notation

The set of $d \times d$ matrices with entries in a set $K \subseteq \mathbb{C}$ is denoted $M_d(K)$.

Matrix multiplication

The vector space $M_d(\mathbb{C})$ is an associative algebra for at least two different products.

If $A = (a_{ij})_{i,j=1}^d$ and $B = (b_{ij})_{i,j=1}^d$ then

$$\begin{aligned} (AB)_{ij} &:= \sum_{k=1}^{d} a_{ik} b_{kj} \quad \text{(standard)} \\ \text{and} \quad (A \circ B)_{ij} &:= a_{ij} b_{ij} \quad \text{(Hadamard).} \end{aligned}$$

Positive definiteness

Definition

A matrix $A \in M_d(\mathbb{C})$ is *positive semidefinite* if

$$\mathbf{x}^* A \mathbf{x} = \sum_{i,j=1}^{d} \overline{x_i} a_{ij} x_j \geq 0 \qquad \text{for all } \mathbf{x} \in \mathbb{C}^d.$$

A matrix $A \in M_d(\mathbb{C})$ is *positive definite* if

$$\mathbf{x}^* A \mathbf{x} = \sum_{i,j=1}^{d} \overline{x_i} a_{ij} x_j > 0 \qquad \text{for all } \mathbf{x} \in \mathbb{C}^d \setminus \{0\}.$$

Remark

The collection $M_d(\mathbb{C})_+$ of $d \times d$ positive semidefinite matrices with complex entries is a *closed cone*: stable under addition, positive homotheties and pointwise limits.

Some consequences – I

Symmetry

If $A \in M_d(K)_+$ then $A^T \in M_d(K)_+$: note that $(CD)^T = D^T C^T$ and so

$$0 \le \mathbf{x}^* A \mathbf{x} = (\mathbf{x}^* A \mathbf{x})^T = \mathbf{y}^* A^T \mathbf{y} \qquad \text{for all } \mathbf{y} = \bar{\mathbf{x}} \in \mathbb{C}^d.$$

Hermitianity

If $A \in M_d(\mathbb{C})_+$ then $A = A^*$: note that $(CD)^* = D^* C^*$ and so

$$\mathbf{x}^* A \mathbf{x} = (\mathbf{x}^* A \mathbf{x})^* = \mathbf{x}^* A^* \mathbf{x} \quad (\mathbf{x} \in \mathbb{C}^d) \quad \Longrightarrow \quad A = A^* \quad \text{by } \textit{polarisation}:$$

look at

$$(\mathbf{x}+\mathbf{y})^* A(\mathbf{x}+\mathbf{y}) - (\mathbf{x}-\mathbf{y})A(\mathbf{x}-\mathbf{y}) \\ + i(\mathbf{x}+i\mathbf{y})^* A(\mathbf{x}+i\mathbf{y}) - i(\mathbf{x}-i\mathbf{y})^* A(\mathbf{x}-i\mathbf{y}).$$

Some consequences – II

Non-negative eigenvalues

If $A \in M_d(\mathbb{C})_+$ then $A = A^*$, so we can find a unitary matrix $U \in M_d(\mathbb{C})$ such that

$$A = U^* \operatorname{diag}(\lambda_1, \ldots, \lambda_d) U.$$

We must have that $\lambda_i = (U^* \mathbf{e}_i)^* A (U^* \mathbf{e}_i) \geq 0$ for all i, so $\sigma(A) \subseteq \mathbb{R}_+$.

Conversely, if $A \in M_d(\mathbb{C})$ is Hermitian and has non-negative eigenvalues then

$$A = B^* B \in M_d(\mathbb{C})_+,$$

where

$$B = A^{1/2} = U^* \operatorname{diag}\big(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_d}\big) U.$$

Some consequences – III

Theorem

Let $A \in M_d(\mathbb{C})$. The following are equivalent.

- $\mathbf{z}^* A \mathbf{z} \geqslant 0$ for all $\mathbf{z} \in \mathbb{C}^d$.
- $A = A^*$ and $\sigma(A) \subseteq \mathbb{R}_+$.
- $A = U \operatorname{diag}(\lambda_1, \ldots, \lambda_d) U^*$, with $\boldsymbol{\lambda} \in \mathbb{R}_+^d$ and $U^* U = UU^* = I$.
- $A = \sum_{i=1}^d \lambda_i \mathbf{v}_i \mathbf{v}_i^*$, where $\lambda_i \geqslant 0$ and $\mathbf{v}_i^* \mathbf{v}_j = \mathbb{1}_{i=j}$ for all i, j.
- $A = B^* B$, where $B \in M_n(\mathbb{C})$.

If these conditions hold, then A is positive semidefinite and we write $A \geqslant 0$.

The Schur product theorem

Theorem 1 (Schur, 1911)

If A, $B \geqslant 0$ then $A \circ B \geqslant 0$, where the Hadamard product is such that

$$(A \circ B)_{ij} := a_{ij} b_{ij} \qquad \text{for all } i, j.$$

Proof 1.

Note that A^T is positive semidefinite whenever A is, and so

$$\begin{aligned} \mathbf{z}^*(A \circ B)\mathbf{z} &= \operatorname{tr}(\operatorname{diag}(\mathbf{z})^* B \operatorname{diag}(\mathbf{z}) A^T) \\ &= \operatorname{tr}\big((A^T)^{1/2} \operatorname{diag}(\mathbf{z})^* B \operatorname{diag}(\mathbf{z}) (A^T)^{1/2}\big). \end{aligned}$$

□

Proof 2.

Note that

$$(\mathbf{u}\mathbf{u}^*) \circ (\mathbf{v}\mathbf{v}^*) = (\mathbf{u} \circ \mathbf{v})(\mathbf{u} \circ \mathbf{v})^*,$$

so the rank-one case holds, and use linearity. □

A question of Pólya and Szegö

Corollary 2

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$*, with* $c_n \geqslant 0$ *for all* n*, then*

$$f[A] := \sum_{n=0}^{\infty} c_n A^{\circ n} = \begin{pmatrix} f(a_{11}) & \cdots & f(a_{1d}) \\ \vdots & \ddots & \vdots \\ f(a_{d1}) & \ldots & f(a_{dd}) \end{pmatrix} \geqslant 0$$

for any $A \in M_d(\mathbb{C})_+$ *such that* $f(a_{ij})$ *is well defined for all* i, j*.*

Observation

A function f with such a power-series expansion is said to be *absolutely monotonic*. Absolutely monotonic functions preserve positivity for square matrices of arbitrary size.

Question (Pólya–Szegö, 1925)

Are there any other functions with this property?

Motivation – isometric embeddings of metric spaces

Theorem 3 (Schoenberg after Fréchet and Menger)

Let (X, ρ) be a metric space containing the points $x_0, \ldots, x_n$. These points can be isometrically embedded into Euclidean space $\mathbb{R}^d$ but not into $\mathbb{R}^{d-1}$ if and only if the matrix

$$D[x_0, \ldots, x_n] := \left[\rho(x_i, x_0)^2 + \rho(x_0, x_j)^2 - \rho(x_i, x_j)^2\right]_{i,j=1}^n$$

is positive semidefinite with rank d.

Corollary 4

A separable metric space (X, ρ) can be isometrically embedded into the Hilbert space ℓ^2 if and only if, given any collection of points $x_0, \ldots, x_n$, where $n \geqslant 2$, the matrix $D[x_0, \ldots, x_n]$ is positive semidefinite.

Motivation – isometric spherical embeddings

Let the $d-1$-dimensional unit sphere

$$\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^T\mathbf{x} = 1\}$$

be equipped with the geodesic distance

$$\angle(\mathbf{x}, \mathbf{y}) = \arccos \mathbf{x}^T\mathbf{y} \in [0, \pi].$$

Theorem 5 (Schoenberg after Fréchet and Menger)

Let (X, ρ) be a metric space containing the points, $x_1, \ldots, x_n$. For any integer $d \geqslant 2$, these points may be isometrically embedded into $\mathbb{S}^{d-1}$ equipped with the geodesic distance but not $\mathbb{S}^{d-2}$ if and only

$$\rho(x_i, x_j) \leqslant \pi \quad (1 \leqslant i, j \leqslant n)$$

and the matrix

$$\big[\cos \rho(x_i, x_j)\big]_{i,j=1}^{n}$$

is positive semidefinite of rank d.

Schoenberg's theorem

Theorem 6 (Schoenberg, 1942)

If $f : [-1,1] \to \mathbb{R}$ is

(i) *continuous and*

(ii) *such that $f[A] \geqslant 0$ for every $A \geqslant 0$ with entries in $[-1,1]$ and of any size,*

then f is absolutely monotonic:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{for all } x \in [-1,1], \quad \text{where } c_n \geqslant 0 \text{ for all } n.$$

Theorem 7 (Rudin, 1959)

On $(-1,1)$, the hypothesis of continuity is not required in Schoenberg's result.

Vasudeva's theorem

Characterising entrywise positivity preservers in fixed dimension is a significant challenge.

The only complete characterisations known hold for dimensions 1 and 2.

Theorem 8 (Vasudeva, 1979)

Let $f : (0, \infty) \to \mathbb{R}$. Then $f[A] \geqslant 0$ for every matrix $A \in M_2\big((0, \infty)\big)_+$ if and only if the function f is

(i) *non-negative,*

(ii) *non-decreasing and*

(iii) *multiplicatively midpoint convex, that is,*

$$f(\sqrt{xy})^2 \leqslant f(x)f(y) \qquad \text{for all } x, y > 0.$$

In particular, such a function f is continuous.

Horn's theorem

Theorem 9 (Horn, 1969)

If $f : (0,\infty) \to \mathbb{R}$ *is continuous and such that* $f[A] \geqslant 0$ *for every* $d \times d$ *positive-semidefinite matrix* A*, where* $d \geqslant 3$*, then*

(i) f *is* $d-3$*-times continuously differentiable and*

(ii) $f^{(k)}(x) \geqslant 0$ *for all* $k = 0, \ldots, d-3$ *and all* $x > 0$*.*

If, further, f *is* $d-1$*-times differentiable, then*

$$f^{(k)}(x) \geqslant 0 \qquad \textit{for } k = 0, \ldots, d-1 \textit{ and all } x > 0.$$

Observation (Guillot–Khare–Rajaratnam)

To obtain Horn's result, it suffices to consider only matrices having the form $A = a\mathbf{1} + \mathbf{u}\mathbf{u}^T$, where $a > 0$ and $u_i \geqslant 0$ for all i.

Positivity of Maclaurin coefficients

Proposition 10 (B–G–K–P, 2016)

If $f : D(0,\rho) \to \mathbb{R}$ is analytic, where $\rho > 0$, and such that $f[A] \geqslant 0$ whenever $A \in M_d\big((0,\rho)\big)_+$ has rank one then the first d non-zero Maclaurin coefficients of the function f are strictly positive.

Proof.

Suppose the first d non-zero Maclaurin coefficents are $c_{n_1}, \ldots, c_{n_d}$. Let $\mathbf{u}^T = (u_1, \ldots, u_d)$ have distinct entries and note that $(u_i^{n_j})$ is a generalised Vandermonde matrix, so invertible. Hence $\{\mathbf{u}^{\circ n_1}, \ldots, \mathbf{u}^{\circ n_d}\}$ is linearly independent; let $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\} \subseteq \mathbb{R}^d$ be such that $\mathbf{v}_i^T \mathbf{u}^{\circ n_j} = \mathbb{1}_{i=j}$. Then, for any i,

$$0 \leqslant \varepsilon^{-n_i} \mathbf{v}_i^T f[\varepsilon \mathbf{u}\mathbf{u}^T] \mathbf{v}_i = c_{n_i} + \sum_{j > n_i} c_j \varepsilon^{j - n_i} (\mathbf{v}_i^T \mathbf{u}^{\circ j})^2 \to c_{n_i} \text{ as } \varepsilon \to 0+. \quad \square$$

Positivity preservers in fixed dimension

Theorem 11 (B–G–K–P, 2016)

Let $\rho > 0$, $d \geqslant 1$, $\mathbf{c} = (c_0, \ldots, c_{d-1})^T \in \mathbb{R}^d$, $c' \in \mathbb{R}$, $m \geqslant 0$ *and*

$$f(z) = \sum_{i=0}^{d-1} c_i z^i + c' z^m,$$

The following are equivalent.

(i) $f[A] \geqslant 0$ *for every* $A \in M_d\big(\overline{D}(0,\rho)\big)_+$.

(ii) *Either* $\mathbf{c} \in \mathbb{R}_+^d$ *and* $c' \in \mathbb{R}_+$, *or* $\mathbf{c} \in (0,\infty)^d$ *and* $c' \geqslant -\mathfrak{C}(\mathbf{c}; m, \rho)^{-1}$, *where*

$$\mathfrak{C}(\mathbf{c}; m, \rho) := \sum_{j=0}^{d-1} \binom{m}{j}^2 \binom{m-j-1}{d-j-1}^2 \frac{\rho^{m-j}}{c_j}.$$

(iii) $f[A] \geqslant 0$ *for every* $A \in M_d\big((0,\rho)\big)_+$ *with rank at most one.*

Steps towards the proof of Theorem 11 – I

Let

$$f(x) = c_0 x^{n_0} + \cdots + c_{d-1} x^{n_{d-1}} + c' x^M,$$

where the non-negative integers $n_0 < \cdots < n_{d-1} < M$ and the real coefficients $c_0, \ldots, c_{d-1} > 0$ and $c' < 0$.

Let $t = -1/c'$ and consider

$$p_t(x) := t \sum_{j=0}^{d-1} c_j x^{n_j} - x^M.$$

What is the smallest t such that $p_t[-]$ preserves positivity on the cone $M_d\big((0,\infty)\big)_+$?

A trick due to FitzGerald and Horn means that we can focus on the rank-one case.

Steps towards the proof of Theorem 11 – II

Proposition 12

Let $\mathbf{u} \in (0, \infty)^d$ *have distinct coordinates. The following are equivalent.*

(i) *The matrix* $p_t[\mathbf{u}\mathbf{u}^T]$ *is positive semidefinite.*

(ii) *The determinant* $\det p_t[\mathbf{u}^T\mathbf{u}] \geqslant 0$*.*

(iii) *We have that*

$$t \geqslant \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u})^2}{c_j s_{\mathbf{n}}(\mathbf{u})^2},$$

where the Schur polynomial

$$s_{\mathbf{n}}(\mathbf{u}) := \frac{\det \mathbf{u}^{\circ \mathbf{n}}}{\det \mathbf{u}^{\circ \mathbf{n}_{\min}}} = \frac{\det(u_i^{n_{j-1}})}{\det(u_i^{j-1})}$$

and

$$\mathbf{n}_j := (n_0, \ldots, \widehat{n_j}, \ldots, n_{d-1}, M).$$

Steps towards the proof of Theorem 11 – III

Proposition 13 (Khare and Tao, 2021)

Let $\mathbf{m} = (m_0 < \cdots < m_{d-1})$ *and* $\mathbf{n} = (n_0 < \cdots < n_{d-1})$ *be* d*-tuples of non-negative integers with* $m_i \leqslant n_i$ *for* $i = 0, \ldots, d-1$*. The function*

$$f : (0, \infty)^d \to \mathbb{R}; \ \mathbf{u} \mapsto \frac{s_{\mathbf{n}}(\mathbf{u})}{s_{\mathbf{m}}(\mathbf{u})}$$

is non-decreasing in each coordinate.

Moreover, the *Weyl dimension formula* gives that

$$s_{\mathbf{n}}\big((1, \ldots, 1)^T\big) = \prod_{0 \leqslant i < j \leqslant d-1} \frac{n_j - n_i}{j - i} = \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})},$$

where $V(\mathbf{n})$ is a Vandermonde determinant.

Hankel matrices

Moment matrices

Let μ be a measure on $\mathbb{R}$ with moments of all orders, and let

$$s_n = s_n(\mu) := \int_R x^n \mu(\mathrm{d}x) \qquad (n \geqslant 0).$$

The *Hankel matrix associated with* μ is

$$H_\mu := \begin{pmatrix} s_0 & s_1 & s_2 & \cdots \\ s_1 & s_2 & s_3 & \cdots \\ s_2 & s_3 & s_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (s_{i+j})_{i,j=0}^{\infty}.$$

The Hamburger moment problem

Theorem 14 (Hamburger, 1920)

A sequence $(s_n)_{n \geqslant 0}$ is the moment sequence for a positive Borel measure on $\mathbb{R}$ if and only if the associated Hankel matrix is positive semidefinite.

Corollary 15

A map f preserves positivity when applied entrywise to Hankel matrices if and only if it maps moment sequences to themselves: given any positive Borel measure μ,

$$f\big(s_n(\mu)\big) = s_n(\nu) \qquad (n \geqslant 0)$$

for some positive Borel measure ν.

Preserving positivity for Hankel matrices - I

Theorem 16 (B–G–K–P, 2016)

Let $f : \mathbb{R} \to \mathbb{R}$. The following are equivalent.

1. *The function f maps the set of moment sequences of measures supported on $[-1, 1]$ into itself.*
2. *$f[A] \geqslant 0$ whenever A is a positive-semidefinite Hankel matrix of any size.*
3. *$f[A] \geqslant 0$ whenever A is a positive-semidefinite matrix of any size.*
4. *The function f is absolutely monotonic:*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{for all } x \in \mathbb{R},$$

with $c_n \geqslant 0$ for all n.

Preserving positivity for Hankel matrices - II

Theorem 17 (B–G–K–P, 2016)

Let $f : \mathbb{R} \to \mathbb{R}$. Then f maps the set of moment sequences of measures supported on $[0,1]$ into itself if and only if f is absolutely monotonic on $(0,\infty)$ and $0 \leqslant f(0) \leqslant \lim_{\varepsilon \to 0+} f(\varepsilon)$.

Theorem 18 (B–G–K–P, 2016)

Let $f : \mathbb{R} \to \mathbb{R}$. Then f maps the set of moment sequences of measures supported on $[-1,0]$ into the set of moment sequences of measures supported on $(-\infty,0]$ if and only if there exists an absolutely monotonic entire function $F : \mathbb{C} \to \mathbb{C}$ such that

$$f(x) = \begin{cases} F(x) & \text{if } 0 < x < \infty, \\ 0 & \text{if } x = 0, \\ -F(-x) & \text{if } -\infty < x < 0. \end{cases}$$

Multi-variable preservers

A function $f : \mathbb{R}^m \to \mathbb{R}$ acts entrywise on m-tuples of matrices as follows: if $B^{(p)} = (b_{ij}^{(p)})$ is an $n \times n$ matrix for $p = 1, \ldots, m$ then the $n \times n$ matrix $f[B^{(1)}, \ldots, B^{(m)}]$ has (i, j) entry

$$f[B^{(1)}, \ldots, B^{(m)}]_{ij} = f(b_{ij}^{(1)}, \ldots, b_{ij}^{(m)}) \qquad \text{for all } i, j \in [1 : n].$$

Theorem 19 (FitzGerald, Micchelli and Pinkus)

The function $f : \mathbb{R}^m \to \mathbb{R}$ acts entrywise to send m-tuples of positive semidefinite matrices with entries in I of arbitrary size to the set of positive semidefinite matrices if and only if f is represented on $\mathbb{R}^m$ by a convergent power series with non-negative coefficients:

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \qquad \textit{for all } \mathbf{x} \in \mathbb{R}^m, \ \textit{where } c_{\boldsymbol{\alpha}} \geqslant 0 \textit{ for all } \boldsymbol{\alpha} \in \mathbb{Z}_+^m.$$

Matrices with negative eigenvalues

Some notation

Let $I \subseteq \mathbb{R}$ and set

$$\mathcal{S}_n^{(k)}(I) := \{A \in M_n(I) : A = A^T \text{ has exactly } k \text{ negative eigenvalues}\}.$$

We count eigenvalues with multiplicity.
The sets

$$\mathcal{S}_n^{(0)}(I), \quad \mathcal{S}_n^{(1)}(I), \quad \ldots, \quad \mathcal{S}_n^{(n)}(I)$$

form a partition the set of $n \times n$ real symmetric matrices with entries in I.
We let

$$\mathcal{S}^{(k)}(I) := \bigcup_{n=k}^{\infty} \mathcal{S}_n^{(k)}(I)$$

be the set of real symmetric matrices of arbitrary size with entries in I and exactly k negative eigenvalues.

Inertia preservers

Theorem 20

Let $I := (-\rho, \rho)$, where $0 < \rho \leqslant \infty$, and let k be a non-negative integer. Given a function $f : I \to \mathbb{R}$, the following are equivalent.

1. *The entrywise transform $f[-]$ preserves the inertia of all matrices in $\mathcal{S}^{(k)}(I)$.*
2. *The function is a positive homothety: $f(x) \equiv cx$ for some constant $c > 0$.*

Thus inertia preservers are very rigid; as soon as an entrywise map preservers inertia for matrices of arbitrary dimension, it preserves eigenvalues up to simultaneous scaling.

In fact, even more is true.

Negativity preservers – I

Theorem 21

Let $I := (-\rho, \rho)$, *where* $0 < \rho \leqslant \infty$, *and let* k *be a positive integer. Given a function* $f : I \to \mathbb{R}$, *the following are equivalent.*

1. *The entrywise transform* $f[-]$ *sends* $\mathcal{S}^{(k)}(I)$ *to* $\mathcal{S}^{(k)}(\mathbb{R})$.
2. *The function* f *is a positive homothety, so that* $f(x) \equiv cx$ *for some* $c > 0$, *or, when* $k = 1$, *we can also have that* $f(x) \equiv -c$ *for some* $c > 0$.

We can weaken the hypotheses by looking at

$$\overline{\mathcal{S}_n^{(k)}}(I) := \bigcup_{j=0}^{k} \mathcal{S}_n^{(j)}(I),$$

the set of $n \times n$ real symmetric matrices with at most k negative eigenvalues.

Negativity preservers – II

Theorem 22

Let $I := (-\rho, \rho)$, *where* $0 < \rho \leqslant \infty$, *and let* k *and* l *be positive integers. Given a function* $f : I \to \mathbb{R}$, *the following are equivalent.*

1. *The entrywise transform* $f[-]$ *sends* $\overline{\mathcal{S}_n^{(k)}}(I)$ *to* $\overline{\mathcal{S}_n^{(l)}}$ *for all* $n \geqslant k$.
2. *The entrywise transform* $f[-]$ *sends* $\mathcal{S}_n^{(k)}(I)$ *to* $\overline{\mathcal{S}_n^{(l)}}$ *for all* $n \geqslant k$.
3. *Exactly one of the following occurs:*
   1. *the function* f *is constant, so that* $f(x) \equiv d$ *for some* $d \in \mathbb{R}$;
   2. *it holds that* $l \geqslant k$ *and* f *is linear, with* $f(x) \equiv f(0) + cx$, *where* $c > 0$ *and also* $f(0) \geqslant 0$ *if* $l = k$.

Negativity preservers – III

Theorem 22 (continued)

The entrywise transform $f[-]$ *sends* $\overline{\mathcal{S}_n^{(k)}}(I)$ *(and so* $\mathcal{S}_n^{(k)}(I)$*) to* $\overline{\mathcal{S}_n^{(0)}} = \mathcal{S}_n^{(0)}$ *for all* $n \geqslant k$ *if and only if* $f(x) \equiv c$ *for some* $c \geqslant 0$*.*

Finally, the entrywise transform $f[-]$ *sends* $\mathcal{S}_n^{(0)}(I)$ *to* $\overline{\mathcal{S}_n^{(l)}}$ *for all* $n \geqslant 1$ *if and only if*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{for all } x \in (-\rho, \rho), \text{ where } c_n \geqslant 0 \text{ for all } n \geqslant 1.$$

Note that setting $k = l = 0$ in Theorem 22(1) (the missing case) gives hypothesis (ii) of Schoenberg's theorem.

Multi-variable negativity preservers – I

In Theorem 22, the parameters k and l control the degree of negativity in the domain and the co-domain, respectively.

In the multi-variable setting, the domain parameter k becomes an m-tuple of non-negative integers $\mathbf{k} = (k_1, \ldots, k_m)$.

Given such a $\mathbf{k}$, we may permute the entries so that any zero entries appear first: more formally, there exists $m_0 \in [0 : m]$ with

$$k_p = 0 \text{ for } p \in [1 : m_0] \text{ and } k_p \geqslant 1 \text{ for } p \in [m_0 + 1 : m].$$

We say that $\mathbf{k}$ is *admissible* in this case and let $k_{\max} := \max\{1, k_p : p \in [1 : m]\}$,

$$\mathcal{S}_n^{(\mathbf{k})}(I) := \mathcal{S}_n^{(k_1)}(I) \times \cdots \times \mathcal{S}_n^{(k_m)}(I)$$

and

$$\overline{\mathcal{S}_n^{(\mathbf{k})}}(I) := \overline{\mathcal{S}_n^{(k_1)}}(I) \times \cdots \times \overline{\mathcal{S}_n^{(k_m)}}(I).$$

Multi-variable negativity preservers – II

Theorem 23

Let $I := (-\rho, \rho)$, where $0 < \rho \leqslant \infty$. Let m and n be non-negative integers, with $m \geqslant 1$ and let $\mathbf{k} \in \mathbb{Z}_+^m$ be an admissible tuple. Given any function $f : I^m \to \mathbb{R}$, the following are equivalent.

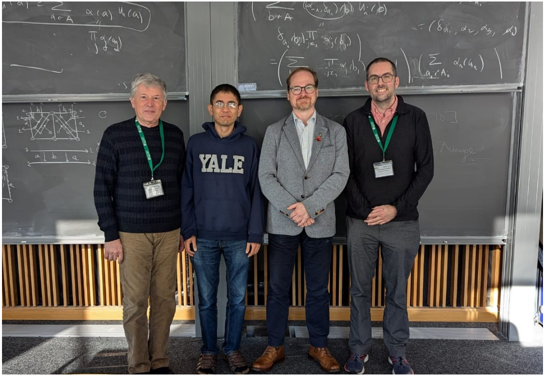
1. *The entrywise transform $f[-]$ sends $\overline{\mathcal{S}_n^{(\mathbf{k})}}(I)$ to $\overline{\mathcal{S}_n^{(l)}}$ for all $n \geqslant k_{\max}$.*
2. *The map $f[-]$ sends $\mathcal{S}_n^{(\mathbf{k})}(I)$ to $\overline{\mathcal{S}_n^{(l)}}$ for all $n \geqslant k_{\max}$.*
3. *There exists a function $F : (-\rho, \rho)^{m_0} \to \mathbb{R}$ and a non-negative constant c_p for each $p \in [m_0 + 1 : m]$ such that*
   1. *we have the representation*

      $$f(\mathbf{x}) = F(x_1, \ldots, x_{m_0}) + \sum_{p=m_0+1}^{m} c_p x_p \qquad \textit{for all } \mathbf{x} \in I^m,$$

   2. *the function $\mathbf{x}' \mapsto F(\mathbf{x}') - F(\mathbf{0})$ is absolutely monotone on $(0, \rho)^{m_0}$ and*
   3. *it holds that $l \geqslant \mathbf{1}_{F(\mathbf{0})<0} + \sum_{p:c_p>0} k_p$.*

The end!

Thank you for your attention



The quartet at the International Centre for Mathematical Sciences, Edinburgh, in November 2024