An analog of multiplier sequences for the set of totally positive sequences

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Totally positive sequences

Definition

A sequence of nonnegative numbers $(a_k)_{k=0}^{\infty}$ is called the totally positive sequence, if all minors of the infinite matrix

$$\left\| \begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ 0 & 0 & 0 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right\| \tag{1}$$

are non-negative.

The class of totally positive sequences is denoted by TP. The class of generating functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is denoted by $\widetilde{\mathrm{TP}}$.

Totally (multiply) positive sequences were introduced by Fekete in 1912 (the problem of exact calculation of the number of positive zeros of a real polynomial). TP (MP) sequences arise in many areas of mathematics and its applications.

Totally positive sequences

The class $\widetilde{\mathrm{TP}}$ was completely described in the classical theorem by Aissen, Schoenberg, Whitney and Edrei.

Theorem

(Aissen, Schoenberg, Whitney and Edrei). Let $(a_k)_{k=0}^{\infty}$ be a given sequence of nonnegative numbers. Then $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \widetilde{\mathrm{TP}}$ if and only if

$$f(z) = Cz^n e^{\gamma z} \prod_{k=1}^{\infty} (1 + \alpha_k z)/(1 - \beta_k z),$$

where $C \geq 0, n \in \mathbb{Z}, \gamma \geq 0, \alpha_k \geq 0, \beta_k \geq 0, \sum(\alpha_k + \beta_k) < \infty$.

Totally positive sequences: trivial examples

Example

Let $a_k = k + 1$. Then $(a_k)_{k=0}^{\infty} \in \mathrm{TP}$, since

$$\sum_{k=0}^{\infty} (k+1) z^k = \frac{1}{(1-z)^2}.$$

Example

Let $b_k = k + 1 + \varepsilon, \varepsilon > 0$. Then $(b_k)_{k=0}^{\infty} \notin \mathrm{TP}$, since

$$\sum_{k=0}^{\infty} (k+1+\varepsilon) z^k = \frac{-\varepsilon z + (1+\varepsilon)}{(1-z)^2}.$$

Totally positive sequences

Simple corollaries of theorem AESW.

Corollary

A polynomial with nonnegative coefficients $P(z) = \sum_{k=0}^{n} a_k z^k$ *has only real zeros if and only if the sequence of its coefficients is totally positive:* $(a_0, a_1, \ldots, a_n, 0, 0, \ldots) \in \mathrm{TP}$.

Corollary

An entire function with nonnegative coefficients $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *of order less than* 1 *has only real zeros if and only if the sequence of its coefficients is totally positive:* $(a_k)_{k=0}^{\infty} \in \mathrm{TP}$.

The Laguerre-Pólya I class

Definition

A real entire function f belongs to the *Laguerre-Pólya class of type I*, written $f \in \mathcal{L}-\mathcal{P}I$, if

$$f(x) = cx^n e^{\beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right), \qquad (2)$$

where $c \in \mathbb{R}, \beta \geq 0, x_k > 0,\ n \in \mathbb{N} \cup \{0\}$, and $\sum_{k=1}^{\infty} x_k^{-1} < \infty$.

Corollary

Let $f = \sum_{k=0}^{\infty} a_k z^k, a_k > 0,$ *be an entire function. Then* $f \in \mathcal{L}-\mathcal{P}I$ *if and only if* $(a_k)_{k=0}^{\infty} \in \mathrm{TP}$.

A remark

Let $(a_k)_{k=0}^\infty$ and $(b_k)_{k=0}^\infty$ be two sequences of positive numbers such that the generating functions are entire functions of order less than one. If $(a_k)_{k=0}^\infty \in \mathrm{TP}$ and $(b_k)_{k=0}^\infty \in \mathrm{TP}$, then $(a_k b_k)_{k=0}^\infty \in \mathrm{TP}$ (the Hadamard convolution) and $(k! a_k b_k)_{k=0}^\infty \in \mathrm{TP}$ (the Schur convolution).

Without the assumption about generating functions this is obviously not true:

1. For $a_k = k+1$ we have $\sum_{k=0}^\infty (k+1) z^k = \frac{1}{(1-z)^2}$, so $(a_k)_{k=0}^\infty \in \mathrm{TP}$. For $b_k = 2 - \frac{1}{2^k}$ we have $\sum_{k=0}^\infty (2 - \frac{1}{2^k}) z^k$ $= \frac{1}{(1-z)(1-z/2)}$, so $(b_k)_{k=0}^\infty \in \mathrm{TP}$. But $(a_k b_k)_{k=0}^\infty \notin \mathrm{TP}$, since $\sum_{k=0}^\infty (k+1)(2-\frac{1}{2^k}) z^k = \frac{2-z^2}{2(1-z)^2(1-z/2)^2}$.

2. For $a_k = b_k \equiv 1$ we have $(a_k)_{k=0}^\infty, (b_k)_{k=0}^\infty \in \mathrm{TP}$, but $(k!)_{k=0}^\infty \notin \mathrm{TP}$.

The Laguerre-Pólya Theorem

Theorem

(E. Laguerre and G. Pólya).

(i) Let $(P_n)_{n=1}^{\infty},\ P_n(0) = 1,$ *be a sequence of real polynomials having only real negative zeros which converges uniformly in the circle* $|z| \leq A, A > 0.$ *Then this sequence converges locally uniformly to an entire function from the class* $\mathcal{L} - \mathcal{P}I$*.*

(ii) For every $f \in \mathcal{L} - \mathcal{P}I$ *there is a sequence of real polynomials with only real negative zeros which converges locally uniformly to* f*.*

The second quotients of Taylor Coefficients

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with positive coefficients. We will use the following notation:

$$q_n = q_n(f) := \frac{a_{n-1}^2}{a_{n-2}a_n}, \; n \geq 2.$$

It is easy to see that

$$a_n = \frac{a_1}{q_2^{n-1} q_3^{n-2} \cdot \ldots \cdot q_{n-1}^2 q_n} \left(\frac{a_1}{a_0}\right)^{n-1}, \; n \geq 2.$$

Sufficient Condition for an Entire Function to belong to the Laguerre-Pólya class

Theorem

(J. I. Hutchinson).

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be an entire function with positive coefficients. Inequalities* $q_n(f) \geq 4, \ \forall n \geq 2$, *are valid if and only if the following two properties hold:*
(i) The zeros of f *are all real, simple and negative and*

(ii) the zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$, $m < n$, *formed by taking any number of consecutive terms of* f, *are all real and non-positive.*

Easy to check: the constant 4 is the smallest possible in both statements.

Sufficient Condition for an Entire Function to belong to the Laguerre-Pólya class

Theorem

(Thu Hien Nguyen and A.V.).
Let $P(x) = \sum_{k=0}^{n} a_k x^k$, $a_k > 0$, *be a polynomial, and* $n \geq 4$. *If there exists* $\alpha, 1 + \sqrt{5} \leq \alpha < 4$, *such that* $q_k(P) \in \left[\alpha, \frac{8}{\alpha(4-\alpha)}\right]$ *for all* $k = 2, 3, \ldots, n$, *then the zeros of* P *are all real, simple and negative.*

Corollary

(Thu Hien Nguyen and A.V.).
Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$, *be an entire function. If there exists* $\alpha, 1 + \sqrt{5} \leq \alpha < 4$, *such that* $q_k(f) \in \left[\alpha, \frac{8}{\alpha(4-\alpha)}\right]$ *for all* $k = 2, 3, \ldots$, *then the zeros of* f *are all real, simple and negative.*

Multiplier sequences

The question about whether or not a given polynomial has only real zeros is of great importance in many areas of mathematics. So, the problem to describe the set of operators that preserve this set of polynomials is of great interest.

Definition

A sequence $(\gamma_k)_{k=0}^{\infty}$ of real numbers is called a multiplier sequence (written $(\gamma_k)_{k=0}^{\infty} \in \mathcal{MS}$) if, whenever a real polynomial $P(x) = \sum_{k=0}^{n} a_k z^k$ has only real zeros, the polynomial $\sum_{k=0}^{n} \gamma_k a_k z^k$ has only real zeros.

Example

Let $\gamma_k = k, k = 0, 1, 2, \ldots$ For $P(x) = \sum_{k=0}^{n} a_k z^k$ with real coefficients and all real zeros, we have $\sum_{k=0}^{n} k a_k z^k = zP'(z)$, and this polynomial has only real zeros.

Multiplier sequences

Theorem

(G. Pólya and J.Schur).

Let $(\gamma_k)_{k=0}^{\infty}$ *be a given real sequence. The following three statements are equivalent.*

1. $(\gamma_k)_{k=0}^{\infty}$ *is a multiplier sequence.*

2. For every $n \in \mathbb{N}$ *the polynomial* $P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k$ *has only real zeros of the same sign.*

3. The power series $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ *converges absolutely in the whole complex plane and the entire function* $\Phi(z)$ *or the entire function* $\Phi(-z)$ *admits the representation*

$$c z^n e^{\beta z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{x_k}\right), \tag{3}$$

where $c \in \mathbb{R}, \beta \geq 0, n \in \mathbb{N} \cup \{0\}, 0 < x_k \leq \infty,\ \sum_{k=1}^{\infty} \frac{1}{x_k} < \infty.$

Multiplier sequences

Corollary

The sequence $(\gamma_0, \gamma_1, \ldots, \gamma_l, 0, 0, \ldots)$ *is a multiplier sequence if and only if the polynomial* $P(z) = \sum_{k=0}^{l} \frac{\gamma_k}{k!} z^k$ *has only real zeros of the same sign.*

The way to construct multiplier sequences gives the following remarkable theorem proved by Laguerre and extended by Pólya.

Theorem

(E. Laguerre). Let f *be an entire function from the Laguerre-Pólya class having only negative zeros. Then* $(f(k))_{k=0}^{\infty} \in \mathcal{MS}$.

As it follows from the theorem above,

$$\left(a^{-k^2}\right)_{k=0}^{\infty} \in \mathcal{MS},\ a \geq 1, \quad \left(\frac{1}{k!}\right)_{k=0}^{\infty} \in \mathcal{MS}.$$

Multiplier sequences

Corollary

1. $f(z) = \sum_{k=0}^{\infty} \frac{z^k}{k! a^{k^2}} \in \mathcal{L} - \mathcal{P}I, a \geq 1.$

2. $g(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^m} \in \mathcal{L} - \mathcal{P}I, m \in \mathbb{N}.$

Open problem. 1. For which $b > 1$

$$h(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^b} \in \mathcal{L} - \mathcal{P}I?$$

2. More generally, fix an arbitrary $b > 1$. For which $a \geq 1$

$$\varphi(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^b a^{k^2}} \in \mathcal{L} - \mathcal{P}I?$$

Convolution operator

Definition

Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ be a nonnegative sequence. We define the linear convolution operator on the set of real sequences

$$\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) = (a_k b_k)_{k=0}^{\infty}.$$

The following problem was posed by Alan Sokal during the inspiring AIM workshop "Theory and applications of total positivity", July 24-July 28, 2023.

Problem

Describe the set of nonnegative sequences $\mathbf{A} = (a_k)_{k=0}^{\infty}$*, such that the corresponding convolution operator* $\Lambda_{\mathbf{A}}$ *preserves the set of* TP*-sequences: for every* $(b_k)_{k=0}^{\infty} \in \mathrm{TP}$ *we have* $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) \in \mathrm{TP}$.

Example

We consider the multiplier sequence $\mathbf{\Gamma} = (k)_{k=0}^{\infty}$ and the corresponding convolution operator $\Lambda_{\mathbf{\Gamma}}((b_k)_{k=0}^{\infty}) = (kb_k)_{k=0}^{\infty}$. This operator preserves the set of finite totally positive sequences (i.e., the set of coefficients of polynomials with nonnegative coefficients and only real zeros). But this operator does not preserve the set of all totally positive sequences. Consider the function

$$f(z) = \frac{1}{(1-z)(2-z)} = \sum_{k=0}^{\infty} b_k z^k$$

(we have $b_k = 1 - \frac{1}{2^{k+1}}$). By Theorem ASWE, $(b_k)_{k=0}^{\infty} \in \mathrm{TP}$. But

$$\sum_{k=0}^{\infty} kb_k z^k = zf'(z) = \frac{z(3-2z)}{(1-z)^2(2-z)^2}.$$

This function has a positive zero, so the sequence of its coefficients is not a TP-sequence.

Generating functions with at least one pole

We will denote by A the generating function of a sequence $\mathbf{A} = (a_k)_{k=0}^{\infty} : A(z) = \sum_{k=0}^{\infty} a_k z^k$.

Suppose that $\mathbf{A}$ has the property that the corresponding convolution operator $\Lambda_{\mathbf{A}}$ preserves the set of TP-sequences. Since the constant sequence of all ones is the TP-sequence, by theorem ASWE, the generating function has the representation

$$A(z) = Cz^n e^{\gamma z} \prod_{k=1}^{\infty} (1 + \alpha_k z)/(1 - \beta_k z),$$

where $C \geq 0, n \in \mathbf{Z}, \gamma \geq 0, \alpha_k \geq 0, \beta_k \geq 0, \sum(\alpha_k + \beta_k) < \infty.$

Theorem

(Olga Katkova and A.V.).
Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ be a nonnegative sequence, and suppose its generating function is a meromorphic function with at least one pole. Then for every $(b_k)_{k=0}^{\infty} \in \mathrm{TP}$ we have $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) \in \mathrm{TP}$ if and only if $A(z) = \frac{C}{1-\beta z}, C > 0, \beta > 0.$

Sufficiency

Suppose that $A(z) = \frac{C}{1-\beta z}, C > 0, \beta > 0$. Then $A(z) = \sum_{k=0}^{\infty} C\beta^k z^k$, so for every $\mathbf{B} = (b_k)_{k=0}^{\infty} \in \mathrm{TP}$ with the generation function B, the generation function of $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty})$ is $\sum_{k=0}^{\infty} C\beta^k b_k z^k = CB(\beta z) \in \widetilde{\mathrm{TP}}$. The sufficiency is proved.

It remains to describe TP-preservers whose generating functions are entire functions.

1. Two term sequences. Let us consider a nonnegative sequence $\mathbf{A} = (a_k)_{k=0}^{\infty}$, such that $a_0 \geq 0, a_1 \geq 0$, and $a_k = 0$ for $k \geq 2$. Then, obviously, for every $(b_k)_{k=0}^{\infty} \in \mathrm{TP}$ we have $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) \in \mathrm{TP}$.

Three term sequences

2. Three term sequences. Obvious statement.

Statement

Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ *be a nonnegative sequence, such that* $a_k > 0$ *for* $k = 0, 1, 2,$ *and* $a_k = 0$ *for* $k \geq 3$. *Then for every* $(b_k)_{k=0}^{\infty} \in \mathrm{TP}$ *we have* $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) \in \mathrm{TP}$ *if and only if* $A(z) = a_0 + a_1 z + a_2 z^2$ *has only real (and negative) zeros. Moreover,* $\Lambda_{\mathbf{A}} : \mathrm{TP} \to \mathrm{TP}$ *if and only if* $\Lambda_{\mathbf{A}} : \mathrm{TP}_2 \to \mathrm{TP}$.

Four and five term sequences

Theorem

(Olga Katkova and A.V.).
Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ *be a nonnegative sequence, such that* $a_k > 0$ *for* $0 \leq k \leq 3$, *and* $a_k = 0$ *for* $k \geq 4$. *Then for every* $(b_k)_{k=0}^{\infty} \in \mathrm{TP}$ *we have* $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) \in \mathrm{TP}$ *if and only if both polynomials* $\sum_{k=0}^{3} a_k x^k$ *and* $\sum_{k=1}^{3} a_k x^k$ *have only real (and nonpositive) zeros. Moreover,* $\Lambda_{\mathbf{A}} : \mathrm{TP} \to \mathrm{TP}$ *if and only if* $\Lambda_{\mathbf{A}} : \mathrm{TP}_3 \to \mathrm{TP}$.

Theorem

Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ *be a nonnegative sequence, such that* $a_k > 0$ *for* $0 \leq k \leq 4$, *and* $a_k = 0$ *for* $k \geq 5$. *Then for every* $(b_k)_{k=0}^{\infty} \in \mathrm{TP}$ *we have* $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) \in \mathrm{TP}$ *if and only if the three polynomials* $\sum_{k=0}^{4} a_k x^k$, $\sum_{k=1}^{4} a_k x^k$ *and* $\sum_{k=2}^{4} a_k x^k$ *have only real (and nonpositive) zeros. Moreover,* $\Lambda_{\mathbf{A}} : \mathrm{TP} \to \mathrm{TP}$ *if and only if* $\Lambda_{\mathbf{A}} : \mathrm{TP}_4 \to \mathrm{TP}$.

Infinite positive sequences

The following example was given by Alan Sokal.

Example

Let f be an entire function of the form $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_0 = a_1 = 1$, $a_k = \frac{1}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$ for $k \geq 2$, where $(q_k)_{k=2}^{\infty}$ is a sequence of arbitrary parameters under the following conditions: $q_k \geq 4$ for all k. Suppose that $(b_k)_{k=0}^{\infty} \in \mathrm{TP}$ is an arbitrary sequence. For an entire function $(A * B)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$ we have $\frac{(a_{n-1} b_{n-1})^2}{(a_{n-2} b_{n-2})(a_n b_n)} = \frac{a_{n-1}^2}{a_{n-2} a_n} \cdot \frac{b_{n-1}^2}{b_{n-2} b_n} \geq 4$ for all $n \geq 2$, since $\frac{a_{n-1}^2}{a_{n-2} a_n} = q_n \geq 4$ by our assumption, and $\frac{b_{n-1}^2}{b_{n-2} b_n} \geq 1$, because every TP-sequence is, in particular, a 2-times positive sequence. Thus, using Hutchinson's Theorem , we get $(A * B)(z) \in \mathrm{TP}$.

Conjecture

We formulate the following conjecture, which is consistent with previous theorems and and example.

Conjecture

Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ *be a nonnegative sequence with entire generating function. Then* $\mathbf{A}$ *is a* TP*-preserver, i.e. for every* $(b_k)_{k=0}^{\infty} \in \mathrm{TP}$ *we have* $\Lambda_{\mathbf{A}}((b_k)_{k=0}^{\infty}) \in \mathrm{TP}$ *if and only if for every* $s \in \mathbb{N} \cup \{0\}$ *the formal power series* $\sum_{k=s}^{\infty} a_k z^k$ *is an entire function from the* $\mathcal{L}$-$\mathcal{P}I$ *class (in particular, it has only real nonpositive zeros).*

Necessity. Let $\mathbf{A} = (a_k)_{k=0}^{\infty}$ be a nonnegative sequence with entire generating function, such that the operator $\Lambda_{\mathbf{A}}$ preserves the set of the TP-sequences. For every $k \in \mathbb{N}$ the sequence $\mathbf{B}_k = (0, 0, \ldots, 0, 1, 1, 1, 1, \ldots) \in \mathrm{TP}$, ($\mathbf{B}_k$ has k zeros and after that all ones). So, $\Lambda_{\mathbf{A}}(\mathbf{B}_k) = (0, 0, \ldots, 0, a_{k+1}, a_{k+2}, a_{k+3}, \ldots)$ $\in \mathrm{TP}$. Hence, by ASWE theorem, the function $\sum_{j=k+1}^{\infty} a_j x^j$ $\in \mathcal{L}$-$\mathcal{P}I$, in particular, it has only real nonpositive zeros.

Entire functions with remainders having only real zeros

Entire functions whose Taylor sections have only real zeros were studied in various works, but entire functions whose remainders have only real zeros have been studied less (some results can be found in the survey by I.V. Ostrovskii).

The entire function $g_a(z) = \sum_{j=0}^{\infty} z^j a^{-j^2}$, $a > 1$, is called the *partial theta-function*. The survey by S.O. Warnaar contains the history of investigation of the partial theta-function and some of its main properties. The paper by O. Katkova, T. Lobova and A.V. answers the question: for which $a > 1$ we have $g_a \in \mathcal{L}\text{-}\mathcal{P}I$. In particular, it is proved that there exists a constant $q_\infty \approx 3.23363666\ldots$, such that $g_a \in \mathcal{L}\text{-}\mathcal{P}I$ if and only if $a^2 \geq q_\infty$.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_0 = a_1 = 1$, $a_k = \frac{1}{q_2^{k-1} q_3^{k-2} \cdot \ldots \cdot q_{k-1}^2 q_k}$ for $k \geq 2$, and $(q_k)_{k=2}^{\infty}$ is an arbitrary sequence such that: $q_2 \geq q_3 \geq q_4 \geq \ldots$ and $\lim_{n\to\infty} q_n \geq q_\infty$. Then, by theorem by Thu Hien Nguen and A.V., f has all remainders from the class $\mathcal{L}\text{-}\mathcal{P}I$.

Entire functions with remainders having only real zeros

Example

Consider a function $f(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \in \mathcal{L}\text{-}\mathcal{P}I$ $((\frac{1}{k!})_{k=0}^{\infty}$ $\in \mathrm{TP})$. Its first remainder $g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k!} = e^z - 1 \notin \mathcal{L}\text{-}\mathcal{P}I$, it has infinitely many nonreal zeros. Moreover, for all $s \in \mathbb{N}$ the remainder $\sum_{k=s}^{\infty} \frac{z^k}{k!}$ has infinitely many nonreal zeros.

Theorem

(Olga Katkova and A.V.).
Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function such that for every $s \in \mathbb{N} \cup \{0\}$ the function $\sum_{k=s}^{\infty} a_k z^k$ belongs to the $\mathcal{L}\text{-}\mathcal{P}I$ class. Then

$$q_n(f) = \frac{a_{n-1}^2}{a_{n-2}a_n} \geq 3, n = 2, 3, 4, \ldots .$$

Entire functions with remainders having only real zeros

Problem

Let us consider the set $\mathcal{M}$ of all entire functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ such that for every $s \in \mathbb{N} \cup \{0\}$ the function $\sum_{k=s}^{\infty} a_k z^k$ belongs to the $\mathcal{L}$-$\mathcal{P}I$ class. Find the following constant

$$c = \inf \left\{ q_n(f) = \frac{a_{n-1}^2}{a_{n-2} a_n} \mid f \in \mathcal{M}, n = 2, 3, 4, \dots \right\}.$$

Conjecture

$c = q_\infty$.
($q_\infty \approx 3.23363666\dots$).

Thank you

for attention!

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