# Pólya frequency sequences and functions

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# 1. Totally positive matrices and Pólya frequency sequences

# Totally positive/nonnegative matrices

**Definition.** A rectangular matrix is *totally positive (TP)* if all minors are positive. (Similarly, totally nonnegative (TN).)

Thus all entries $> 0$, all $2 \times 2$ minors $> 0$, $\ldots$

These matrices occur widely in mathematics:

# Totally positive matrices in mathematics

TP and TN matrices occur in

- analysis and differential equations (Aissen, Edrei, Schoenberg, Pólya, Loewner, Whitney)
- probability and statistics (Efron, Karlin, Pitman, Proschan, Rinott)
- interpolation theory and splines (Curry, Schoenberg)
- Gabor analysis (Gröchenig, Romero, Stöckler)
- interacting particle systems (Gantmacher, Krein)
- matrix theory (Ando, Cryer, Fallat, Garloff, Holtz, Johnson, Pinkus, Sokal)

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- representation theory and the Grassmannian (Lusztig, Postnikov, Lam)
- cluster algebras (Berenstein, Fomin, Zelevinsky)
- integrable systems (Kodama, Williams)
- quadratic algebras (Borger, Davydov, Grinberg, Hô Hai)
- combinatorics (Branden, Brenti, Skandera, Sturmfels, Wagner, . . . )

⋮

# Examples of TP/TN matrices

1. Generalized Vandermonde matrices are TP: if $0 < x_1 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ are real, then
$$\det(x_j^{y_k})_{j,k=1}^n > 0.$$
2. (Pólya:) The *Gaussian kernel* is TP: given $\sigma > 0$ and scalars
$$x_1 < x_2 < \cdots < x_n, \qquad y_1 < y_2 < \cdots < y_n,$$
the matrix $G[\mathbf{x};\mathbf{y}] := (e^{-\sigma(x_j - y_k)^2})_{j,k=1}^n$ is TP.
3. The lower-triangular matrix $A = (\mathbf{1}_{j \geq k})_{j,k=1}^n$ is TN.
4. *Submatrices* and *Limits* of TN matrices are TN.
5. *Products* of TN/TP matrices are TN/TP, by the Cauchy–Binet formula.

# Pólya frequency sequences

A real sequence $(a_n)_{n \in \mathbb{Z}}$ is a *Pólya frequency sequence* if for any integers

$$l_1 < l_2 < \cdots < l_n, \qquad m_1 < m_2 < \cdots < m_n,$$

the determinant $\det(a_{l_j - m_k})_{j,k=1}^n \geq 0$.

In other words, these are semi-infinite Toeplitz matrices

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which are totally nonnegative (TN).

# Generating functions of Pólya frequency sequences

- Two remarkable results (1950s) say that finite and one-sided Pólya frequency sequences are simply products of "atoms"!
- The "atoms" are explained next. For now: why products?

Suppose $\mathbf{a} = (\dots, 0, 0, a_0, a_1, a_2, a_3, \dots)$ is one-sided. Its *generating function* is

$$\Psi_{\mathbf{a}}(s) := a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \cdots, \qquad a_0 \neq 0.$$

Now if $\mathbf{a}, \mathbf{b}$ are one-sided PF sequences, then their Toeplitz "matrices" are TN:

$$T_{\mathbf{a}} := \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad T_{\mathbf{b}} := \begin{pmatrix} b_0 & 0 & 0 & \cdots \\ b_1 & b_0 & 0 & \cdots \\ b_2 & b_1 & b_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By the Cauchy–Binet formula, so also is $T_{\mathbf{a}} T_{\mathbf{b}} \rightsquigarrow$ **(Miracle 1?)** Toeplitz matrix.

**(Miracle 2?)** This product matrix corresponds to the coefficients of the power series $\Psi_{\mathbf{a}}(s)\Psi_{\mathbf{b}}(s)$! I.e., $\mathcal{L} : T_{\mathbf{a}} \mapsto \Psi_{\mathbf{a}}(s)$ is an $\mathbb{R}$-*algebra* map.

# Finite Pólya frequency sequences – and real-rootedness

"Atomic" **finite** PF sequences:

- The sequence $(\dots, 0, 0, a_0, 0, 0, \dots)$ and $(\dots, 0, 0, 1, \alpha, 0, 0, \dots)$ are PF sequences if $a_0, \alpha > 0$.
  Indeed, every "square submatrix" drawn from these sequences either has a zero row/column, or is triangular with positive diagonal entries.
- The "atom" $(\dots, 0, 0, 1, \alpha, 0, 0, \dots)$ corresponds to $\Psi_{\mathbf{a}}(x) = 1 + \alpha x$.
- By previous slide, $a_0(1 + \alpha_1 x)(1 + \alpha_2 x) \cdots (1 + \alpha_m x)$ generates a PF sequence $\mathbf{a}_m$, when all $\alpha_j > 0$. In fact, these are *all finite PF sequences:*

**Theorem (Aissen–Schoenberg–Whitney and Edrei, *J. d'Analyse Math.*, 1950s; and Schoenberg, *Ann. of Math.*, 1955)**

*Suppose $a_0, \dots, a_m > 0$. The following are equivalent.*

1. $\mathbf{a} = (\dots, 0, 0, a_0, \dots, a_m, 0, 0, \dots)$ *is a PF sequence.*
2. *The generating function $\Psi_{\mathbf{a}}(x)$ has $m$ negative real roots (i.e., the above form).*
3. *The generating function $\Psi_{\mathbf{a}}(x)$ has $m$ real roots.*

# Infinite one-sided Pólya frequency sequences

For "infinite" one-sided PF sequences, only one other "atom" – and limits:

Recall, the lower-triangular matrix $A = (\mathbf{1}_{j \geq k})_{j,k=1}^n$ is TN (direct proof). Hence $\mathbf{a}_1 := (\dots, 0, 0, 1, 1, \dots)$ is a one-sided PF sequence, with generating function:

$$\Psi_{\mathbf{a}_1}(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

**Claim:** The function $\mathbf{a}_\beta := (\dots, 0, 0, 1, \beta, \beta^2, \dots)$ is a PF sequence for $\beta > 0$.

*Proof:* Given increasing tuples of integers $(l_j), (m_k)$ for $1 \leq j, k \leq n$,

$$((\mathbf{a}_\beta)_{l_j - m_k}) = \mathrm{diag}(\beta^{l_j})_{j=1}^n \cdot (\mathbf{1}_{l_j \geq m_k})_{j,k=1}^n \cdot \mathrm{diag}(\beta^{-m_k})_{k=1}^n,$$

and this has a nonnegative determinant since $\mathbf{a}_1$ is PF. □

- Therefore $(1 - \beta x)^{-1}$ is a PF sequence for $\beta > 0$.

**Limits:** If $\mathbf{a}_m$ are PF sequences, converging "pointwise" to $\mathbf{a}$, then $\mathbf{a}$ is a PF sequence.

- *Example:* Since $(1 + \delta x/m)^m$ generates a PF sequence for all $m \geq 1$, so does $e^{\delta x}$. (E.g. **Fekete:** $(\dots, 0, 0, 1, \frac{\delta}{1!}, \frac{\delta^2}{2!}, \dots)$ is a PF sequence.)

# Infinite one-sided Pólya frequency sequences (cont.)

- More examples: if $\alpha_j, \beta_j \geq 0$ for all $j \geq 0$ are summable, then

$$\prod_{j=1}^{\infty} (1 + \alpha_j x), \qquad \prod_{j=1}^{\infty} (1 - \beta_j x)^{-1}$$

  both generate PF sequences.
- Hence so does their product:

$$e^{\delta x} \frac{\prod_{j=1}^{\infty} (1 + \alpha_j x)}{\prod_{j=1}^{\infty} (1 - \beta_j x)}.$$

Remarkably, these are *all* of the PF sequences!

**Theorem (Aissen–Schoenberg–Whitney and Edrei, *J. d'Analyse Math.*, 1950s)**

*A one-sided sequence $\mathbf{a} = (\dots, 0, 0, a_0 = 1, a_1, \dots)$ is a PF sequence, if and only if it is of the above form.*

(Uses Hadamard's thesis (1892) and Nevanlinna's refinement (1929) of Picard's theorem.)

# From Pólya–Schur multipliers to Ramanujan graphs

What if $\Psi_{\mathbf{a}}(x)$ is an entire function? It must be $e^{\delta x}\prod_{j\geq 1}(1+\alpha_j x)$. Thus:

**Theorem (Pólya–Schur, *Crelle*, 1914)**

*An entire function* $\Psi(x) = \sum_{n\geq 0} a_n x^n$ *with* $\Psi(0) = 1$ *generates a one-sided PF sequence,*
*if and only if* $\Psi(x)$ *is in the first Laguerre–Pólya class* $\mathcal{LP}_1$*,*
*if and only if the sequence* $n!a_n$ *is a multiplier sequence of the first kind.*

In other words, if $\sum_{j\geq 0} c_j x^j$ is a real-rooted *polynomial*, so is $\sum_{j\geq 0} j!a_j c_j x^j$.

- This circle of ideas – and classification of Pólya–Schur type multiplier sequences – has found far-reaching generalizations in work of Brändén with Borcea (late 2000s) and others.
- Taken forward by Marcus–Spielman–Srivastava (2010s):
  - Kadison–Singer conjecture.
  - Existence of bipartite Ramanujan (expander) graphs of every degree and every order.

# 2. Pólya frequency functions

# Toeplitz TN kernels

Above: the Gaussian kernel $K_-(x,y) := \exp(-(x-y)^2)$ is TP.
More generally, a *totally nonnegative (TN) function* is $\Lambda : \mathbb{R} \to \mathbb{R}$ such that its Toeplitz kernel is TN:

$$T_\Lambda(x,y) := \Lambda(x-y), \qquad x,y \in \mathbb{R}.$$

**"Representative" examples:**

1. $\Lambda(x) = e^{-x^2}$.
2. $\Lambda(x) = c\mathbf{1}(x=a)$ for $a \in \mathbb{R}$ and $c \geq 0$. (Draw any submatrix from $T_\Lambda$; it is either a diagonal matrix or has a zero row/column.)
3. $\Lambda(x) = e^{ax+b}$ is TN. Indeed,
$$T_\Lambda((x_j,y_k)) = (e^{ax_j - ay_k + b})_{j,k\geq 1}$$
and this has rank-one, so all "larger" minors vanish (hence are $\geq 0$).
4. $\Lambda(x) = \mathbf{1}_{x\geq 0}$. (Can be verified to be TN by explicit computation.)

Note: the last two examples are not integrable functions.

# Pólya frequency functions

**Definition:** A function $\Lambda : \mathbb{R} \to \mathbb{R}$ is a *Pólya frequency function* if
(a) it is integrable,
(b) it is nonzero at two points, and
(c) the associated Toeplitz kernel $T_\Lambda$ is TN.

Pólya Frequency Functions (PFFs) have a beautiful structure theory,[1] developed by Schoenberg and others. They connect to real function theory, PDEs, approximation theory (splines), Gabor analysis, ...

*Consequences of the definition:* All Pólya frequency functions $\Lambda$ are

- nonzero on a semi-axis, or nonzero on $\mathbb{R}$;
- continuous except at most at one point $a$ (where $\Lambda(a^+), \Lambda(a^-)$ exist).
- All TN functions are an exponential $e^{ax+b} \times$ a Pólya frequency function.

[1] When I first studied these fascinating objects, PFFs were my BFFs!

# Pólya frequency functions – examples

1. The Gaussian kernel $e^{-x^2}$.

2. While $\mathbf{1}_{x \geq 0}$ is not integrable, $e^{-x}\mathbf{1}_{x \geq 0}$ is a Pólya frequency function.

   Indeed, if $\Lambda(x)$ is a TN function, then so is $\Delta(x) := e^{ax+b}\Lambda(x)$ because

   $$(T_\Delta(x_j, y_k))_{j,k=1}^n = \operatorname{diag}(e^{ax_j+b})_{j=1}^n (T_\Lambda(x_j, y_k))_{j,k=1}^n \operatorname{diag}(e^{-ay_k})_{k=1}^n,$$

   and so the determinant is $\geq 0$.

3. If $\Lambda(x)$ is a PF function, so is $c\Lambda(ax+b)$ for $a \neq 0,\ c > 0,\ b \in \mathbb{R}$. ("*Change of origin and scale*")

4. *Limits* of PF functions (if nonzero and integrable) are PF functions.

# Convolution

Another way to construct new examples of TP/TN kernels from old ones:

- In the matrix/"discrete" case: given two matrices $A_{m \times n}$ and $B_{n \times p}$ which are both TN, *their product is also TN* – by Cauchy–Binet.

- The Cauchy–Binet formula has a *continuous* version $\rightsquigarrow$ *Basic composition formula* (Pólya–Szegő). This implies:

**Corollary:** *If* $\Lambda_1, \Lambda_2 : \mathbb{R} \to [0, \infty)$ *are integrable Pólya frequency functions, then so is their convolution*

$$(\Lambda_1 * \Lambda_2)(x) := \int_{\mathbb{R}} \Lambda_1(t)\Lambda_2(x - t)\ dt, \qquad x \in \mathbb{R}.$$

This will help construct additional examples of Pólya frequency functions.

# Pólya frequency functions and Laplace transforms

The bilateral Laplace transform of a PF function $\Lambda$ is

$$\mathcal{L}(\Lambda)(s) := \int_{\mathbb{R}} e^{-sx}\Lambda(x)\,dx, \qquad s \in \mathbb{C}.$$

**Fact:** $\mathcal{L}$ is an algebra map: $\mathcal{L}(\Lambda_1 * \Lambda_2) = \mathcal{L}(\Lambda_1)\mathcal{L}(\Lambda_2)$!

Now consider *one-sided* PF functions: $\varphi_a(x) := \frac{1}{a}e^{-x/a}\mathbf{1}_{x\geq 0} \rightsquigarrow$ Laplace transform $\mathcal{L}(\varphi_a)(s) = 1/(1+as)$.

- Let $a_j \geq 0$ with $\sum_{j=1}^\infty a_j < \infty$. Then for each $n$, the convolution $\varphi_{a_1} * \cdots * \varphi_{a_n}$ is a one-sided PF function (Hirschman–Widder density), with Laplace transform

$$\mathcal{L}(\varphi_{a_1} * \cdots * \varphi_{a_n})(s) = \frac{1}{\prod_{j=1}^n (1+a_j s)}.$$

# Laguerre–Pólya class and Schoenberg's results: I. One-sided

- Shifting the origin of $\varphi_{a_1} * \cdots * \varphi_{a_n}$ to $\delta \geq 0$ yields a one-sided PF function with Laplace transform $e^{-\delta s} / \prod_{j=1}^{n} (1 + a_j s)$.
- Taking limits of PF functions gives a PF function $\rightsquigarrow$ a PF function with Laplace transform $e^{-\delta s} / \prod_{j=1}^{\infty} (1 + a_j s)$.
- Its reciprocal is the analytic (entire) function $e^{\delta s} \prod_{j=1}^{\infty} (1 + a_j s)$.

Remarkably, *every* one-sided PF function shares this property:

**Theorem (Schoenberg, *J. d'Analyse Math.*, 1951)**

*A function $\Lambda : \mathbb{R} \to \mathbb{R}$, continuous on $(0, \infty)$ and with $\int_{\mathbb{R}} \Lambda(x)\, dx = 1$, is a one-sided PF function vanishing on $(-\infty, 0)$, if and only if*

$$\frac{1}{\mathcal{L}(\Lambda)(s)} = e^{\delta s} \prod_{j=1}^{\infty} (1 + a_j s), \quad \textit{where} \quad \delta, a_j \geq 0, \ \sum_j a_j < \infty.$$

This is the limit of the polynomials $(1 + \frac{\delta s}{n})^n \prod_{j=1}^{n} (1 + a_j s)$, with negative roots.

# Laguerre–Pólya class and Schoenberg's results: II. Two-sided

Similarly, using the Gaussian kernel and "oppositely directed" variants of $e^{-x}\mathbf{1}_{x\geq 0}$, Schoenberg proved:

**Theorem (Schoenberg, *J. d'Analyse Math.*, 1951)**

*A function $\Lambda : \mathbb{R} \to \mathbb{R}$ with $\int_{\mathbb{R}} \Lambda(x)\, dx = 1$ is a PF function, if and only if*

$$\frac{1}{\mathcal{L}(\Lambda)(s)} = e^{-\gamma s^2 + \delta s} \prod_{j=1}^{\infty} (1 + a_j s) e^{-a_j s},$$

*where $\gamma \geq 0$ and $\delta, a_j \in \mathbb{R}$ are such that $0 < \gamma + \sum_j a_j^2 < \infty$.*

These two classes of entire functions were very well-studied by Laguerre, Pólya, and Schur in the early 20th century:

1. The first class of entire functions are limits – uniform on compact sets – of real polynomials with real non-positive roots. ("One-sided")
2. The second class $\rightsquigarrow$ limits of real polynomials with real roots.

$\rightsquigarrow$ *Laguerre–Pólya functions* (allowing for a factor of $cs^m$, $c \geq 0, m \in \mathbb{Z}^{\geq 0}$).

# From the Laguerre–Pólya class to the Riemann Hypothesis

Pólya initiated the study of functions $\Lambda(t)$ such that $\mathcal{L}(\Lambda)(s)$ has only pure imaginary zeros. His work alludes to the following result:

**Theorem (Pólya, *J. reine angew. Math.*, 1927)**

*The following statements are equivalent:*

1. *The Riemann Xi-function $\Xi(s) = \xi(1/2 + iz)$ is in the Laguerre–Pólya class, where $\xi(s) := \binom{s}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$.*
2. *The Riemann Hypothesis is true.*

Combined with Schoenberg's result above, this yields:

**Theorem (Gröchenig, *Appl. Numer. Harm. Anal.*, 2020)**

*Let $\xi(s) = \binom{s}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$ be the Riemann xi-function. If*

$$\Lambda(x) := \int_{\mathbb{R}} \xi(u + 1/2)^{-1} e^{-ixu}\, du$$

*is a Pólya frequency function, then the Riemann Hypothesis is true.*

The Laguerre–Pólya class is thus a distinguished one in several areas.

# The Riemann Hypothesis

For the same reason, Pólya frequency sequences connect to number theory:

**Theorem** (Katkova, *Comput. Meth. Funct. Th.*, 2007)

*Let* $\xi(s) = \binom{s}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$ *be the Riemann xi-function. If*

$$\xi_1(s) := \xi(1/2 + \sqrt{s})$$

*generates a PF sequence, then the Riemann Hypothesis is true.*

Katkova proved that $\xi_1$ is PF of order at least $43$, and is "asymptotically PF" of all orders.

# Pólya frequency functions are probability densities

**Reformulation via probability:**

- Traditionally: "frequency functions" $\longleftrightarrow$ *densities* of (continuous) random variables.
- Convolutions of these $\longleftrightarrow$ adding the (independent) random variables.

Thus, Schoenberg's theorems reformulate (B–G–K.–P, *MRR* 2022) and say:

**(1) One-sided variant:**
Pólya frequency functions vanishing on $(-\infty, 0)$ and positive on $(0, \infty)$ are precisely the densities of $\sum_{j \geq 1} \alpha_j X_j$,
where $\alpha_j \geq 0$ are summable and $X_j$ are i.i.d. $\exp(1)$ variables (these are TN).

**(2) Two-sided variant:**
For Pólya frequency functions not vanishing on a semi-axis, one simply needs to
(a) allow negative $\alpha_j$, and/or
(b) add one more normal variable (recall, Gaussian densities $G_\sigma$ are TP).

**(3) "General" PF functions:** are the above, up to shift of origin and scale.

# 3. Total positivity preservers: Joint with Belton, Guillot, Putinar

# 1. Preservers of $2 \times 2$ TN matrices

**Question (Deift, 2017):** Which transforms preserve total positivity?

**Setting 1:** *Preservers of* $2 \times 2$ *TN matrices*

Fix $x, y \geq 0$, and let $A = \begin{pmatrix} x & xy \\ 1 & y \end{pmatrix}, \quad B = \begin{pmatrix} xy & x \\ y & 1 \end{pmatrix}$. These are TN.
Applying $0 \leq \det F[-]$, we get:

$$F(x)F(y) \geq F(xy)F(1), \qquad F(xy)F(1) \geq F(x)F(y),$$

and so denoting $G(x) := F(x)/F(1)$,

$$G(xy) = G(x)G(y).$$

This gives that $G(x)$ is either a power function $x^\alpha$ ($\alpha \geq 0$) or the Heaviside function $\mathbf{1}_{x>0}$. (Conversely, these are preservers.) □

# 2. Preservers of TN kernels of order 2

**Setting 2:** *Preservers of TN kernels of order 2*

- Can define TN of "finite order":

  *Let $X, Y$ be nonempty totally ordered sets. A kernel $K : X \times Y \to \mathbb{R}$ is totally nonnegative of order $k$, denoted TN$^{(k)}_{X \times Y}$, if all minors of $K(\cdot,\cdot)$ of size $\leq k$ are nonnegative.*

- What are the preservers of such kernels? I.e., if $K$ is TN$^{(2)}_{X \times Y}$, so is $F \circ K$.

**Theorem (Belton–Guillot–K.–Putinar, *J. d'Analyse Math.*, 2023)**

*For all $X, Y$ of size $\geq 2$, the transform $F \circ -$ preserves the class of TN$^{(2)}_{X \times Y}$ kernels if and only if $F(x) = cx^\alpha$ for some $c, \alpha \geq 0$, or $F(x) = c\mathbf{1}_{x>0}$ for some $c > 0$.*

**Proof:** These functions are all preservers. Conversely, act by any preserver on every $2 \times 2$ TN matrix *padded on $X \times Y$ by zeros.* □

# 3. Preservers of $2 \times 2$ TP matrices

**Setting 3:** *Preservers of* $2 \times 2$ *TP matrices*

Here we use *Whitney's density theorem*: $\text{TP}_{m \times n}$ matrices are dense in $\text{TN}_{m \times n}$ matrices. Thus,

1. First prove $F$ is increasing, by applying $F[-]$ to $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ for $x > y > 0$.
2. Using this, show that $F$ is continuous on $(0, \infty)$, hence extends to a continuous function on $[0, \infty)$. Also call this $F$.
3. Hence by Whitney density, now $F[-]$ preserves $2 \times 2$ TN matrices.

Thus, $F$ is as above; and cannot be constant on an interval. So $F(x) = x^\alpha$ for some $\alpha > 0$. (Conversely, all such powers are preservers.) □

# 4. Preservers of TP kernels of order 2

**Setting 4:** *Preservers of TP kernels of order 2*

- Can define TP of "finite order" similar to the TN version.
- What are the preservers of these kernels? I.e., if $K$ is $\mathrm{TP}^{(2)}_{X \times Y}$ then so is $F \circ K$.

**Theorem (Belton–Guillot–K.–Putinar, *J. d'Analyse Math.*, 2023)**

*For all $X, Y$ of size $\geq 2$, the transform $F \circ -$ preserves the class of $\mathrm{TP}^{(2)}_{X \times Y}$ kernels, if and only if $F(x) = cx^{\alpha}$ for some $c, \alpha > 0$.*

Note: now we cannot use "padding by zeros" (since the kernels are TP).

# 4. Preservers of TP kernels of order 2 (cont.)

Thus we make two observations [B–G–K.–P, 2023] :

1. If there exist TP$^{(2)}$ (or TP) kernels on $X \times Y$, then how big can such totally ordered sets $X, Y$ be?

   **Answer:** *They must embed into* $(0, \infty)$*!*

2. Can we "embed" every $2 \times 2$ TP matrix into a TP$^{(2)}$ kernel on $X \times Y$? Or more ambitiously, into a TP kernel on arbitrary $X, Y$?
   - Unlike the TN-case, we cannot pad by zeros.
   - Nevertheless, the answer is: *Yes!* Because. . .

# 4. Preservers of TP kernels of order 2 (cont.)

. . . we come back full circle – to our very first example of TP matrices:

**Lemma (Belton–Guillot–K.–Putinar, *J. d'Analyse Math.*, 2023)**

*Every $2 \times 2$ TP matrix is – up to rescaling by some $c > 0$ – a generalized Vandermonde matrix.*

Hence, it embeds into the TP *kernel* $ce^{xy}$ on $(0, \infty)^2$ – so on $X \times Y$.

Therefore, any preserver in $\mathrm{TP}^{(2)}_{X \times Y}$ must preserve $2 \times 2$ TP matrices. By above, it is a power function. (Conversely, all powers are $\mathrm{TP}^{(2)}$ preservers.) $\square$

In fact, in our paper we *classified the preservers in $TP^{(k)}_{X \times Y}$, for all integers $1 \leq k \leq \infty$, and all nonempty partially ordered sets $X, Y$.*

# Preservers of Pólya frequency functions

**Question:** If $\Lambda(x)$ is a PF function, for which $F : [0,\infty) \to [0,\infty)$ is $F \circ \Lambda$ also one?

- Choose $\Lambda_1(x)$ = exponential density = PF function. First show $F \circ \Lambda_1$ is also discontinuous, so by Schoenberg's classification again an exp-density:

$$F \circ \mathbf{1}_{x \geq 0} e^{-x} = \mathbf{1}_{x \geq 0} c e^{-b_0 x}, \qquad c, b_0 > 0 \quad \text{(except at 0).}$$

- So $F(t) = ct^{b_0}$ for $0 < t < t_0$. Now show that $F(t) = ct^{b_0}$ for all $t > 0$.
- Apply $F \circ -$ to the PF function $\Lambda_2(x) := x \cdot \mathbf{1}_{x \geq 0} e^{-x} = (\Lambda_1 * \Lambda_1)(x)$ = density of sum of two i.i.d. exp-variables $\rightsquigarrow$
  Also a PF function, so $G(s) := 1/\mathcal{L}(F \circ \Lambda_2)(s)$ is in the Laguerre–Pólya class. But $G(s) = (s + b_0)^{1+b_0}/\Gamma(b_0 + 1)$, so $b_0 > 0$ must be an integer.
- Applying $F$ to $\Lambda_0(x) = 2e^{-|x|} - e^{-2|x|}$, conclude: $b_0 = 1$. In summary:

**Theorem (Belton–Guillot–K.–Putinar, *J. d'Analyse Math.*, 2023)**

*The transform $F \circ -$ preserves the class of (one-sided) Pólya frequency functions, if and only if $F(x) = cx$ for some $c > 0$.*

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