Positivity preservers over finite fields

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Abstract:

We resolve an algebraic version of Schoenberg's celebrated theorem characterizing the functions f with the property that the matrix (f(a_{ij})) is positive definite for any positive definite matrix (a_{ij}). Compared to the classical real and complex settings, we consider matrices with entries in a finite field. Here, we say that such a matrix is positive definite if all its leading principal minors are non-zero quadratic residues. We obtain a complete characterization of entrywise positivity preservers in that setting for matrices of a fixed dimension. When the dimension of the matrices is at least 3, we prove that, surprisingly, the positivity preservers are precisely the positive multiples of the field's automorphisms. We also obtain several characterizations of preservers in the more challenging dimension 2 case. Our proofs build on several novel connections between positivity preservers and field automorphisms via the works of Weil, Carlitz, and Muzychuk-Kovács, and via the structure of cliques in Paley graphs.

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