Matrix positivity preservers over finite fields

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Positive definite matrices (real case)

Let A be a real symmetric matrix.

Theorem

The following are equivalent for a symmetric matrix $A \in M_n(\mathbb{R})$:

1. *A is positive definite ($x^T A x > 0 \ \forall x \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$.).*
2. *All the eigenvalues of A are positive.*
3. *There exist a non-singular matrix $B \in M_n(\mathbb{R})$ such that $A = B^2$.*
4. *There exist a full rank matrix $B \in M_{n,m}(\mathbb{R})$ such that $A = BB^T$.*
5. *The matrix A admits a Cholesky factorization $A = LL^T$ (L is lower triangular with positive diagonal entries).*
6. *All the principal minors of A are positive.*
7. **The leading principal minors of A are positive.**

Moreover, the entrywise product $A \circ B = (a_{ij} b_{ij})$ of two positive definite matrices is positive definite.

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- Positive elements in $\mathbb{F}_q$ (non-zero quadratic residues):

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Definition: (see Cooper, Hanna, and Whitlatch, 2022) A matrix $A \in M_n(\mathbb{F}_q)$ is *positive definite* if it is symmetric and its leading principal minors are **positive**.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

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- However,

$$\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

is not positive definite since $\det A = 3 \notin \mathbb{F}_7^+$.

(Lack of) Equivalent definitions

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5. *Only if q is even or $q \equiv 3 \pmod 4$ The matrix A admits a Cholesky factorization $A = LL^T$*
   (L is lower triangular with positive diagonal entries).
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In particular, the quadratic form approach does not yield a useful notion of matrix positivity.

Proposition (Cooper, Hanna, and Whitlatch, 2022)

Let $\mathbb{F}_q$ be a finite field, let $n \geq 3$, and let $A \in M_n(\mathbb{F}_q)$. Then there exists a non-zero vector $x \in \mathbb{F}_q^n$ so that $x^T Ax = 0$.

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Proposition (Guillot, Gupta, Vishwakarma, Yip, 2024)

Let $n \geq 2$ and let $A \in M_n(\mathbb{F}_q)$ be a positive definite matrix. Then

$$\{x^T A x : x \in \mathbb{F}_q^n\} = \mathbb{F}_q.$$

Non-linear entrywise transformers

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Lots of variants considered (for matrices in $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$).
For more details, see e.g.:

- A. Belton et al, *A panorama of positivity. I, II.*, 2019, 2020.
- A. Khare, *Matrix analysis and entrywise positivity preservers*, London Math Society Lecture Notes Series, 2022.

Other preserver problems over finite fields

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Theorem (Dieudonné, 1949)

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Let $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be an invertible linear map over a field $\mathbb{F}$. Suppose ϕ maps the set of singular matrices into itself. Then

$$\phi(A) = MAN \quad or \quad \phi(A) = MA^T N$$

for some $M, N \in M_n(\mathbb{F})$ with $\det(MN) \neq 0$.

See e.g. Marko Orel, "Preserver problems over finite fields" for more details.

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- In particular, in $\mathbb{F}_q$, we have $(x+y)^p = x^p + y^p$.

Theorem (Guillot, Gupta, Vishwakarma, Yip, 2024)

Let $q = p^k$. Then all the positive multiples of the field automorphisms of $\mathbb{F}_q$ preserve positivity on $M_n(\mathbb{F}_q)$ for all $n \geq 1$.

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\det f[A] &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)}^{p^\ell} a_{2,\sigma(2)}^{p^\ell} \cdots a_{n,\sigma(n)}^{p^\ell} \\
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The result follows by applying the above to all leading principal minors of A. $\square$

Paley graphs

- The *quadratic character* $\eta : \mathbb{F}_q \to \{-1, 0, 1\}$ is:

$$\eta(x) = x^{\frac{q-1}{2}} = \begin{cases} 1 & \text{if } x \in \mathbb{F}_q^+ \\ -1 & \text{if } x \notin \mathbb{F}_q^+ \text{ and } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

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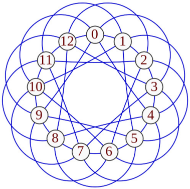
- Let $q = p^k$ where p is odd. The *Paley graph* $P(q) = (V, E)$ is the graph such that
  1. $V = \mathbb{F}_q$ and
  2. $(a, b) \in E$ if and only if $\eta(a - b) = 1$.



The Paley graph $P(13)$.

Credits: David Eppstein – Wikipedia.

- A function f is an *automorphism of the Paley graph* $P(q)$ if

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Theorem (Carlitz, 1960)

Suppose $q = p^k$ where p is odd. Let $f : \mathbb{F}_q \to \mathbb{F}_q$ such that $f(0) = 0$, $f(1) = 1$ and $\eta(f(a) - f(b)) = \eta(a - b)$ for all $a, b \in \mathbb{F}_q$. Then $f(x) = x^{p^\ell}$ for some $0 \leq \ell \leq k - 1$.

Main result: $n \geq 3$

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Moreover, when p *is odd, the above are equivalent to*

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- When $q \equiv 3 \pmod 4$, -1 is not a square, $\mathbb{F}_q = \{0\} \sqcup \mathbb{F}_q^+ \sqcup (-\mathbb{F}_q^+)$.

Key ingredient: bijectivity on $\mathbb{F}_q^+$

Lemma

Let $\mathbb{F}_q$ be a finite field with q even or $q \equiv 3 \pmod 4$ and let $f : \mathbb{F}_q \to \mathbb{F}_q$. Suppose f preserves positive definiteness on $M_2(\mathbb{F}_q)$. Then:

1. *The restriction of f to $\mathbb{F}_q^+$ is a bijection of $\mathbb{F}_q^+$ onto itself.*
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which is not PD, a contradiction. A similar argument can be used if $f(0) \in -\mathbb{F}_q^+$. Thus $f(0) = 0$. $\square$

Characteristic 2: preservers on $M_2(\mathbb{F}_q)$

- Assume $q = 2^k$ for some $k \geq 1$.
- Since $f(x) = x^2$ is bijective, every $x \in \mathbb{F}_q$ has a unique square root $\sqrt{x}$.
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Idea of proof

$(1) \implies (2)$. For $x, y \neq 0$, consider

$$A(z) = \begin{pmatrix} x & \sqrt{xy}z \\ \sqrt{xy}z & y \end{pmatrix} \qquad (z \in \mathbb{F}_q). \quad \det A(z) = xy(1 - z^2).$$

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$(3) \implies (1)$. Trivial.

Theorem

Let $q = 2^k$ *and let* $f : \mathbb{F}_q \to \mathbb{F}_q$ *preserve positivity on* $M_3(\mathbb{F}_q)$*. Then* $f(x) = cx^{2^l}$ *for some* $0 \leq l \leq k-1$ *and* $c \in \mathbb{F}_q^+$*.*

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Characteristic 2: dimension ≥ 3

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Observe:

$$\det A = 0 \iff x^2 + y^2 = (x+y)^2 = 1 \iff x + y = 1$$
$$\det f[A] = 0 \iff x^{2n} + y^{2n} = (x^n + y^n)^2 = 1 \iff x^n + y^n = 1.$$

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$$S_1 := \{(x,y) \in \mathbb{F}_q^2 : x + y = 1\}, \qquad S_2 := \{(x,y) \in \mathbb{F}_q^2 : x^n + y^n = 1\}.$$

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7. Thus $x \mapsto x^n$ is a field automorphism and so $n = 2^l$ for some $0 \leq l \leq k-1$. $\square$

Theorem (Main Result, Guillot, Gupta, Vishwakarma, Yip, 2024)

Let $q = p^k$ and $f : \mathbb{F}_q \to \mathbb{F}_q$. Then the following are equivalent:

1. *f preserves positivity on $M_n(\mathbb{F}_q)$ for some $n \geq 3$.*
2. *f preserves positivity on $M_n(\mathbb{F}_q)$ for all $n \geq 3$.*
3. *$f(x) = cx^{p^\ell}$ for some $c \in \mathbb{F}_q^+$ and $0 \leq \ell \leq k-1$.*

Moreover, when p is odd, the above are equivalent to

4. *$f(0) = 0$ and f is an automorphism of the Paley graph associated to $\mathbb{F}_q$, i.e., $\eta(f(a) - f(b)) = \eta(a-b)$ for all $a, b \in \mathbb{F}_q$.*

- The key idea for resolving the $p \neq 2$ cases is to show that the positivity preservers are automorphisms of the associated Paley graph, i.e.,

$$\eta(f(a) - f(b)) = \eta(a-b) \text{ for all } a, b \in \mathbb{F}_q.$$

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Thus, $\eta(f(a) - f(b)) = 1$ since $\eta(f(b)) = 1$.

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where $c \in \mathbb{F}_q^+$ and $\eta(b-c) = 1$. Consider the linear map $g : \mathbb{F}_q \to \mathbb{F}_q$ given by $g(x) = x + b$.

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We know that $\eta(f(c)) = 1$, and using the previous case applied with $a' = b$ and $b' = c$, we conclude that $\eta(f(b) - f(c)) = 1$. Thus, $\eta(f(a) - f(b)) = 1$.

Finally, if $\eta(a-b) = -1$, then $\eta(b-a) = 1$. Hence, by the above argument $\eta(f(b) - f(a)) = 1$. That implies $\eta(f(a) - f(b)) = -1$. Thus, $(1) \implies (3)$ and the result follows.

For 2×2 matrices...

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- When $q \equiv 3 \pmod 4$, all positivity preservers are $f(x) = cx^{p^\ell}$ for some $c \in \mathbb{F}_q^+$ and $0 \leq \ell \leq k-1$. Proof is much more complicated for $M_2(\mathbb{F}_q)$!

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- When $q \equiv 1 \pmod 4$, we resolved the case $q = r^2$. Otherwise, this is an open problem.

Proposition

Let $q = p^k$ be a prime power with $q \equiv 1 \pmod 4$ and let f be a positivity preserver over $M_2(\mathbb{F}_q)$ with $f(1) = 1$. **Assume additionally that f is injective on $\mathbb{F}_q^+$.** *Then there exists $0 \leq l \leq k-1$ such that $f(x) = x^{p^l}$ for all $x \in \mathbb{F}_q$.*

The proof relies on the following result of Muzychuk and Kovács.

General approach when $q \equiv 1 \pmod 4$

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- With (quite a bit of) extra work, we rule out the ax^{-p^l} case.

When $q \equiv 1 \pmod 4$,

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Open problem: If f preserves positivity on $M_2(\mathbb{F}_q)$ where $q \equiv 1 \pmod 4$ is not a square, does f have to be injective on $\mathbb{F}_q^+$?

The $q = r^2$ case

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In the Paley graph $P(q)$, the clique number of $P(q)$ is r. Moreover, all maximum cliques are of the form $\alpha\mathbb{F}_r + \beta$, where $\alpha \in \mathbb{F}_q^+$ and $\beta \in \mathbb{F}_q$ (squares translates of the subfield $\mathbb{F}_r$).

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- Note that $\mathbb{F}_q^*/\mathbb{F}_r^*$ is a well-defined group.
- We can thus write $\mathbb{F}_q^* = a_1\mathbb{F}_r^* \sqcup a_2\mathbb{F}_r^* \sqcup \cdots \sqcup a_{r+1}\mathbb{F}_r^*$.

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- We can thus write $\mathbb{F}_q^* = a_1\mathbb{F}_r^* \sqcup a_2\mathbb{F}_r^* \sqcup \cdots \sqcup a_{r+1}\mathbb{F}_r^*$.
- We say that a coset of the form $a\mathbb{F}_q^*$ with $a \in \mathbb{F}_q^+$ is a *square coset*.

Outline of proof for $q = r^2$

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4. We conclude $f(x) = ax^{p^j}$ for all $x \in \mathbb{F}_q$.

The above steps are highly non-trivial and exploit the known maximal clique structure of $P(r^2)$.

Ongoing work

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Possible research directions:

- New connections to other areas/problems? Applications?
- Applications of positive definite matrices over $\mathbb{F}_q$?

Reference

D. Guillot, H. Gupta, P.K. Vishwakarma, and C.H. Yip. Positivity preservers over finite fields. arXiv:2404.00222 (2024) 32 pages.

Thank you!