Gabor Frames of Totally Positive Functions and Estimates of their Frame Bounds

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joint work with

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Outline

- Gabor families and their pre-Gramian matrix
- Matrix analysis for “painless” frames
- Matrix analysis for rational lattice parameters
- Matrix analysis for Gabor frames of totally positive functions

1. Gabor families and their pre-Gramian matrix

Definition: Gabor family, Gabor frame

Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$. The set

$$\mathcal{G}(g, \alpha, \beta) = \{M_{l\beta} T_{k\alpha} g := e^{2\pi i \beta l \cdot} g(\cdot - \alpha k) : k, l \in \mathbb{Z}\}$$

is called a *Gabor family*. If there exist constants $A, B > 0$, such that

$$A\|f\|_2^2 \leq \sum_{k,l\in\mathbb{Z}} |\langle f, M_{l\beta} T_{k\alpha} g\rangle|^2 \leq B\|f\|_2^2 \qquad \text{for all } f \in L^2(\mathbb{R}),$$

then $\mathcal{G}(g, \alpha, \beta)$ is a *Gabor frame*, and A, B are called *lower* and *upper frame bound*.

Result: For every Gabor frame, there exists another Gabor frame $\mathcal{G}(\gamma, \alpha, \beta)$ such that

$$f = \sum_{k,l\in\mathbb{Z}} \langle f, M_{l\beta} T_{k\alpha} g\rangle \, M_{l\beta} T_{k\alpha} \gamma$$

holds for all $f \in L^2(\mathbb{R})$. The function $\gamma \in L^2(\mathbb{R})$ is called a *dual window* of g.

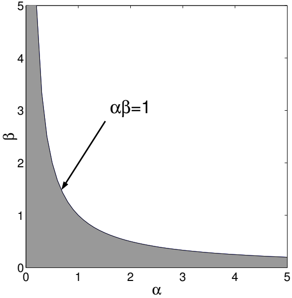
Problems addressed in this talk:

P1 Describe a class of functions $g \in L^2(\mathbb{R})$ such that

$$\mathcal{G}(g,\alpha,\beta) = \{M_{l\beta} T_{k\alpha} g := e^{2\pi i \beta l \cdot} g(\cdot - \alpha k) : k, l \in \mathbb{Z}\}$$

constitutes a frame of $L^2(\mathbb{R})$, for all lattice parameters

$$(\alpha,\beta) \in \mathcal{F} = \{(x,y) \in \mathbb{R}_+^2 : xy < 1\}.$$



The *maximal set* $\mathcal{F}$:
Daubechies 1992
Benedetto, Heil, Walnut, 1995

Problems addressed in this talk:

P2 If $\mathcal{G}(g, \alpha, \beta)$ is a frame of $L^2(\mathbb{R})$,

find explicit dual windows $\gamma \in L^2(\mathbb{R})$.

P3 If $\mathcal{G}(g, \alpha, \beta)$ is a frame of $L^2(\mathbb{R})$ for all $0 < \beta < \alpha^{-1}$,

find the rate at which the lower frame bound decreases near the *critical density* $\beta \nearrow \alpha^{-1}$.

Link between Gabor frames and matrix analysis

Theorem [Janssen 1993, Ron and Shen 1997]

The set $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R})$ with bounds $A, B > 0$

if and only if

the (pre-Gramian) matrices

$$P_g(x) = \left(g(x + j\alpha - \tfrac{k}{\beta}) \right)_{j,k \in \mathbb{Z}}$$

satisfy

$$\beta A \|c\|^2 \leq \|P_g(x)c\|^2 \leq \beta B \|c\|^2$$

for almost all $x \in [0, \alpha)$ and all $c \in \ell_2(\mathbb{Z})$.

- $P_g(x)$ is a bi-infinite matrix which defines a bounded operator on $\ell_2(\mathbb{Z})$, which is also bounded from below, with bounds independent of $x \in [0, \alpha)$.

Link between Gabor frames and matrix analysis

Moreover, $\mathcal{G}(\gamma, \alpha, \beta)$ is a dual Gabor frame, if the pre-Gramian matrices

$$P_\gamma(x) = \left(\gamma(x + j\alpha - \frac{k}{\beta})\right)_{j,k\in\mathbb{Z}}$$

satisfy

$$P_\gamma(x)^* P_g(x) = \mathrm{id}_{\ell^2(\mathbb{Z})} \qquad \text{for a.e. } x \in (0, \alpha),$$

$$\operatorname*{ess\,sup}_x \|P_\gamma(x)\|_{\ell^2\to\ell^2} < \infty;$$

that is, $P_\gamma(x)^*$ is a uniformly bounded set of left-inverses of $P_g(x)$.

Link between Gabor frames and matrix analysis

Formulation in terms of **sampling in shift-invariant spaces**:

- The columns of

$$P_g(x) = \Big(g(x + j\alpha - \tfrac{k}{\beta})\Big)_{j,k\in\mathbb{Z}}$$

  refer to a shift-invariant subspace

$$V^2(g) = \operatorname{clos}\operatorname{span}\{g(\cdot - k/\beta)\} \subset L^2(\mathbb{R}).$$

- Its rows refer to sampling points $\{x_j = x + j\alpha;\ j \in \mathbb{Z}\}$.
- The frame bounds $A, B > 0$ are characterized by

$$\beta A\|c\|_2^2 \le \sum_{j\in\mathbb{Z}} |f_c(x + \alpha j)|^2 \le \beta B\|c\|^2 \quad \text{for all } c \in \ell_2(\mathbb{Z}),$$

  where we set $f_c = \sum_{k\in\mathbb{Z}} c_k g(\cdot - k/\beta) \in V^2(g)$.

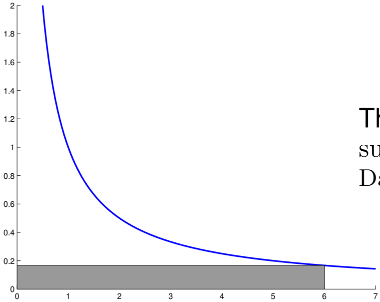
2. Matrix analysis for "painless" frames

g compactly supported, $\beta < (\mathrm{length}(\mathrm{supp}\, g))^{-1}$, $\alpha\beta < 1$

$\mathcal{G}(g,\alpha,\beta)$ has the frame bounds $A, B > 0$, where

$$\beta A = c(g,\alpha) := \operatorname*{ess\,inf}_x \sum_{j\in\mathbb{Z}} |g(x+j\alpha)|^2,$$

$$\beta B = C(g,\alpha) := \operatorname*{ess\,sup}_x \sum_{j\in\mathbb{Z}} |g(x+j\alpha)|^2.$$



The "painless" frame bounds, if $\mathrm{supp}\, g = [0, 6]$:
Daubechies, Grossmann, Meyer 1986

Proof:

The assumption $1/\beta > \mathrm{length}(\mathrm{supp}\, g)$ implies that the nonzero entries in the columns of $P_g(x)$ do not overlap. It is a simple task to write down the Moore-Penrose pseudoinverse $\Gamma(x)$ of such a matrix:

$$P_g(x) = \begin{pmatrix} \ddots & & & & \\ & \times & & & \\ & \vdots & & & \\ & \times & & & \\ & & a_1 & & \\ & & \vdots & & \\ & & a_r & & \\ & & & \times & \\ & & & \vdots & \\ & & & \times & \\ & & & & \ddots \end{pmatrix}, \qquad \Gamma(x)^T = \begin{pmatrix} \ddots & & & & \\ & \times & & & \\ & \vdots & & & \\ & \times & & & \\ & & b_1 & & \\ & & \vdots & & \\ & & b_r & & \\ & & & \times & \\ & & & \vdots & \\ & & & \times & \\ & & & & \ddots \end{pmatrix}$$

with $b_j = a_j/\|a\|^2$.

3. Matrix analysis for rational lattice parameters

- For a rational lattice density

$$\alpha\beta = \frac{p}{q} \in \mathbb{Q}, \qquad p, q \in \mathbb{N}, \;\; 0 < p < q, \;\; gcd(p, q) = 1$$

and all pairs $(j, k) = (qm, pm)$ with $m \in \mathbb{Z}$ we have

$$x + j\alpha - \frac{k}{\beta} = x + \frac{1}{\beta}\,\frac{jp - kq}{q} = x.$$

- The pre-Gramian satisfies $P_g(x + j\alpha) = P_g(x)$. In other words, it is a block-Toeplitz matrix

$$P_g(x) = \begin{pmatrix} \ddots & & & & & & \\ & P_1 & P_0 & P_{-1} & & & \\ & \cdots & P_1 & P_0 & P_{-1} & \cdots & \\ & & & P_1 & P_0 & P_{-1} & \\ & & & & & & \ddots \end{pmatrix}$$

$$P_m = P_m(x) = \left(g(x + qm\alpha + j\alpha - k/\beta)\right)_{\substack{0 \le j \le q-1 \\ 0 \le k \le p-1}} \quad \text{for} \quad m \in \mathbb{Z}.$$

Matrix analysis for rational lattice parameters

- Optimal frame bounds can be obtained from the symbol of the corresponding Laurent operator

$$\sigma_g(x,\omega) = \sum_{m\in\mathbb{Z}} P_m(x) e^{-2\pi i m\omega},$$

namely

$$(\beta A)^{-1} = \operatorname*{ess\,sup}_x \left(\operatorname*{ess\,sup}_\omega \left(\sigma_g(x,\omega)\right)^{\dagger} \right)$$

$$\beta B = \operatorname*{ess\,sup}_x \left(\operatorname*{ess\,sup}_\omega \left(\sigma_g(x,\omega)\right) \right)$$

- This observation can be translated into the Zibulski-Zeevi condition, which uses a $q \times p$-matrix of Zak-transforms of g.

4. Methods for Gabor frames of totally positive functions

I. J. Schoenberg started an extensive investigation of *totally positive functions* in 1947:

Definition

A non-constant measurable function $g : \mathbb{R} \to \mathbb{R}$ is *totally positive*, if it satisfies the following condition: For every two sets of increasing real numbers

$$x_1 < x_2 < \cdots < x_N, \qquad y_1 < y_2 < \cdots < y_N, \qquad N \in \mathbb{N},$$

we have the inequality

$$D = \det \left[g(x_j - y_k) \right]_{1 \le j,k \le N} \ge 0.$$

Methods for Gabor frames of totally positive functions

- Schoenberg showed that g is totally positive and integrable, if and only if its Fourier transform is

$$\hat{g}(\omega) = Ce^{-\gamma\omega^2 + 2\pi i\delta\omega} \prod_{\nu=1}^{\infty} \frac{e^{2\pi i\omega/a_\nu}}{1 + 2\pi i\omega/a_\nu},$$

with real parameters C, γ, δ, real $a_\nu \neq 0$ satisfying

$$C > 0, \quad \gamma \geq 0, \quad 0 < \gamma + \sum_{\nu=1}^{\infty} a_\nu^{-2} < \infty.$$

- We consider the sub-class of totally positive functions of finite type:

$$\hat{g}(\omega) = C \prod_{\nu=1}^{m} (1 + 2\pi i\omega/a_\nu)^{-1},$$

with real $a_1, \ldots, a_m \neq 0$, $C > 0$.

Examples of totally positive functions of finite type

- sums of one-sided exponentials:

$$0 \le g(x) = \sum_{\nu=1}^{m} c_\nu e^{-a_\nu x} \, \chi_{[0,\infty)}(x) \in C^{m-2}(\mathbb{R}),$$

  with $a_1, \ldots, a_m > 0$; coefficients c_ν come from divided difference

$$g(x) = [a_1, \ldots, a_m \mid e^{-x\cdot}] \, \chi_{[0,\infty)}.$$

- two-sided exponentials, e.g.

$$g(x) = e^{ax}\chi_{(-\infty,0)} + e^{-bx}\chi_{[0,\infty)} \in C(\mathbb{R}),, \quad a, b > 0;$$

- Variants including polynomial factors, e.g.

$$g(x) = x^m e^{-x} \, \chi_{[0,\infty)} \in C^{m-1}(\mathbb{R}).$$

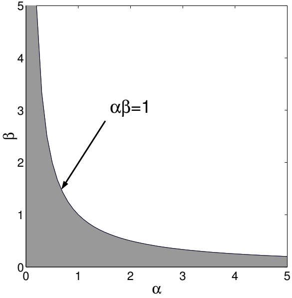
Observation: The functions decay exponentially. The set of TP functions of finite type is closed under translation, dilation and convolution.

Problem P1: frame-set

Theorem (Gröchenig, St. 2011)

Assume that g is a totally positive function of finite type $m \geq 2$.

Then $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame, if and only if $\alpha\beta < 1$.



TP functions of finite type and the maximal set

Problem P2: dual windows

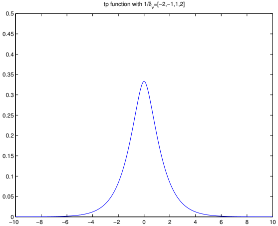
Furthermore, let $r := \lfloor \frac{1}{1-\alpha\beta} \rfloor$, and assume that in the definition of the Fourier transform $\hat{g}$,

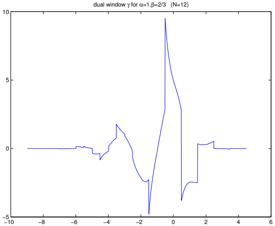
- n_1 is the number of positive a_ν's,
- n_2 is the number of negative a_ν's.

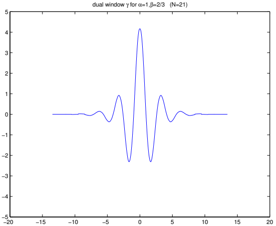
Then we construct, for each $L \in \mathbb{N}$, a dual window γ_L with **compact support**

$$\text{supp}\ \gamma_L \subset [-\tfrac{r\,n_1+L}{\beta} - \alpha, \tfrac{r\,n_2+L}{\beta} + \alpha].$$

Example:







Conjecture: the sequence of duals γ_L converges to the canonical dual

Proof by matrix analysis of the pre-Gramiam

- Choose g with

$$\hat{g}(\omega) = \prod_{\nu=1}^{n} (1 + 2\pi i\omega/a_\nu)^{-1}, \qquad a_1, \dots, a_n \in \mathbb{R} \setminus \{0\}.$$

- Fix $\alpha = 1$. (All other cases by scaling of g.)
- The pre-Gramian

$$P_g(x) = (g(x + j - k/\beta))_{j,k\in\mathbb{Z}}$$

is a bi-infinite **totally positive** matrix. It is fully populated, if some a_ν's are positive and some are negative.

Matrix product with invertible bidiagonal matrices:

In a first step, we obtain a slant-banded matrix by the following operations:

- The function N_g with

$$\hat{N}_g(\omega) = \prod_{\nu=1}^{n} \left(1 - e^{-(a_\nu + 2\pi i\omega)}\right) \hat{g}(\omega)$$

is an *exponential B-spline* with compact support $[0, n]$:

$$N_g(x) = Ce^{-a_1(\cdot)}\chi_{[0,1)} * e^{-a_2(\cdot)}\chi_{[0,1)} * \ldots * e^{-a_n(\cdot)}\chi_{[0,1)}$$

- The pre-Gramians of g and N_g are related by

$$P_{N_g}(x) = B_1 \cdots B_n \, P_g(x),$$

where B_ν is a bidiagonal (biinfinite) invertible Toeplitz matrix

$$B_\nu = I - e^{-a_\nu} D_1, \qquad D_1 = (\delta_{k,j+1})_{j,k\in\mathbb{Z}}.$$

New pre-Gramian P_{N_g}:

- The pre-Gramian $P_{N_g}(x)$ has at most n nonzero entries per column.
- The sequence of row indices j_k of the first nonzero entry of column k is strictly increasing with gaps; more precisely

$$j_{k+r} - j_k \geq r + 1 \quad \text{with} \quad r := \lfloor \tfrac{1}{1-\alpha\beta} \rfloor.$$

$$P_{N_g}(x) = \begin{pmatrix} \ddots & \times & \times & & k & & & & \\ & & \times & \times & \downarrow & & & & \\ & & \times & \times & \times & \leftarrow & j_k & & \\ & & & \times & \times & \times & & & \\ & & & & \times & \times & & & \\ & & & & & \times & \times & & \\ & & & & & & \times & \times & \\ & & & & & & \times & \times & \\ & & & & & & & \times & \ddots \end{pmatrix}$$

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$$P_{N_g}(x) = \begin{pmatrix} \ddots & \times & \times & & & & & & \\ & & {\color{red}\times} & {\color{red}\times} & & & & & \\ & & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & & & & \\ & & & {\color{red}\times} & {\color{red}\times} & {\color{red}\times} & & & \\ & & & & {\color{red}\times} & {\color{red}\times} & & & \\ & & & & & \times & \times & & \\ & & & & & & \times & \times & \\ & & & & & & \times & \times & \\ & & & & & & & \times & \ddots \end{pmatrix}$$

New pre-Gramian P_{N_g}:

- Results in Approximation Theory (Karlin 1968, Schumaker 1981, Gasca, Pena et al. 1992): Every finite block of $P_{N_g}(x)$ is almost strictly totally positive, i.e.
  - every minor is non-negative,
  - the minor is strictly positive iff its diagonal entries are positive.
- A left-inverse Γ_{N_g} of $P_{N_g}(x)$ is constructed by
  - choosing a finite block $P_{N_g}(j1 : j2,\ k1 : k2)$ of full column rank, such that only zeros appear to the left and right in the same rows of P_{N_g},
  - taking rows from the Moore-Penrose pseudoinverse of this block as the nonzero entries in corresponding rows of $\Gamma_{N_g}(x)$.

Gabor frames with window function N_g

Theorem (Kloos, St. 2014)

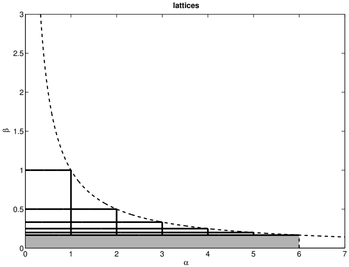
Let N_g be an exponential B-spline of finite order n. Then $\mathcal{G}(N_g, 1, \beta)$ is a Gabor frame for all $0 < \beta < 1$.

Furthermore, $\mathcal{G}(N_g, \alpha, \beta)$ is a Gabor frame in the following cases:

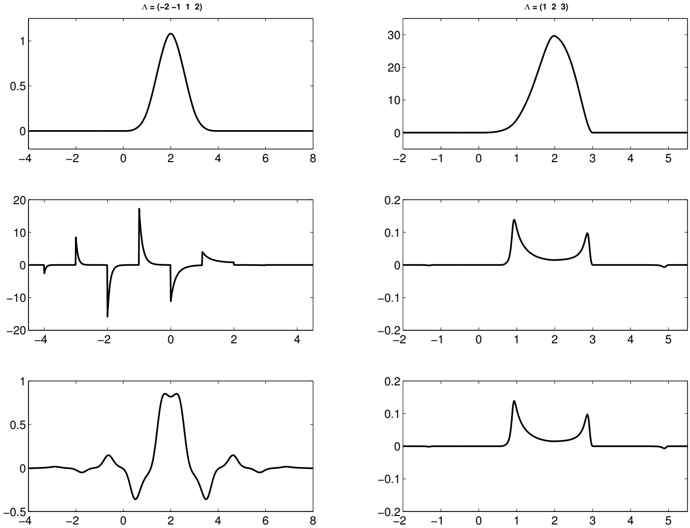
(1) $0 < \alpha < m$ and $0 < \beta \leq m^{-1}$ ("painless"),

(2) $\alpha \in \{1, 2, \ldots, m-1\}$, $\beta > 0$ and $\alpha\beta < 1$,

(3) $\alpha > 0$, $\beta \in \{1, 2^{-1}, \ldots, (m-1)^{-1}\}$ and $\alpha\beta < 1$.



Example: Exponential B-splines (top) with two duals



Previous work:

Explicit duals $\gamma \in C^{m-2}(\mathbb{R})$ with compact support were constructed, if $\beta < (2m)^{-1}$ (Christensen, Massopust 2012, Nielsen 2019)

Matrix analysis for explicit frame bounds

- Nonsingular totally positive matrices $P \in \mathbb{R}^{m\times m}$ can be factorized in terms of $m-1$ lower (and $m-1$ upper) bidiagonal matrices of the form

$$B_\nu = I + D_\nu \quad \text{with} \quad D_\nu = (d_{\nu,j}\delta_{j+1,k})_{j,k=1,\ldots,m}$$

(and their transpose), combined with a diagonal matrix with positive entries. See Gasca, Pena, 1995.

- Here, $d_j \geq 0$ are factors in the complete Neville-elimination, first transforming P into an upper triangular matrix U and then transforming U^T into a diagonal matrix, by subsequent row-operations.

- If P has bandwidth s, the number of factors is reduced from $m-1$ to s.

Matrix analysis for explicit frame bounds

- The simple relation

$$(I + D_\nu)^{-1} = \sum_{j=0}^{m-1} (-D_\nu)^j$$

allows us to obtain the following result:

If $0 < d_j \leq 1 - \epsilon$ for all $1 \leq j \leq m-1$, then $\|(I + D_\nu)^{-1}\|_2 \leq \frac{1}{\epsilon}$.

- Take a finite block $P \in \mathbb{R}^{p \times m}$ of $P_{N_g}(x)$ with $p > m$ with the following properties:
  - P has full rank.
  - P has a slanted band-structure as in $P_{N_g}(x)$.
  - P contains all nonzero entries of P_{N_g} in the corresponding rows.

  Find a factorization with s invertible bidiagonal matrices $B_\nu = I + D_\nu$ such that $0 \leq d_{\nu,j} \leq \alpha\beta < 1$.

  Then the lower frame bound satisfies

$$A^{-1} = O\left((1 - \alpha\beta)^{-s}\right).$$

A first example

The even exponential B-spline of order 2 is defined by

$$B_2(x) = (e^{\lambda(\cdot)}\chi_{[0,1]} * e^{-\lambda(\cdot)}\chi_{[0,1]})(x) = \begin{cases} \dfrac{\sinh(\lambda x)}{\lambda}, & 0 \le x \le 1, \\[2ex] \dfrac{\sinh(\lambda(2-x))}{\lambda}, & 1 < x \le 2. \end{cases}$$

Theorem (Kloos, St. 2014)

The lower frame bound of $\mathcal{G}(B_2, 1, \beta)$ satisfies

$$c_\lambda(1-\beta) \le A \qquad \text{for} \quad 1/2 \le \beta < 1$$

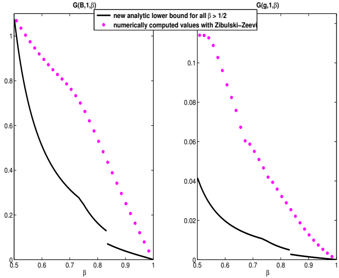
with explicit constant $c_\lambda > 0$.

Example:

The exponential B-spline of order 2 with exponents $\Lambda = (-1, 1)$ is

$$B_2(x) = \begin{cases} \sinh x, & x \in [0,1], \\ \sinh(2-x), & x \in (1,2], \\ 0 & \text{otherwise.} \end{cases}$$

The bounds for A are shown on the left, the bound for the related TP function $g(x) = e^{-\lambda|x|}$ are shown on the right. (right figure).



Explicit frame bounds by other methods:

- The Gaussian window $g(x) = e^{-\pi x^2}$ satisfies the same asymptotic relation

$$A^{-1} = O\left((1-\alpha\beta)^{-1}\right) \text{ for } \quad \alpha\beta \to 1.$$

  (Borichev, Gröchenig, Lyubarskii 2010; methods of proof from complex analysis)

- Upper bounds of both frame bounds A, B for more general Gabor frames in $\mathbb{R}^d$ (without the requirement of a lattice structure for time-frequency shifts) were recently obtained by K. Gröchenig, J. L. Romero and M. Speckbacher.

Ongoing research

Quantitative results for the decomposition of full-rank TP matrices would be of great benefit. They are useful for

- Gabor frames: sharp estimates of the frame bounds
- theory of sampling in shift-invariant spaces generated by TP functions and (exponential) B-splines.