Gabor frames with totally positive windows

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Applied Matrix Positivity

The sampling matrix

$g \in L^2(\mathbb{R})$ generator/template, $x \in \mathbb{R}$,
$\alpha, \beta > 0$ lattice parameters

Central object — sampling matrix, collocation matrix

$$P(x) = \left(g(x + \alpha j - \frac{k}{\beta})\right)_{j,k\in\mathbb{Z}}$$

More general versions use $P_{jk} = g(x_j - y_k)$.

Relevance: shift-invariant spaces

Shift-invariant space with generator g in $L^p(\mathbb{R})$

$$V^p(g) = \{f \in L^p(\mathbb{R}) : f(x) = \sum_{k\in\mathbb{Z}} c_k g(x-k), c \in \ell^p(\mathbb{Z})\}$$

$X = (x_j)_{j\in\mathbb{Z}} \subseteq \mathbb{R}$ is *sampling set* (set of stable sampling), if exist $A, B > 0$ such that

$$A\|f\|_p^p \leq \sum_{j\in\mathbb{Z}} |f(x_j)|^p \leq B\|f\|_p^p \qquad \text{for all } f \in V^p(g)\,.$$

For $g(x) = \frac{\sin \pi x}{\pi x}$ (with $\hat{g} = \chi_{[-1/2,1/2]}$) we obtain the Paley-Wiener space

$$V^2(g) = \mathrm{PW} = \{f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subseteq [-1/2, 1/2]\}$$

Shift-invariant spaces are substitutes for Paley-Wiener space.

Connection

X is sampling for $V^p(g)$, if and only if the infinite matrix

$$P = \Big(g(x_j - k)\Big)_{j,k\in\mathbb{Z}}$$

is left-invertible (and bounded) on $\ell^p(\mathbb{Z})$:

$$\sum_{j\in\mathbb{Z}} |f(x_j)|^p = \sum_{j\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} c_k g(x_j - k)\Big|^p = \|Pc\|_p^p \geq A\|c\|_p^p \asymp \|f\|_p^p$$

Uniform sampling: $X = x_0 + \alpha\mathbb{Z}$ leads to $P(x_0)$ with entries $g(x_0 + \alpha j - k)$.

Gabor Frames

Time-frequency shifts $M_{\beta l} T_{\alpha k} g(x) = e^{2\pi i \beta l x} g(x - \alpha k)$

Collection of time-frequency shifts $\mathcal{G}(g, \alpha, \beta) = \{M_{\beta l} T_{\alpha k} g : k, l \in \mathbb{Z}\}$ is Gabor frame for $L^2(\mathbb{R})$, if $\exists A, B > 0$, such that

$$A\|f\|_2^2 \leq \sum_{k,l\in\mathbb{Z}} \Big|\langle f, M_{\beta l} T_{\alpha k} g\rangle\Big|^2 \leq B\|f\|_2^2 \qquad \text{for all } f \in L^2(\mathbb{R})$$

Time-frequency analysis

Connection

Theorem

Assume that $\sum_{k\in\mathbb{Z}} \sup_{|x|\leq 1/2} |g(x+k)| < \infty$ *and g is continuous. TFAE:*

(i) *For all* $x \in \mathbb{R}$

$$P(x) = \Big(g(x + \alpha j - \frac{k}{\beta})\Big)_{j,k\in\mathbb{Z}}$$

is left-invertible on ℓ^2.

(ii) $x + \alpha\beta\mathbb{Z}$ *is sampling for* $V^2(g_\beta)$ *for all* $x \in \mathbb{R}$ *with* $g_\beta(x) = g(x/\beta)$.

(iii) $\{M_{\beta l} T_{\alpha k} g : k, l \in \mathbb{Z}\}$ *is a Gabor frame for* $L^2(\mathbb{R})$.

Enter total positivity

A function $g : \mathbb{R} \to \mathbb{R}$ is called totally positive (TP)[1], if for all $n \in \mathbb{N}$ and choices $x_1 < x_2 < \cdots < x_{n-1} < x_n$ and $y_1 < y_2 < \cdots < y_{n-1} < y_n$ the truncated matrix has non-negative determinant:

$$\det \Big(g(x_j - y_k) \Big)_{j,k=1,\dots,n} \geq 0$$

Consequence: Morally, finite sections of $P(x)$ with entries $g(x + \alpha j - \frac{k}{\beta})$ are invertible.

[1] Polya frequency function, if $g \in L^1(\mathbb{R})$

Examples

Main examples ($a_j \in \mathbb{R}, \gamma > 0$):

- $\eta_a(x) = e^{-ax}\chi_{[0,\infty)}(ax)$ (one-sided exponential)
- $g = \eta_{a_1} * \eta_{a_2} * \cdots * \eta_{a_n}$ TP of finite type, e.g., $e^{-|x|} = \eta_1 * \eta_{-1}$
- $g(x) = e^{-\gamma x^2}$, $\gamma > 0$
- $g = e^{-\gamma x^2} * \eta_{a_1} * \eta_{a_2} * \cdots * \eta_{a_n}$ TP of Gaussian type
- $g(x) = (e^{ax} + e^{-ax})^{-1}$ hyperbolic secant
- $g(x) = e^{-x-e^{-x}}$

Remark: If g is TP and in $L^1(\mathbb{R})$, then g has exponential decay.

State-of-the-art

Theorem

$\alpha, \beta > 0$

(i) *Let* $g = \eta_{a_1} * \eta_{a_2} * \cdots * \eta_{a_n}$ *with* $n > 1$ *or*
$$g = e^{-\gamma x^2} * \eta_{a_1} * \eta_{a_2} * \cdots * \eta_{a_n} \text{ with } n \geq 0 \text{ and } \gamma > 0.$$
Then $\{M_{\beta l} T_{\alpha k} g : k, l \in \mathbb{Z}\}$ *is a Gabor frame, if and only if* $\alpha\beta < 1$.

(ii) *Let* $g \in L^1(\mathbb{R})$ *be arbitrary totally positive and* $\alpha\beta$ *rational.*
Then $\{M_{\beta l} T_{\alpha k} g : k, l \in \mathbb{Z}\}$ *is a Gabor frame, if and only if* $\alpha\beta < 1$.

History:

- Finite type (J. Stöckler, KG, 2013) with definition of TP
- Gaussian type (J.L. Romero, J. Stöckler, KG, 2018) with Schoenberg's characterization of TP and complex analysis
- Rational Gabor frames (KG, 2023): new approach

Invertibility of TP matrices

Lemma (de Boor's lemma, 1967)

Assume that A is $n \times n$ totally positive matrix (all minors have non-negative determinant). If there exists $u \in \mathbb{R}^n, \|u\|_\infty = 1$, such that Au is alternating, $(-1)^j (Au)_j \geq \delta > 0$, then

$$\|A^{-1}\|_{\infty \to \infty} \leq \frac{1}{\delta} .$$

Theorem (de Boor, Friedland, Pinkus, 1982)

Let $J = \{1, \ldots, n\}, \mathbb{N}$ or $\mathbb{Z}$, and A be a totally positive matrix indexed by J, such that $A : \ell^\infty(J) \to \ell^\infty(J)$. If there exists $u \in \ell^\infty(J)$ such that

$$(-1)^j (Au)_j > 0 \quad \textit{and} \quad \inf_{j \in J} |u(j)| > 0 ,$$

then A is onto $\ell^\infty(J)$.

Example – uniform, critical sampling

Let $X = x_0 + \mathbb{Z}$ ($\alpha = 1$), then

$$(P(x_0)c)(j) = \sum_{k \in \mathbb{Z}} c_k g(x_0 + j - k) = (c * g_{x_0})(j) \quad j \in \mathbb{Z}$$

with $g_{x_0}(j) = g(x_0 + j)$.

Convolution $c \mapsto c * g_{x_0}$ is invertible, if and only if $\inf_\xi |\widehat{g_{x_0}})(\xi)| > 0$.

$$\hat{g_{x_0}}(\xi) = \sum_{k \in \mathbb{Z}} g(x_0 + k) e^{2\pi i k \xi} := Zg(x_0, -\xi)$$

Zg is called **Zak transform** (Weil-Brezin transform, Floquet transform, kq-transform etc.)

Zak transform of TP functions

Example: If $g(x) = e^{-\pi x^2}$, Zg is a version of a theta function, and

$$Zg(x,\xi) = 0 \;\Leftrightarrow\; x,\xi = \tfrac{1}{2} \mod 1$$

$\Rightarrow$ $x_0 + \mathbb{Z}$ is sampling, if and only if$x_0 \neq \frac{1}{2}$, but $\frac{1}{2} + \mathbb{Z}$ is *not* sampling.
$\{e^{2\pi i l x} e^{-\pi(x-k)^2} : k, l \in \mathbb{Z}\}$ is **not** a frame (but is dense in $L^2(\mathbb{R})$).

Theorem

If $g \in L^1(\mathbb{R})$ is totally positiv $\neq \eta_a$, then its Zak transform has a single zero at some point $(x_0, 1/2) \in [0,1) \times [0,1)$.

Kloos-Stöckler, Kloos, Vinogradov-Ulitskaya (2014-2017).
Conclusion: for some $x_0 \in [0,1)$ the set $x_0 + \mathbb{Z}$ is not sampling in $V^p(g)$.

Recall

Theorem

$\alpha, \beta > 0$

Let $g \in L^1(\mathbb{R})$ be arbitrary totally positive and $\alpha\beta$ rational. Then $\{M_{\beta l} T_{\alpha k} g : k, l \in \mathbb{Z}\}$ is a Gabor frame, if and only if $\alpha\beta < 1$.

Proof strategy

Without loss of generality $\beta = 1$ and $\alpha < 1$ (otherwise consider $\tilde{g}(x) = g(x/\beta)$).
Assume that $Zg(x_0, \frac{1}{2}) = 0$ is the only zero of Zg in $[0, 1)^2$.

- **Step 1**: Since $\alpha < 1$, we may extract a set X from $x + \alpha\mathbb{Z}$ such that

$$\#X \cap [j, j+1) = 1 \quad \text{and} \quad \mathrm{dist}(X, x_0 + \mathbb{Z}) = \delta > 0$$

Then $X = \{j + \delta_j : j \in \mathbb{Z}\}$, $\delta_j \in [x_0 + \delta, x_0 + 1 - \delta]$.
Consider the submatrix [2]

$$P = P(x) = \Big(g(j + \delta_j - k)\Big)_{j,k\in\mathbb{Z}}$$

Goal: show that the extracted matrix P is invertible on $\ell^\infty(\mathbb{Z})$ and ℓ^1.

[2] Note that δ_j depends on x.

- **Step 2**: Apply Theorem of de Boor, Friedlander, Pinkus with vector $u \in \ell^\infty(\mathbb{Z}), u_k = (-1)^k$. Then

$$\begin{aligned}(Pu)_j &= \sum_{k\in\mathbb{Z}} g(j+\delta_j-k)(-1)^k \\ &= \sum_{k'\in\mathbb{Z}} g(\delta_j-k')(-1)^{k'+j} \\ &= (-1)^j Zg(\delta_j, \tfrac{1}{2})\end{aligned}$$

Since we stay away from the zero $Zg(x_0, \frac{1}{2}) = 0$ by assumption, Pu is uniformly alternating.
Conclusion: P maps $\ell^\infty(\mathbb{Z})$ onto $\ell^\infty(\mathbb{Z})$.
Spectral invariance arguments[3] imply that P also maps $\ell^1(\mathbb{Z})$ onto $\ell^1(\mathbb{Z})$.

[3] Highly non-trivial! Makes use of decay of g.

Injectivity?

Proposition

(i) If $\alpha\beta = \frac{p}{q} \in \mathbb{Q}$*, then* $\delta_j = \delta_{j+pn}$ *for all* $n \in \mathbb{Z}$ *and* P *commutes with the shifts* [a] T_p*.*
(ii) If $PT_p = PT_p$ *and* $P : \ell^1 \to \ell^1$ *is onto, then* P *is one-to-one and thus invertible on* $\ell^1(\mathbb{R})$*.*

[a] Recall $(T_p c)_k = c_{k-p}$.

Proof idea: represent P as a $p \times p$-matrix-valued function. For $p \times p$-matrices, surjectivity is equivalent to injectivity!
End of proof of main theorem: use spectral invariance again to transfer invertibility on ℓ^∞ to invertibility on ℓ^2 and

$$\sum_{j\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} c_k g(x + \alpha j - k)|^2 \geq \sum_{j\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} c_k g(x + j + \delta_j - k)|^2$$
$$= \|P(x)c\|_2^2 \geq A\|c\|_2^2$$

Many new problems for infinite TP matrices

- invertibility and injectivity of TP matrices
- variation diminishing property of infinite TP matrices:

Let

$$V(c) = \{j \in \mathbb{Z} : c_j c_{j+1} < 0\}$$

be the set of sign changes and

$$D(c) = \liminf_{n\to\infty} \inf_{a\in\mathbb{Z}} \frac{\# V(c) \cap [a, a+n]}{n+1}$$

the density of sign changes [4]. Let A be a totally positive, **infinite** matrix.
Is it true that

$$D(Ac) \leq D(c)$$

[4] Perhaps just lim or lim sup?

Total positivity and the Riemann Hypothesis

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Riemann xi-function for functional equation

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

Theorem

The Riemann hypothesis holds, if and only if

$$\Lambda(x) = \int_{-\infty}^{\infty} \frac{1}{\xi(\frac{1}{2} + 2\pi\tau)} e^{-2\pi i x\tau}\, d\tau$$

is a totally positive function.

References

C. de Boor, S. Friedland, and A. Pinkus.
Inverses of infinite sign regular matrices.
Trans. Amer. Math. Soc., 274(1):59–68, 1982.

K. Gröchenig.
Totally positive functions and Gabor frames over rational lattices.
Adv. Math., 427:Paper No. 109113, 12, 2023.

K. Gröchenig, J. L. Romero, and J. Stöckler.
Sampling theorems for shift-invariant spaces, Gabor frames, and totally positive functions.
Invent. Math., 211(3):1119–1148, 2018.

K. Gröchenig and J. Stöckler.
Gabor frames and totally positive functions.
Duke Math. J., 162(6):1003–1031, 2013.

Spectral invariance

Proposition (Stability Theorem)

Let $G = (G_{kl})_{k,l\in\mathbb{Z}}$ be a bi-infinite matrix with off-diagonal decay

$$|G_{kl}| \leq C(1+|k-l|)^{-\sigma} \qquad k,l\in\mathbb{Z}\,,$$

for some $\sigma > 1$.
*(i) Spectral invariance: If G is invertible on **some** $\ell^{p_0}(\mathbb{Z})$, then G is invertible on **all** $\ell^p(\mathbb{Z})$ for $1 \leq p \leq \infty$.*
(ii) Stability: If G is stable on some $\ell^{p_0}(\mathbb{Z})$, i.e., if G satisfies

$$\|Gc\|_{p_0} \geq A\|c\|_{p_0} \qquad \text{for all } c\in\ell^{p_0}(\mathbb{Z}),$$

then G is stable on all $\ell^p(\mathbb{Z})$ for $1 \leq p \leq \infty$.