Multivariate Higher Order Monotonicity – and its Preservation

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1. Basic notions

$I_1, \ldots, I_d \subseteq \mathbb{R}$ intervals (always non-degenerate)

$I := I_1 \times \cdots \times I_d$, $f : I \to \mathbb{R}$

For $s \in I$, $h \in \mathbb{R}_+^d$ such that also $s + h \in I$

$$(E_h f)(s) := f(s+h)$$

$$\Delta_h := E_h - E_0, \text{ i.e. } (\Delta_h f)(s) := f(s+h) - f(s)$$

Since $\{E_h \mid h \in \mathbb{R}_+^d\}$ is commutative, so is $\{\Delta_h \mid h \in \mathbb{R}_+^d\}$.

For any h, $\Delta_h^0 f := f$ (also for $h = 0$), but $\Delta_0 f = 0 \; \forall f$.

For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ and $h = (h_1, \ldots, h_d) \in \mathbb{R}_+^d$

$$\Delta_h^{\mathbf{n}} := \Delta_{h_1 e_1}^{n_1} \Delta_{h_2 e_2}^{n_2} \ldots \Delta_{h_d e_d}^{n_d}$$

(where $e_1, \ldots, e_d$ are standard unit vectors), so that $(\Delta_h^{\mathbf{n}} f)(s)$ is defined for $s, s + \sum_{i=1}^{d} n_i h_i e_i \in I$.
We first consider $d = 1$, $\mathbf{n} = n \in \mathbb{N}$, $I \subseteq \mathbb{R}$ and put

$$\sigma_n(x_1, \ldots, x_n) := x_1 + \cdots + x_n.$$

Assume $I = [0,1]$, hence $f : [0,1] \to \mathbb{R}$. For $t \in [0,1[$ and $h > 0$, $t + nh \leq 1$,

$$\left(\Delta_h^n f\right)(t) = f(t+nh) - \binom{n}{1} f(t+(n-1)h) \pm \cdots + (-1)^n f(t)$$
$$= \left[\Delta_{(h,\dots,h)}^{(1,\dots,1)} (f \circ \sigma_n)\right] \left(\frac{t}{n}, \dots, \frac{t}{n}\right).$$

If a univariate f is C^∞, then

$$\Delta_h^n(f) \geq 0 \ \forall h > 0 \quad \Leftrightarrow \quad f^{(n)} \geq 0.$$

If a multivariate f is C^∞, then

$$\Delta_h^{\mathbf{n}}(f) \geq 0 \ \forall h \in \mathbb{R}_+^d \quad \Leftrightarrow \quad f_{\mathbf{n}} := \frac{\partial^{|\mathbf{n}|} f}{\partial x_1^{n_1} \dots \partial x_d^{n_d}} \geq 0$$

Definition 1.

$I \subseteq \mathbb{R}^d$, $\mathbf{n} \in \mathbb{N}_0^d \setminus \{0\}$, $f : I \to \mathbb{R}$ is $\mathbf{n}$-$\uparrow$ (read "$\mathbf{n}$-*increasing*") iff

$$\left(\Delta_h^{\mathbf{p}} f\right)(s) \geq 0 \ \forall s \in I,\ \ h \in \mathbb{R}_+^d,\ \mathbf{p} \in \mathbb{N}_0^d,\ 0 \lneqq \mathbf{p} \leq \mathbf{n}$$

such that $s_j + p_j h_j \in I_j \ \forall j \in [d] := \{1, \ldots, d\}$.

For a C^∞ function f then

$$f \text{ is } \mathbf{n}\text{-}\uparrow \quad \Leftrightarrow \quad f_{\mathbf{p}} \geq 0 \text{ for } 0 \lneqq \mathbf{p} \leq \mathbf{n}.$$

- For $d = 1$, $\mathbf{n} = n \in \mathbb{N}$, f is 1-$\uparrow$ iff f is (weakly) increasing, and f is 2-$\uparrow$ iff f is increasing and convex.
- $d = 2$, $f(s_1, s_2) := (s_1 s_2 - a)_+$, $a > 0$
  f is $(1,0)$-$\uparrow$, $(0,1)$-$\uparrow$, $(1,1)$-$\uparrow$ (will be shown later), but not $(2,2)$-$\uparrow$:
  $$\left(\Delta^{(2,2)}_{(\sqrt{a},\sqrt{a})} f\right)(0) = -a.$$

2. The univariate case

Theorem 1.

$f : [0,1[\to \mathbb{R}_+$ is n-$\uparrow$ ($n \geq 2$) iff $\exists!\ a_0, \ldots, a_{n-2} \geq 0$ and a measure μ on $[0,1[$ such that

$$f(t) = a_0 + a_1 t + \cdots + a_{n-2} t^{n-2} + \int (t-a)_+^{n-1}\, d\mu(a).$$

f is continuous and for $n > 2$ $(n-2)$ times continuously differentiable.

Let $K_n := \{f : [0,1] \to \mathbb{R}_+ \mid f \text{ is } n\text{-}\uparrow, f(1) = 1\}$, and
$f_a(t) := (t-a)_+/(1-a)$ for $0 \leq a < 1$, $f_1 := \mathbf{1}_{\{1\}}$,
$E_n := \{1, t, \ldots, t^{n-2}\} \cup \{f_a^{n-1} \mid a \in [0,1]\}$, $n \geq 2$

Corollary 1.

K_n is a Bauer simplex, $ex(K_n) = E_n$ for $n \geq 2$.

Corollary 2.

If $f : [0,1[\to \mathbb{R}_+$ is "absolutely monotone", i.e. n-$\uparrow$ $\forall\ n \in \mathbb{N}$, then

$$f(t) = \sum_{j \geq 0} a_j t^j \quad \textit{where } a_j \geq 0\ \forall j.$$

Corollary 3.

$K_\infty := \bigcap_{n \geq 1} K_n$ is a Bauer simplex with
$ex(K_\infty) = \{1, t, t^2, \ldots\} \cup \{\mathbf{1}_{\{1\}}\}$.

3. The multivariate case

First some special situations and examples.

1. $f_j : I_j \to \mathbb{R}$, $1 \le j \le d$, $f := f_1 \otimes \cdots \otimes f_d$, i.e.

$$f(s) = \prod_{i=1}^{d} f_j(s_j)$$

Then

$$\left(\Delta_h^{\mathbf{p}} f\right)(s) = \prod_{j=1}^{d} \left(\Delta_{h_j}^{p_j} f_j\right)(s_j)$$

for $\mathbf{p} \in \mathbb{N}_0^d$, $h \in \mathbb{R}_+^d$, implying for $f_j \geq 0$

$$f \text{ is } \mathbf{n}\text{-}\uparrow \quad \Leftrightarrow \quad f_j \text{ is } n_j\text{-}\uparrow \ \forall j.$$

2. $f(s) := \sum_{j=1}^{d} f_j(s_j)$ (a "tensor sum")

Then

$$\Delta_h^{\mathbf{p}} f(s) = \begin{cases} \Delta_{h_i}^{p_i} f_i(s_i) & \text{if } \mathbf{p} = p_i e_i,\, p_i \geq 1 \\ = 0 & \text{if } \mathbf{p} \text{ has more than one positive entry.} \end{cases}$$

$$f \text{ is } \mathbf{n}\text{-}\uparrow \quad \Leftrightarrow \quad f_j \text{ is } n_j\text{-}\uparrow \ \forall j$$

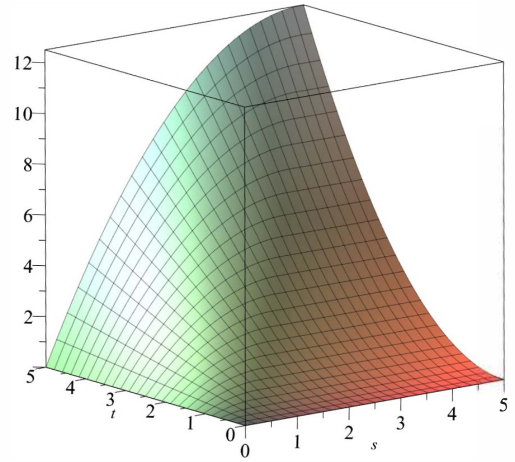
($f_j \geq 0$ **not** necessary)

3. $d = 2$,

$$f(s,t) := \int_0^\infty \underbrace{\underbrace{\mathbf{1}_{[a,\infty[}(s)}_{1\text{-}\uparrow}\,\underbrace{(t-a)_+}_{2\text{-}\uparrow}}_{(1,2)\text{-}\uparrow}\,da \text{ on } \mathbb{R}_+^2$$

$$= \int_0^{s\wedge t} (t-a)_+\,da = (s\wedge t)\cdot\left(t - \frac{1}{2}(s\wedge t)\right)$$

f is $(1,2)$-$\uparrow$ (and not more!)



$$f(s,t) := (s \wedge t) \cdot \left(t - \frac{1}{2}(s \wedge t) \right)$$

For $\mathbf{n} \in \mathbb{N}^d$, $\mathbf{n} \geq \mathbf{2}_d$ we consider

$$K_{\mathbf{n}} := \{f : [0,1]^d \to \mathbb{R}_+ \,|\, f \text{ is } \mathbf{n}\text{-}\uparrow, f(\mathbf{1}_d) = 1\},$$

obviously convex and compact, and

$$E_{\mathbf{n}} := E_{n_1} \otimes \cdots \otimes E_{n_d} = \{f_1 \otimes \cdots \otimes f_d \,|\, f_i \in E_{n_i} \,\forall i\}.$$

Theorem 2.

For $\mathbf{n} \geq \mathbf{2}_d$ $K_{\mathbf{n}}$ is a Bauer simplex, and $ex(K_{\mathbf{n}}) = E_{\mathbf{n}}$.

For example, if $d = 2$, any $f : [0,1]^2 \to \mathbb{R}_+$ which is $(2,2)$-$\uparrow$, has the form

$$f(s,t) = \int_{E_2 \times E_2} \rho_1(s)\rho_2(t)\, d\mu(\rho_1, \rho_2),$$

hence in case $f(s,0) = f(0,t) = 0\ \forall s,t$,

$$f(s,t) = \int_{[0,1]^2} f_a(s) f_b(t)\, d\mu(a,b)$$

for some measure μ on $[0,1]^2$.

Let us now consider the special important case
$\mathbf{n} = \mathbf{1}_d = (1, \ldots, 1)$.
For $d = 2$, $h = (h_1, h_2) \in \mathbb{R}^2_+$

$$\left(\Delta_h^{(1,1)} f\right)(s,t) = f(s+h_1, t+h_2) - f(s, t+h_2) - f(s+h_1, t) + f(s,t)$$

$$\left(\Delta_h^{(1,0)} f\right)(s,t) = f(s+h_1, t) - f(s,t)$$

$$\left(\Delta_h^{(0,1)} f\right)(s,t) = f(s, t+h_2) - f(s,t).$$

For general $d \in \mathbb{N}$, $h \in \mathbb{R}_+^d$

$$\left(\Delta_h^{\mathbf{1}_d} f\right)(s) = f(s+h) - f(s_1, s_2+h_2, \ldots, s_d+h_d) - \cdots$$
$$- f(s_1+h_1, \ldots, s_{d-1}+h_{d-1}, s_d)$$
$$+ f(s_1, s_2, s_3+h_3, \ldots, s_d+h_d) + \cdots$$
$$\cdots + (-1)^d f(s).$$

If f is the distribution function ("d.f.") of some measure μ, say on $\mathbb{R}_+^d$, i.e.

$$f(s) = \mu([0, s]),$$

then

$$\left(\Delta_h^{\mathbf{1}_d} f\right)(s) = \mu(]s, s+h]) \geq 0.$$

Theorem 3.

Let $\emptyset \neq I_j \subseteq \overline{\mathbb{R}}$ be arbitrary (not necessarily intervals), $j \leq d$, $I = I_1 \times \cdots \times I_d$, $f : I \to \mathbb{R}_+$. Then

$$f \text{ is the d.f. of some measure on } \overline{I}$$

$$\Leftrightarrow f \text{ is } \mathbf{1}_d\text{-}\uparrow \text{ and right-continuous.}$$

The proof relies on the fact, that $f(\geq 0)$ is $\mathbf{1}_d$-$\uparrow$ iff f is completely monotone on the semigroup $(I, \wedge)$ iff f is positive definite on $(I, \wedge)$.

In dimension one any 2-$\uparrow$ function is right-continuous; if a multivariate f is (only) $\mathbf{1}_d$-$\uparrow$, this need **not** be the case:

$$f = \mathbf{1}_{]0,1]^2} \text{ on } [0,1]^2.$$

However, for $\mathbf{n} \in \mathbb{N}^d$ each $\mathbf{n}$-$\uparrow$ f is the pointwise limit of some net of $\mathbf{n}$-$\uparrow$ right-continuous functions.
(If $\mathbf{n} \geq \mathbf{2}_d$, then f itself is right-continuous.)

Application. Mean values as distribution functions

For $x \in]0,\infty[^d$ and $t \in \mathbb{R}$ consider

$$M_t(x) := \left(\frac{1}{d}\sum_{i=1}^{d} x_i^t\right)^{\frac{1}{t}} \qquad \text{for } t \neq 0$$

$$M_0(x) := \left(\prod_{i=1}^{d} x_i\right)^{\frac{1}{d}} \qquad \left(= \lim_{t\to 0} M_t(x)\right)$$

$$M_\infty(x) := \max_{i\le d} x_i \qquad \left(= \lim_{x\to\infty} M_t(x)\right)$$

$$M_{-\infty}(x) := \min_{i\le d} x_i \qquad \left(= \lim_{t\to-\infty} M_t(x)\right).$$

(If $t < 0$ and $x_i = 0$ for some i, then $M_t(x) = 0$.)

The function $\mathbb{R} \ni t \mapsto M_t(x)$ (for non-constant x) is continuous and strictly increasing from $\min x_i$ to $\max x_i$.
Since $M_t(1, \ldots, 1) = 1$, these mean values are candidates for d.f.s of probability measures on $[0, 1]^d$.

Theorem 4.

$M_t|_{[0,1]^d}$ is a d.f. $\Leftrightarrow t \in [-\infty, \frac{1}{d-1}] \cup \{\frac{1}{d-2}, \ldots, \frac{1}{2}, 1\}$
($t \in [-\infty, 1]$ for $d = 2$)

How to prove this?

$$M_t = f_t \circ \left(\frac{1}{d} \sum_{i=1}^{d} x_i^t \right), \quad f_t(s) := s^{1/t} \text{ on }]0, \infty[$$

For $t > 0$, $\sum_{i=1}^{d} x_i^t$ is a tensor sum of increasing functions, hence $\mathbf{1}_d$ -$\uparrow$.

We need to know which functions on $\mathbb{R}_+$ preserve this property!

The following result was essentially shown by P. M. Morillas (2005):

Theorem 5.

Let $I \subseteq \mathbb{R}^d$ and $J \subseteq \mathbb{R}$ be intervals, $g : I \to J$, $f : J \to \mathbb{R}$. Then, if g is $\mathbf{1}_d$ -$\uparrow$ and f is d -$\uparrow$, $f \circ g$ is again $\mathbf{1}_d$ -$\uparrow$.

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Let's apply this to the mean values M_t (for $t > 0$):

- for $t \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ we have $f_t(s) = s^k$ for some $k \in \mathbb{N}$, hence f_t is absolutely monotone, and M_t a d.f.

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- for $t \in]0, 1] \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$:

$$\left(x^{\frac{1}{t}}\right)^{(k)} = \frac{1}{t}\left(\frac{1}{t} - 1\right)\cdots\left(\frac{1}{t} - (k-1)\right) x^{\frac{1}{t}-k}$$
$$\frac{1}{t} > k - 1 \;\Rightarrow\; \ldots > 0$$
$$\Leftrightarrow t < \frac{1}{k-1}$$

hence $t < \frac{1}{d-1}$ is sufficient for f_t to be d -$\uparrow$. (in fact also necessary; $t > 1$ will be dealt with later)

Before considering M_t for $t < 0$, this remark: Let $\alpha, \beta > 0$

then $x_i \mapsto -x_i^{-\alpha}$ is increasing on $]0, \infty[$

$\Rightarrow x \mapsto -\sum x_i^{-\alpha}$ is $\mathbf{1}_d$ -$\uparrow$ on $]0, \infty[^d$

$s \mapsto s^{-\beta}$ is completely monotone on $]0, \infty[$

$\Leftrightarrow s \mapsto (-s)^{-\beta}$ is absolutely monotone on $]-\infty, 0[$

Now let $t < 0$, $\alpha := -t$, $\beta := -\frac{1}{t}$

$$\Rightarrow M_t(x) = d^{-1/t}\left[-\left(-\sum x_i^{-\alpha}\right)\right]^{-\beta} \text{ is } \mathbf{1}_d \text{ -}\uparrow,$$

again by Theorem 5.

An important supplement to Theorem 5:

Theorem 6.

Let $I \subseteq \mathbb{R}^d$ be an interval (always non-degenerate), $\sigma_d(x) := \sum_{i=1}^d x_i$, $J := \sigma_d(I)$, $f : J \to \mathbb{R}$. Then

$$f \text{ is } d\text{-}\uparrow \quad \Leftrightarrow \quad f \circ \sigma_d \text{ is } \mathbf{1}_d\text{-}\uparrow .$$

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$$f \text{ is } d\text{-}\uparrow \quad \Leftrightarrow \quad f \circ \sigma_d \text{ is } \mathbf{1}_d\text{-}\uparrow .$$

Here "$\Rightarrow$" follows from Theorem 5. For "$\Leftarrow$" it is sufficient to consider $I = [0, \frac{1}{d}]^d$ and $J = [0, 1]$. Then for $t \in [0, 1[$, $h > 0$, $t + k \cdot h \leq 1$ $(k \leq d)$

$$\left(\Delta_h^k f\right)(t) = f(t + kh) - \binom{k}{1} f(t + (k-1)h) + \cdots + (-1)^k f(t)$$

$$= \left(\Delta_{(h,\ldots,h)}^{\mathbf{1}_k, \mathbf{0}_{d-k}}(f \circ \sigma_d)\right)\left(\frac{t}{d}, \ldots, \frac{t}{d}\right)$$

However: attention!

We'd need $\frac{t}{d} + h \leq \frac{1}{d}$, or $t + dh \leq 1$, but only know $t + kh \leq 1$.

Lemma 1.

$J \subseteq \mathbb{R}$ interval, $f : J \to \mathbb{R}$, $k \in \mathbb{N}$. If $\exists h_0 > 0$ such that $\left(\Delta_h^k f\right)(t) \geq 0 \ \forall t \in J,\ h \in]0, h_0]$ with $t + kh \in J$, then the same holds $\forall h > 0$ such that $t + kh \in J$.

To finish the proof of Theorem 6, choose $h_0 := \frac{1-t}{d}$.

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Corollary 4.

$f : [0,1] \to \mathbb{R}$ is d -$\uparrow$ iff $f \circ M_1$ is $\mathbf{1}_d$ -$\uparrow$.

A natural question:

Suppose F is a two-dimensional d.f., G a three-dimensional d.f., for which functions f on $[0,1]^2$ is then always $f \circ (F \times G)$ a five-dimensional d.f.?

Apart from normalisation, when is $f\,(F(x), G(y))$ $\mathbf{1}_5$ -$\uparrow$?

Theorem 7.

Let $I_1 \subseteq \mathbb{R}^{n_1}, \ldots, I_d \subseteq \mathbb{R}^{n_d}$ *be intervals,* $g_1 : I_1 \to [0,1]$ $\mathbf{1}_{n_1}$ -$\uparrow, \ldots, g_d : I_d \to [0,1]$ $\mathbf{1}_{n_d}$ -$\uparrow$, $\mathbf{n} := (n_1, \ldots, n_d)$*. If* $f : [0,1]^d \to \mathbb{R}_+$ *is* $\mathbf{n}$ -$\uparrow$*, then* $f \circ (g_1 \times \cdots \times g_d)$ *is* $\mathbf{1}_{|\mathbf{n}|}$ -$\uparrow$.

Theorem 7.

Let $I_1 \subseteq \mathbb{R}^{n_1}, \ldots, I_d \subseteq \mathbb{R}^{n_d}$ be intervals, $g_1 : I_1 \to [0,1]$ $\mathbf{1}_{n_1}$ -$\uparrow, \ldots, g_d : I_d \to [0,1]$ $\mathbf{1}_{n_d}$ -$\uparrow$, $\mathbf{n} := (n_1, \ldots, n_d)$. If $f : [0,1]^d \to \mathbb{R}_+$ is $\mathbf{n}$ -$\uparrow$, then $f \circ (g_1 \times \cdots \times g_d)$ is $\mathbf{1}_{|\mathbf{n}|}$ -$\uparrow$.

Proof.

We may assume $f(\mathbf{1}_d) = 1$, and also $\mathbf{n} \geq \mathbf{2}_d$. Then (Theorem 2)

$$f(s) = \int_{E_{\mathbf{n}}} \prod_{i=1}^{d} \rho_i(s_i)\, d\mu(\rho_1, \ldots, \rho_d)$$

for some probability measure μ on $E_{\mathbf{n}}$. So

$$f \circ (g_1 \times \cdots \times g_d) = \int \bigotimes_{i=1}^{d} (\rho_i \circ g_i)\, d\mu(\rho_1, \ldots, \rho_d)$$

where each $\rho_i \circ g_i$ is $\mathbf{1}_{n_i}$ -$\uparrow$ (Theorem 5), therefore $\bigotimes_{i=1}^{d}(\rho_i \circ g_i)$ is $\mathbf{1}_{|\mathbf{n}|}$ -$\uparrow$, and so is then $f \circ (g_1 \times \cdots \times g_d)$ as a mixture of those. □

A special case would be $g_i = \sigma_{n_i}$ $(\forall i)$ on suitable n_i-dimensional intervals. As a generalisation of Theorem 6 we have

$(\sigma_{\mathbf{n}} := \sigma_{n_1} \times \cdots \times \sigma_{n_d})$

Theorem 8.

Let $I_1 \subseteq \mathbb{R}^{n_1}, \ldots, I_d \subseteq \mathbb{R}^{n_d}$ be non-degenerate intervals, $J_i := \sigma_{n_i}(I_i)$, $J := J_1 \times \cdots \times J_d$ and $f : J \to \mathbb{R}$. Then

$$f \text{ is } \mathbf{n}\text{-}\!\uparrow \quad \Leftrightarrow \quad f \circ \sigma_{\mathbf{n}} \text{ is } \mathbf{1}_{|\mathbf{n}|}\text{-}\!\uparrow$$

(similar proof!)

Finally, we get a natural generalisation of Theorem 5, our **first main result:**

Theorem 9.

Let $g_i : I_i \to [0,1]$ *be* $\mathbf{m_i}$ *-*$\uparrow$*, where* $\mathbf{m}_i \in \mathbb{N}^{n_i}$*. Put* $g := g_1 \times \cdots \times g_d : I_1 \times \cdots \times I_d \to [0,1]^d$*. If* $f : [0,1]^d \to \mathbb{R}$ *is* $(|\mathbf{m_1}|, \ldots, |\mathbf{m_d}|)$ *-*$\uparrow$ *then* $f \circ g$ *is* $(\mathbf{m_1}, \ldots, \mathbf{m_d})$ *-*$\uparrow$*.*

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Let $g_i : I_i \to [0,1]$ *be* $\mathbf{m_i}$ -$\uparrow$, *where* $\mathbf{m}_i \in \mathbb{N}^{n_i}$. *Put* $g := g_1 \times \cdots \times g_d : I_1 \times \cdots \times I_d \to [0,1]^d$. *If* $f : [0,1]^d \to \mathbb{R}$ *is* $(|\mathbf{m_1}|, \ldots, |\mathbf{m_d}|)$ -$\uparrow$ *then* $f \circ g$ *is* $(\mathbf{m_1}, \ldots, \mathbf{m_d})$ -$\uparrow$.

Example.

$d = 2$, $n_1 = 2$, $n_2 = 3$, $\mathbf{m_1} = (2,4)$, $\mathbf{m_2} = (3,3,2)$. If g_1 (bivariate) is $(2,4)$-$\uparrow$, g_2 (trivariate) is $(3,3,2)$-$\uparrow$, and f is $(6,8)$-$\uparrow$, then $f \circ (g_1 \times g_2)$ is $(2,4,3,3,2)$-$\uparrow$ (as a function of 5 variables).

Proof.

$f(s) = \int \bigotimes_{i=1}^d \rho_i(s_i)\, d\mu(\rho)$

μ on $E_{(|\mathbf{m_1}|,\ldots,|\mathbf{m_d}|)}$

$\Rightarrow f \circ g = \int \bigotimes_{i=1}^d \rho_i \circ g_i \, d\mu(\rho)$

g_i is $\mathbf{m_i}$ -↑, equiv. $g_i \circ \sigma_{\mathbf{m_i}}$ is $\mathbf{1}_{|\mathbf{m_i}|}$ -↑

$$f \circ g \circ \sigma_{(\mathbf{m_1},\ldots,\mathbf{m_d})} = \int \underbrace{\bigotimes_{i=1}^d \rho_i \circ g_i \circ \sigma_{\mathbf{m_i}} \, d\mu(\rho)}_{\text{is } \mathbf{1}_{|\mathbf{m_1}|+\cdots+|\mathbf{m_d}|} \text{ -}\uparrow}$$

$\Rightarrow f \circ g$ is $(\mathbf{m_1}, \ldots, \mathbf{m_d})$ -↑. □

Special cases:

$$g \ \mathbf{n}\text{-}\uparrow, \ f \ |\mathbf{n}|\text{-}\uparrow \Rightarrow f \circ g \ \mathbf{n}\text{-}\uparrow$$

$$g \ n\text{-}\uparrow, \ f \ n\text{-}\uparrow \Rightarrow f \circ g \ n\text{-}\uparrow$$

(if defined...)

4. k-increasing functions ($k \in \mathbb{N}$!) in any dimension

Let $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ continuous, $k \in \mathbb{N}$. Then

$$\Delta_h^k(f)(t) \geq 0 \quad \forall\, t \in I,\; h > 0,\; t + kh \in I$$

is equivalent with

$$(\Delta_{h_1}\Delta_{h_2}\dots\Delta_{h_k} f)(t) \geq 0 \quad \forall\, t \in I,\; h_i > 0,\; t + \sum_{i=1}^{k} h_i \in I.$$

(Boas-Widder (1940), easy to see)

The following notion now seems natural:

Definition 2.

$I \subseteq \mathbb{R}^d$ interval, $f : I \to \mathbb{R}$, $k \in \mathbb{N}$. Then f is called *k-increasing* ("k-$\uparrow$") iff $\forall\, j \in [k]$, $\forall\, h^{(1)}, \dots, h^{(j)} \in \mathbb{R}_+^d$, $\forall\, s \in I$ such that $s + h^{(1)} + \dots + h^{(j)} \in I$

$$(\Delta_{h^{(1)}} \dots \Delta_{h^{(j)}} f)(s) \geq 0.$$

- For $k = 2$ these functions are known as *ultramodular*.
- For $d = 1$ this definition is the known one.
- Already for $d = 2$ increasing convexity and being 2-$\uparrow$ are incomparable properties: on $\mathbb{R}_+^2$ the product is 2-$\uparrow$, but not convex; and the Euclidean norm is convex, however not 2-$\uparrow$:

$$\left(\Delta_{e_1}\Delta_{e_2}\sqrt{x^2+y^2}\right)(0) = \sqrt{2} - 2$$

There is a surprisingly close connection to $\mathbf{n}$-$\uparrow$ functions:

Theorem 10.

Let $I \subseteq \mathbb{R}^d$ be an interval, $d, k \in \mathbb{N}$, $f : I \to \mathbb{R}$. Then

$$f \text{ is } k\text{-}\uparrow \quad \Leftrightarrow \quad f \text{ is } \mathbf{n}\text{-}\uparrow \ \forall\, \mathbf{n} \in \mathbb{N}_0^d \text{ with } 0 < |\mathbf{n}| \leq k.$$

Furthermore:

$$\forall\, m \in \mathbb{N},\ \forall \text{ interval } J \subseteq \mathbb{R}^m,\ \forall \text{ positive affine } \varphi : \mathbb{R}^m \to \mathbb{R}^d$$

$$\text{such that } \varphi(J) \subseteq I, \text{ also } f \circ \varphi \text{ is } k\text{-}\uparrow.$$

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Corollary 5.

$I \subseteq \mathbb{R}^d$, $B \subseteq \mathbb{R}$ intervals, $g : I \to B$ and $f : B \to \mathbb{R}$ both k-$\uparrow$, then so is $f \circ g$.

Because: $0 < |\mathbf{n}| \leq k \Rightarrow f\ |\mathbf{n}|\text{-}\uparrow,\ g\ \mathbf{n}\text{-}\uparrow \Rightarrow f \circ g\ \mathbf{n}\text{-}\uparrow$.

Lemma 2.

$I \subseteq \mathbb{R}^{d_1}$, $J \subseteq \mathbb{R}^{d_2}$ intervals, $f : I \to \mathbb{R}_+$, $g : J \to \mathbb{R}_+$ both k-$\uparrow$ $\Rightarrow f \otimes g$ k-$\uparrow$ on $I \times J$. In case $I = J$ the product $f \cdot g$ is also k-$\uparrow$.

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Proof.

$\left[\Delta^{\mathbf{m},\mathbf{n}}_{(h^{(1)},h^{(2)})}(f \otimes g)\right](x,y) = \left(\Delta^{\mathbf{m}}_{h^{(1)}} f\right)(x) \cdot \left(\Delta^{\mathbf{n}}_{h^{(2)}} g\right)(y)$. For $|(\mathbf{m},\mathbf{n})| = |\mathbf{m}| + |\mathbf{n}| \leq k$ both factors are ≥ 0 ($\mathbf{m} = 0$ or $\mathbf{n} = 0$ is possible, therefore $f \geq 0$, $g \geq 0$).

For $I = J$, let $\varphi : \mathbb{R}^d \to \mathbb{R}^{2d}$ be given by $\varphi(x) := (x, x)$, a linear positive map, with $\varphi(I) \subseteq I \times I$. Therefore

$$(f \otimes g) \circ \varphi = f \cdot g \text{ is also } k\text{-}\uparrow .$$

□

Examples.

- Each monomial $f(x) = \prod_{i=1}^{d} x_i^{n_i}$ ($n_i \in \mathbb{N}$) is k-$\uparrow$ on $\mathbb{R}_+^d$ $\forall k \in \mathbb{N}$. $\prod_{i=1}^{d} x^{c_i}$ ($c_i > 0$) is k-$\uparrow$ on $\mathbb{R}_+^d$ at least for $k \leq c_i + 1$ $\forall i$.

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- For $a > 0$ the function $f(x, y) := (xy - a)_+$ is 2-$\uparrow$, since $(t - a)_+$ is 2-$\uparrow$ on $\mathbb{R}_+$. So, by Theorem 10, f is $(1, 1)$-$\uparrow$, but not $(2, 2)$-$\uparrow$ as we saw earlier. It is even not $(1, 2)$-$\uparrow$: $\left(\Delta_{\frac{1}{2},1}^{(1,2)} f\right)\left(\frac{1}{2}, 1\right) = -\frac{1}{2}$, for $a = 1$.

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- The tensor product $g(x, y) := (x - a)_+ \cdot (y - b)_+$, where $a, b > 0$, is $(2, 2)$-$\uparrow$, hence certainly 2-$\uparrow$, but not 3-$\uparrow$, since $x \mapsto (x - a)_+$ is not.

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- $(xyz - a)_+^2$ is 3-↑ on $\mathbb{R}_+^3$, $(xy - a)_+^2$ 3-↑ on $\mathbb{R}_+^2$.

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- $(xyz - a)_+^2$ is 3-$\uparrow$ on $\mathbb{R}_+^3$, $(xy - a)_+^2$ 3-$\uparrow$ on $\mathbb{R}_+^2$.
- $f(x, y, z) := xy + xz + yz - xyz$ on $[0, 1]^3$
  Then $f_1 = y + z - yz \geq 0$, $f_{(1,1)} = 0$, $f_{(1,2)} = 1 - z \geq 0$, $f_{(1,2,3)} = -1$ i.e. f is 2-$\uparrow$, but not 3-$\uparrow$.

Intermezzo: **Bernstein polynomials**

$$b_{i,r}(t) := \binom{r}{i} t^i (1-t)^{r-i}, \quad r \in \mathbb{N},\ i \in \{0,1,\ldots,r\},\ t \in \mathbb{R}$$

For $\mathbf{i} = (i_1, \ldots, i_d) \in \{0,1,\ldots,r\}^d$

$$B_{\mathbf{i},r} := b_{i_1} \otimes \ldots \otimes b_{i_d}.$$

For any $f : [0,1]^d \to \mathbb{R}$ the associated Bernstein polynomials $f^{(1)},\ f^{(2)},\ \ldots$ are defined by

$$f^{(r)} := \sum_{\mathbf{0} \le \mathbf{i} \le \mathbf{r}_d} f\left(\frac{\mathbf{i}}{r}\right) B_{\mathbf{i},r}$$

For each continuity point x of f we have

$$f^{(r)}(x) \to f(x), \quad r \to \infty.$$

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In the following, the "upper right boundary" of $[0,1]^d$ will play a role.

For $\alpha \subseteq [d]$ let $T_\alpha := \{x \in [0,1]^d \mid x_i < 1 \Leftrightarrow i \in \alpha\}$. Then

$$[0,1]^d = \bigcup_{\alpha \subseteq [d]} T_\alpha \text{ is a disjoint union}$$

$$T_\emptyset = \{\mathbf{1}_d\}, \; T_{[d]} = [0,1[^d$$

and $\bigcup_{\alpha \subsetneqq [d]} T_\alpha$ is called the *upper right boundary* of $[0,1]^d$.

It is easy to show, that on each part T_α ($\alpha \subsetneqq [d]$) of this boundary the restriction $f|T_\alpha$ has as its Bernstein polynomials the restrictions $f^{(r)}|T_\alpha$. Thus we have the

Lemma 3.

Let $f : [0,1]^d \to \mathbb{R}$ *have the property that each restriction* $f|T_\alpha$ *for* $\emptyset \neq \alpha \subseteq [d]$ *is continuous. Then* $\lim_{r\to\infty} f^{(r)}(x) = f(x)\ \forall x \in [0,1]^d$, *i.e.* $f^{(r)}$ *converges pointwise to* f.

(Note that $f^{(r)}(\mathbf{1}_d) = f(\mathbf{1}_d)\ \forall r$.)

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(Note that $f^{(r)}(\mathbf{1}_d) = f(\mathbf{1}_d)\ \forall r$.)

Lemma 4.

Let $f : [0,1]^d \to \mathbb{R}$ be 2 -$\uparrow$. Then

1. *f is continuous iff f is continuous in $\mathbf{1}_d$.*
2. *f is right-continuous and on $[0,1[^d$ continuous.*

Theorem 11.

$f : [0,1]^d \to \mathbb{R}$, $\mathbf{2}_d \leq \mathbf{n} \in \mathbb{N}_0^d$, $2 \leq k \in \mathbb{N}$.

(i) f $\mathbf{n}$ -$\uparrow$ $\Rightarrow$ *each* $f^{(r)}$ *is* $\mathbf{n}$ -$\uparrow$ *and* $f^{(r)} \to f$ *pointwise*

(ii) f k -$\uparrow$ $\Rightarrow$ *each* $f^{(r)}$ *is* k -$\uparrow$ *and* $f^{(r)} \to f$ *pointwise*

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We can now tackle another natural question on the preservation of monotonicity, related but different to the previous one.

If $g_1, \ldots, g_m : I \to [0,1]$ are d.f.s on some d-dimensional interval,

$$g = (g_1, \ldots, g_m) : I \to [0,1]^m, \text{ i.e. } g(s) = (g_1(s), g_2(s), \ldots),$$

for which functions f on $[0,1]^m$ is $f \circ g$ again a d.f.?

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for which functions f on $[0,1]^m$ is $f \circ g$ again a d.f.?

For $d = 1$ f has just to be increasing (and right-cont.), for $d = 2$ this is not sufficient:

$$g_1(s,t) := \frac{s+t}{2},\ g_2(s,t) := st,\ f = 1_{[(\frac{1}{2},\frac{1}{2}),(1,1)]}$$
$$\Rightarrow \left[\Delta^{(1,1)}_{(\frac{1}{2},\frac{1}{2})} f \circ (g_1, g_2)\right]\left(\frac{1}{2},\frac{1}{2}\right) = -1.$$

Here is our **second main result**:

Theorem 12.

Let $f : [0,1]^m \to \mathbb{R}_+$ *be* d *-*$\uparrow$ *(*$d \geq 2$*), and let* $g_1, \ldots, g_m : \mathbb{R}^d \to [0,1]$ *be d.f.s of (subprobability) measures. Then, also* $f \circ (g_1, \ldots, g_m)$ *is a d.f. on* $\mathbb{R}^d$*.*

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Idea of proof:

$g := (g_1, \ldots, g_m)$, $h := f \circ g$

h is right-continuous (since f is by Lemma 1).

To show: h is $\mathbf{1}_d$ -$\uparrow$!

Because of Theorem 11 we may assume f to be C^∞.

Idea of proof:

1. Also $g_1, \ldots, g_d$ are C^∞.

   An explicite and rather complicated generalization of the usual multivariate chain rule and of Faà di Bruno's formula leads to the result (Constantine & Savits, TAMS 1996).

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Idea of proof:

1. Also $g_1, \ldots, g_d$ are C^∞.

   An explicite and rather complicated generalization of the usual multivariate chain rule and of Faà di Bruno's formula leads to the result (Constantine & Savits, TAMS 1996).

2. To show: for $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d_+$

$$\left(\Delta_\xi^{\mathbf{1}_d} h\right)(x) = h(x+\xi) \mp \ldots + (-1)^d h(x) \geq 0$$

   $\exists C^\infty$ d.f.s $\tilde{g}_1, \ \ldots, \ \tilde{g}_m$ such that $\tilde{g}_i | B = g_i | B \ \forall i \leq d$, where $B := \{x + \sum_{i \in \alpha} \xi_i e_i | \alpha \subseteq [d]\}$.

$\Rightarrow 0 \leq \Delta_\xi^{\mathbf{1}_d}(f \circ \tilde{g})(x) = \left(\Delta_\xi^{\mathbf{1}_d} h\right)(x)$. □

For $d = 2$ this result was proved in 2011 (Klement et al., Inf. Sc.).

Corollary 6.

Let $m, d, k \in \mathbb{N}$, $J \subseteq \mathbb{R}^m$ and $I \subseteq \mathbb{R}^d$ intervals,

$g = (g_1, \ldots, g_m) : I \to J$, $f : J \to \mathbb{R}$, $\mathbf{n} \in \mathbb{N}^d$.

1. *If each g_i is $\mathbf{n}$ -$\uparrow$, and f is $|\mathbf{n}|$ -$\uparrow$, then $f \circ g$ is $\mathbf{n}$ -$\uparrow$*
2. *If each g_i and f are k -$\uparrow$, then so is $f \circ g$.*

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Let $m, d, k \in \mathbb{N}$, $J \subseteq \mathbb{R}^m$ and $I \subseteq \mathbb{R}^d$ intervals,
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(i) *If each g_i is $\mathbf{n}$-$\uparrow$, and f is $|\mathbf{n}|$-$\uparrow$, then $f \circ g$ is $\mathbf{n}$-$\uparrow$*

(ii) *If each g_i and f are k-$\uparrow$, then so is $f \circ g$.*

Proof.

(i) By Theorem 8 each $g_i \circ \sigma_{\mathbf{n}}$ is $\mathbf{1}_{|\mathbf{n}|}$-$\uparrow$, hence so is by Theorem 12

$$f \circ (g_1 \circ \sigma_{\mathbf{n}}, \ldots, g_m \circ \sigma_{\mathbf{n}}) = (f \circ g) \circ \sigma_{\mathbf{n}},$$

and again Theorem 8 shows $f \circ g$ to be $\mathbf{n}$-$\uparrow$.

(ii) For any $\mathbf{n} \in \mathbb{N}^d$ with $|\mathbf{n}| \leq k$ each g_i is $\mathbf{n}$-$\uparrow$, hence $f \circ g$ is $\mathbf{n}$-$\uparrow$. By Theorem 10 $f \circ g$ is k-$\uparrow$. $\square$

5. Two related notions of monotonicity

$\nabla f := -\Delta f$, i.e. $(\nabla_h f)(s) := f(s) - f(s+h)$

Definition 3.

$f : I \to \mathbb{R}$ is $\mathbf{n}$-$\downarrow$ ("$\mathbf{n}$-decreasing") iff

$$\left(\nabla_h^{\mathbf{p}} f\right)(s) \geq 0 \ \forall s \in I, \ h \in \mathbb{R}_+^d, \ 0 \lneqq \mathbf{p} \leq \mathbf{n}.$$

And f is $\mathbf{n}$-$\updownarrow$ ("$\mathbf{n}$-alternating") iff

$$\left(\nabla_h^{\mathbf{p}} f\right)(s) \leq 0 \ \forall s \in I, \ h \in \mathbb{R}_+^d, \ 0 \lneqq \mathbf{p} \leq \mathbf{n}.$$

Remarks.

(a) f is $\mathbf{n}$-$\downarrow$ on I $\Leftrightarrow$ $f(-\cdot)$ is $\mathbf{n}$-$\uparrow$ on $-I$

(b) f is $\mathbf{n}$-$\updownarrow$ on I $\Leftrightarrow$ $-f(-\cdot)$ is $\mathbf{n}$-$\uparrow$ on $-I$

(c) A C^∞ function f is $\mathbf{n}$-$\downarrow$ iff $(-1)^{|\mathbf{p}|} f_{\mathbf{p}} \geq 0$ $\forall 0 \lneqq \mathbf{p} \leq \mathbf{n}$, and f is $\mathbf{n}$-$\updownarrow$ iff $(-1)^{|\mathbf{p}|} f_{\mathbf{p}} \leq 0$ instead.

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For $d = 1$:

f is 2-$\uparrow$ $\Leftrightarrow$ f is increasing and convex

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For $d = 1$:

$$f \text{ is } 2\text{-}\uparrow \ \Leftrightarrow f \text{ is increasing and convex}$$

$$f \text{ is } 2\text{-}\downarrow \ \Leftrightarrow f \text{ is decreasing and convex}$$

$$f \text{ is } 2\text{-}\updownarrow \ \Leftrightarrow f \text{ is increasing and concave}$$

$f \geq 0$ is n-$\updownarrow \forall n \in \mathbb{N}$ (“∞-$\updownarrow$”) $\Leftrightarrow$ f is a Bernstein function

An easy consequence of Theorem 1 is

Williamson's theorem.

If $f:]0,\infty[\to\mathbb{R}_+$ *is* n*-*$\downarrow$, $n\geq 2$, *then*

$$f(s)=\int(1-cs)_+^{n-1}\,d\mu(c)$$

where μ *is a measure on* $\mathbb{R}_+$*.*

(n-$\downarrow$ functions are often called "*n*-monotone")

A (more recent) generalization reads:

If $f:]0,\infty[^d\to\mathbb{R}_+$ is $\mathbf{n}$-$\downarrow$, $\mathbf{n}\geq\mathbf{2}_d$, then

$$f(s)=\int\prod_{i=1}^{d}(1-c_is_i)_+^{n_i-1}\,d\mu(c)$$

with μ a measure on $\mathbb{R}_+^d$.

An interesting appearance of 3-$\updownarrow$ functions:
For $x, y, z \in \mathbb{R}$ we have always

$$|x+y| + |y+z| + |z+x| \le |x| + |y| + |z| + |x+y+z|$$

the socalled *Hornich-Hlawka inequality*. This can be generalized as follows:

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be 3-$\updownarrow$, then $\forall x, y, z \in \mathbb{R}$

$$f(|x+y|)+f(|y+z|)+f(|z+x|) \le f(|x|)+f(|y|)+f(|z|)+f(|x+y+z|).$$

$f = \mathrm{id}$ gives the original inequality, which also holds for vectors x, y, z. The above generalization for $x, y, z \in \mathbb{R}^d$ can be shown for $f(t) = \sqrt{t}$, $f(t) = \sqrt[4]{t}$, $f(t) = \sqrt[8]{t}$, ..., but is open for other (Bernstein) functions.

The remarks given before are useful even in dimension one, as shown in the following

Example.

$$\varphi(t) := -\log(1-e^{-t}), \quad t \in]0,\infty[$$

Then φ is *completely montone*, i.e. n-$\downarrow$ for each $n \in \mathbb{N}$, and this was shown in an article from 2018 by using so-called Eulerian numbers (of permutations). It follows also from

$$\varphi(-\cdot) = \underbrace{[-\log(1-\cdot)]}_{n\text{-}\uparrow} \circ \underbrace{\exp}_{n\text{-}\uparrow} \quad (\text{on }]-\infty,0[)$$

using the Bernstein function $\log(1+t)$:

$\varphi(-\cdot)$ is n-$\uparrow$ as composition of two such functions, hence φ is n-$\downarrow$ $\forall n \in \mathbb{N}$.

Of special importance is again $\mathbf{n} = \mathbf{1}_d$. Non-negative $\mathbf{1}_d$ -$\downarrow$ functions are (essentially) *survival-functions*, i.e. of the form $\mu([s, \infty])$ for some measure μ.

Non-negative $\mathbf{1}_d$ -$\updownarrow$ are (essentially) *co-survival functions*, i.e. of the form $\mu([s, \infty]^{\complement})$ for some μ.

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A particular subclass of the latter is of special interest:

A d.f. F on $\mathbb{R}_+^d$ is called a *simple multivariate extreme value distribution* iff

$$(F(tx))^t = F(x) \quad \forall x \in \mathbb{R}_+^d, \ \forall t > 0$$

and if F has *standard Fréchet margins*, defined by the (one-dimensional) d.f. $\exp\left(-\frac{1}{u}\right)$ for $u > 0$. Then $F(x) = 0$ if $x_i = 0$ for some i, and $0 < F(x) < 1$ else.

$$f(x) := -\log F\left(\frac{1}{x}\right) \quad \text{where } \frac{1}{x} := \left(\frac{1}{x_1}, \frac{1}{x_2}, \dots\right)$$

is called a *stable tail dependence function* (STDF). It is a function $f : \mathbb{R}_+^d \to \mathbb{R}_+$ with the properties

1. f is homogeneous, i.e. $f(tx) = tf(x)\ \forall t > 0,\ \forall x$
2. $f(e_i) = 1\ \forall i = 1, \dots, d$
3. $\max_{i \le d} x_i \le f(x) \le \sum_{i=1}^d x_i$
4. f is convex

but this is a full characterization of STDFs only for $d = 2$. In higher dimensions, this had been an open problem for some time.

The answer I could give is as follows:

Theorem 13.

A function $f : \mathbb{R}_+^d \to \mathbb{R}$ *is a STDF if and only if* f *is homogeneous,* $\mathbf{1}_d$ -$\updownarrow$, *and* $f(e_1) = \cdots = f(e_d) = 1$.

In this case f *is the co-survival function of a homogeneous Radon measure* μ *on* $[0, \infty]^d \setminus \{\infty\}$, *i.e.*

$$f(x) = \mu\left([x, \infty]^{\complement}\right) \quad , \; x \in \mathbb{R}_+^d.$$

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In this case f has the unique integral representation

$$f(x) = f(\mathbf{1}_d) \cdot \int \max_{i \le d}(c_i x_i) \, d\nu(c),$$

ν being a probability measure on $\{c \in \mathbb{R}_+^d \mid \max_{i \le d} c_i = 1\}$.

Examples.

- The classical norms $f_p(x) := \left(\sum_{i=1}^d x_i^p\right)^{\frac{1}{p}}$ for $p \geq 1$, up to $f_\infty(x) := \max_{i \leq d} x_i$ (GUMBEL)
- $f(x, y) = \frac{x^2+xy+y^2}{x+y}$ on $\mathbb{R}_+^2$
- $\sum_i x_i - \sum_{i<j}(x_i^p + x_j^p)^{\frac{1}{p}} \pm \cdots + (-1)^{d-1}\left(\sum_i x_i^p\right)^{\frac{1}{p}}$ for $p < 0$ (GALAMBOS)

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Which (univariate) functions preserve $\mathbf{n}$-$\downarrow$ (multivariate) functions?

Answer: exactly those preserving $\mathbf{n}$-$\uparrow$ ones, because of Remark (a).

And which ones preserve $\mathbf{n}$-$\updownarrow$ functions:

- g $\mathbf{n}$-$\updownarrow$, f $|\mathbf{n}|$-$\updownarrow$ $\Rightarrow$ $f \circ g$ is $\mathbf{n}$-$\updownarrow$

since $-(f \circ g)(-\,\cdot\,) = [-f(-\,\cdot\,)] \circ [-g(-\,\cdot\,)]$,

- g $\mathbf{n}$-$\updownarrow$, f $|\mathbf{n}|$-$\downarrow$ $\Rightarrow$ $f \circ g$ is $\mathbf{n}$-$\downarrow$

since $f \circ g(-\,\cdot\,) = [f(-\,\cdot\,)] \circ [-g(-\,\cdot\,)]$.

An open problem:

$$K := \{f : [0,1]^2 \to \mathbb{R}_+ \mid f \text{ is } 2\text{-}\uparrow,\ f(1,1) = 1\}$$

Then K is compact and convex, multiplicatively stable.

- Is K a Bauer simplex?
- Determine $\mathrm{ex}(K)$!

I could prove that

$$f_c \circ (f_a \otimes f_b) \in \mathrm{ex}(K)$$

$\forall a, b, c \in [0,1]$ $(f_a(t) = (t-a)_+/(1-a),\ f_1 := 1_{\{1\}})$.

Are there other extreme points?