Completion Problem for Totally Positive Matrices

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HE WHO IS WITHOUT MATHEMATICS SHALL NOT ENTER

Definitions

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- In the set of $n \times n$ matrices:

$$\mathrm{TP} \subset \mathrm{TP}_{n-1} \subset \ldots \subset \mathrm{TP}_2 \subset \mathrm{TP}_1.$$

Theorem (Fekete, 1912)

A matrix is TP_k if and only if it is TP_k contiguous.

Definition

A minor is said to be an initial minor if it is formed of consecutive rows and columns, one of which being the first row or the first column.

Theorem (Gasca and Peña, 1992)

The $n \times n$ matrix A is totally positive if and only if each of its initial minors is positive.

TP_2 matrices are both interesting and important.

Theorem (Fallat & Johnson, 2007)

Suppose A is an $m \times n$ matrix. Then the following are equivalent

- *A is eventually TP under Hadamard exponent.*
- *A is TP_2.*
- *A is positive and all of the 2×2 contiguous minors are positive.*

Notations

- $M_\pi = [m_{ij}] \in M_n$: the permutation matrix of the permutation $\pi \in S_n$, that is,

$$m_{ij} = \begin{cases} 1, & \text{if } j = \pi(i) \\ 0, & \text{otherwise} \end{cases}$$

- $M_\pi(p, q)$: number of the ones in the submatrix lying in the rows $1, 2, \ldots, p$ and columns $1, 2, \ldots, q$.

$M_\pi =$

1					
					1
	1				
			1		
				1	
		1			

$\to M_\pi(3, 4) = 2$

Bruhat partial order

Definition

For two permutations $\sigma \neq \pi \in S_n$, $\sigma <_{Br} \pi$ if π is obtained from $\sigma \in S_n$ by a sequence of transpositions of i and j when $i < j$ and $\pi(i) < \pi(j)$.

Example

$\sigma = 3\underline{1}\underline{5}24 \to \underline{3}512\underline{4} \to 451\underline{2}\underline{3} \to 45132 = \pi$

Or, equivalently:

$$\sigma <_{Br} \pi \iff M_\sigma(p,q) \geq M_\pi(p,q)$$

for all $p, q \in \{1, 2, \ldots, n\}$.

$$M_\sigma = \begin{array}{|c|c|c|c|c|} \hline & & 1 & & \\ \hline 1 & & & & \\ \hline & & & & 1 \\ \hline & 1 & & & \\ \hline & & & 1 & \\ \hline \end{array} \geq \begin{array}{|c|c|c|c|c|} \hline & & & 1 & \\ \hline & & & & 1 \\ \hline 1 & & & & \\ \hline & & 1 & & \\ \hline & 1 & & & \\ \hline \end{array} = M_\pi$$

TP_2 Partial Order

- A_π : For $A \in M_n$ and a permutation $\pi \in S_n$,

$$A_\pi = \prod_{i=1}^{n} a_{i\pi(i)}.$$

$$\text{A}= \begin{array}{|c|c|c|}\hline a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \\ \hline \end{array}\,,\ A_\pi = a_{13}a_{21}a_{32}$$

- *TP_2 Partial Order:* If $\pi, \sigma \in S_n$ are such that

$$A_\pi < A_\sigma,\ \forall A \in \mathrm{TP}_2(n, n),$$

then π precedes σ in the *TP_2 partial order*. We denote this as

$$\pi <_{TP_2} \sigma.$$

TP_2=Bruhat

Theorem (Johnson & Nasserasr, 2010)

For $\pi, \sigma \in S_n$, $\pi \neq \sigma$,

$$\sigma <_{Br} \pi \iff \pi <_{TP_2} \sigma.$$

Proof: $\pi <_{TP_2} \sigma \Rightarrow \sigma <_{Br} \pi$: Suppose $\sigma \not<_{Br} \pi \Rightarrow \exists(p,q)$ such that $M_\sigma(p,q) < M_\pi(p,q)$

$$K = \left[\begin{array}{c|c} 2J & J \\ \hline J & J \end{array}\right]$$

$$2^{M_\sigma(p,q)} = \prod_{l=1}^{n} k_{l\sigma(l)} < \prod_{l=1}^{n} k_{l\pi(l)} = 2^{M_\pi(p,q)}$$

Characterization of TP_2 Matrices

Theorem (Johnson & Nasserasr, 2010)

A matrix $A > 0$ is TP_2 if and only if $A_\pi < A_\sigma$ whenever $\sigma <_{Br} \pi$.

Partial TP_2 Matrix:

- *Partial TP_2 Matrix:* An $m \times n$ matrix in which some entries are specified and the remaining entries are unspecified, and every minor of specified entries of size at most 2 is positive.
- *Example:*

$$\begin{pmatrix} ? & 1 & 2 & 1 & ? & ? \\ 1 & ? & 4 & 3 & ? & 4 \\ ? & 2 & 8 & ? & 4 & ? \\ ? & 4 & ? & 6 & 9 & 9 \\ 1 & ? & ? & ? & 10 & ? \end{pmatrix}$$

Completion Problem

- *TP_2 completion of a partial TP_2 matrix T:* A choice of values for the unspecified entries of T that results in a TP_2 matrix.
- *Example:*

$$\begin{pmatrix} ? & 1 & 2 & 1 & ? & ? \\ 1 & ? & 5 & 3 & ? & 4 \\ ? & 3 & 8 & ? & 12 & ? \\ ? & 4 & ? & 7 & 17 & 12 \\ 1 & ? & ? & ? & 22 & ? \end{pmatrix}$$

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TP_2 Completable Pattern:

- *TP_2 Completable Pattern:* Pattern P for which every partial TP_2 matrix with pattern P has a TP_2 completion.

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- *TP_2 Completion Problem:* Which patterns are TP_2 completable?

$$\begin{pmatrix} ? & 1 & 2 & 1 & ? & ? \\ 1 & ? & 4 & 3 & x & 4 \\ ? & 2 & 8 & ? & 4 & ? \\ ? & 4 & ? & 6 & 9 & 9 \\ 1 & ? & ? & ? & 10 & ? \end{pmatrix} \begin{matrix} \Rightarrow & 4 < x \\ \Rightarrow & x < 2 \end{matrix}$$

Not TP_2 completable

An Application the Completion Problem:

Collaborative filtering in recommendation systems can be modeled as a matrix completion problem:

- The list of m users: $\{u_1, u_2, \ldots, u_m\}$.
- The list of n items: $\{i_1, i_2, \ldots, i_n\}$.
- $m \times n$ matrix A with a_{ij} represents the preference (e.g., rating) of the user u_i for the item i_j.
- The matrix A may be a partial matrix (some entries are unknown).
- Complete the matrix using the ratings given by users on scales 1-5 on items.

Examples of Classes of Matrices for which the Completion Problem has been studied:

- Nonsingular matrices, and Rank completion
- PD matrices, and PSD matrices
- M-matrices, and Inverse M-matrices
- TN matrices, and TP matrices

References:

- H.J. Woerdeman, Minimal rank completions of partial banded matrices, Linear and Multilinear Algebra 36(1):59–68 (1993)
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- Johnson, Smith, The completion problem for M-Matrices and Inverse M-Matrices. Linear Algebra and its Applications 241–243 (1996), 655–667.
- S. Fallat, C. R. Johnson, R. Smith, The general totally positive matrix completion problem with few unspecified entries, ELA. The Electronic Journal of Linear Algebra 7 (2000): 1–20

Double majorization partial order for $A, B \in M_{m,n}$

Let $A, B \in M_{m,n}$. Then A doubly majorizes B if

- $A, B \geq 0$
- *row (column) sums of A and B are equal*
- $A \geq_{DM} B$, *if* $\sum_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} a_{ij} \geq \sum_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} b_{ij}$, *for all* p, q *with* $1 \leq p < m, 1 \leq q < n$.

- *Example:*

$$\begin{pmatrix} \color{red}{1} & \color{red}{1} & 2 & 1 & 2 & 1 \\ 1 & 2 & 5 & 3 & 7 & 4 \\ 1 & 3 & 8 & 5 & 12 & 8 \\ 1 & 4 & 11 & 7 & 17 & 12 \\ 1 & 5 & 14 & 9 & 22 & 16 \end{pmatrix} >_{DM} \begin{pmatrix} \color{red}{1} & \color{red}{1} & 1 & 1 & 3 & 1 \\ 1 & 1 & 5 & 3 & 8 & 4 \\ 1 & 3 & 8 & 5 & 12 & 8 \\ 1 & 5 & 11 & 7 & 16 & 12 \\ 1 & 5 & 15 & 9 & 21 & 16 \end{pmatrix}$$

-

$$\begin{pmatrix} \color{red}{1} & \color{red}{1} & \color{red}{2} & 1 & 2 & 1 \\ 1 & 2 & 5 & 3 & 7 & 4 \\ 1 & 3 & 8 & 5 & 12 & 8 \\ 1 & 4 & 11 & 7 & 17 & 12 \\ 1 & 5 & 14 & 9 & 22 & 16 \end{pmatrix} >_{DM} \begin{pmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & 1 & 3 & 1 \\ 1 & 1 & 5 & 3 & 8 & 4 \\ 1 & 3 & 8 & 5 & 12 & 8 \\ 1 & 5 & 11 & 7 & 16 & 12 \\ 1 & 5 & 15 & 9 & 21 & 16 \end{pmatrix}$$

General Conditions for TP_2-completion

Theorem (Johnson & Nasserasr, 2014)

A partial positive matrix T has a TP_2 completion if and only if

$$\prod_{t_{ij} \text{ specified}} t_{ij}^{a_{ij}} > \prod_{t_{ij} \text{ specified}} t_{ij}^{b_{ij}}$$

for every pair of nonnegative matrices $A, B \in P_T$ with $A >_{DM} B$.

- *Example:*

$$A >_{DM} B : \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} >_{DM} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

-

$$T = \begin{pmatrix} ? & 1 & 2 & 1 & ? & ? \\ 1 & ? & 4 & 3 & x & 4 \\ ? & 2 & 8 & ? & 4 & ? \\ ? & 4 & ? & 6 & 9 & 9 \\ 1 & ? & ? & ? & 10 & ? \end{pmatrix} \qquad (4)(4)(9) > (4)(8)(9)$$

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for every pair of nonnegative matrices $A, B \in P_T$ with $A >_{DM} B$.

C_T:

For an $m \times n$ partial matrix $T = (t_{ij})$, the set C_T consists of all $m \times n$ matrices $X = (x_{ij})$ such that

- $x_{ij} = 0$ if t_{ij} is unspecified,
- $Xe = 0$, and $e^t X = 0$, with $e = (1, 1, \dots, 1)^t$,
- $\sum_{\substack{1 \le i \le p \\ 1 \le j \le q}} x_{ij} \ge 0$, for all $1 \le p \le m, 1 \le q \le n$.

Example:

$$\mathcal{P} = \begin{pmatrix} ? & \times & \times & \times & ? & ? \\ \times & ? & \times & \times & ? & \times \\ ? & \times & \times & ? & \times & ? \\ ? & \times & ? & \times & \times & \times \\ \times & ? & ? & ? & \times & ? \end{pmatrix}$$

$$\begin{pmatrix} 0 & x_1 & x_2 & -x_1 - x_2 & 0 & 0 \\ x_3 & 0 & x_4 & x_5 & 0 & -x_3 - x_4 - x_5 \\ 0 & x_6 & -x_2 - x_4 & 0 & x_2 + x_4 - x_6 & 0 \\ 0 & -x_1 - x_6 & 0 & x_1 + x_2 - x_5 & x_6 - x_2 - x_3 - x_4 & x_3 + x_4 + x_5 \\ -x_3 & 0 & 0 & 0 & x_3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & x_1 & x_2 & -x_1 - x_2 & 0 & 0 \\ x_3 & 0 & x_4 & x_5 & 0 & -x_3 - x_4 - x_5 \\ 0 & x_6 & -x_2 - x_4 & 0 & x_2 + x_4 - x_6 & 0 \\ 0 & -x_1 - x_6 & 0 & x_1 + x_2 - x_5 & x_6 - x_2 - x_3 - x_4 & x_3 + x_4 + x_5 \\ -x_3 & 0 & 0 & 0 & x_3 & 0 \end{pmatrix}$$

- $x_1 \geq 0$
- $x_1 + x_2 \geq 0$
- $x_3 \geq 0$
- $x_1 + x_3 \geq 0$
- $x_1 + x_2 + x_3 + x_4 \geq 0$
- $x_3 + x_4 + x_5 \geq 0$
- $x_1 + x_3 + x_6 \geq 0$
- $-x_2 + x_3 + x_5 + x_6 \geq 0$

C_T is a cone.

Theorem

For an $m \times n$ partial TP_2 matrix T, the set C_T is a polyhedral cone and pointed. Thus it has a unique minimal integral set of generators.

Conditions in terms of C_T:

Theorem (Johnson & Nasserasr, 2014)

A partial positive matrix T has a TP_2 completion if and only if it satisfies

$$\prod_{t_{ij} \text{ specified}} t_{ij}^{m_{ij}} > 1 \qquad \forall M \in C_T.$$

Theorem (Johnson & Nasserasr, 2014)

Let T be a partial positive matrix and $C_T = cone\{G_1, G_2, \ldots, G_r\}$. Then T is TP_2 completable if and only if it satisfies the finitely many polynomial inequalities on the specified entries of T of the form

$$\prod_{t_{ij} \text{ specified}} t_{ij}^{g_{ij}^k} > 1 \qquad \text{for } k = 1, 2, \ldots, r.$$

An Algorithm to Compute the Minimal Conditions:

Half space representation of the cone:

$$\begin{aligned} &x_1 \geq 0 \\ &x_1 + x_2 \geq 0 \\ &x_3 \geq 0 \\ &x_1 + x_3 \geq 0 \\ &x_1 + x_2 + x_3 + x_4 \geq 0 \\ &x_3 + x_4 + x_5 \geq 0 \\ &x_1 + x_3 + x_6 \geq 0 \\ &-x_2 + x_3 + x_5 + x_6 \geq 0 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \geq 0$$

The Generators Computed by cdd+:

0	0	0	1	-1	1
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1
0	0	1	-1	0	-1
1	-1	0	0	0	-1
0	1	0	-1	1	0

Reference: K. Fukuda. cdd+ reference manual. Institute for Operations Research, Swiss Federal Institute of Technology, Zurich, Switzerland, 1995.

$$\begin{pmatrix} 0 & x_1 & x_2 & -x_1-x_2 & 0 & 0 \\ x_3 & 0 & x_4 & x_5 & 0 & -x_3-x_4-x_5 \\ 0 & x_6 & -x_2-x_4 & 0 & x_2+x_4-x_6 & 0 \\ 0 & -x_1-x_6 & 0 & x_1+x_2-x_5 & x_6-x_2-x_3-x_4 & x_3+x_4+x_5 \\ -x_3 & 0 & 0 & 0 & x_3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \to G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T = \begin{pmatrix} ? & t_{12} & t_{13} & t_{14} & ? & ? \\ t_{21} & ? & t_{23} & t_{24} & ? & t_{26} \\ ? & t_{32} & t_{33} & ? & t_{35} & ? \\ ? & t_{42} & ? & t_{44} & t_{45} & t_{46} \\ t_{51} & ? & ? & ? & t_{55} & ? \end{pmatrix}$$

$$\prod_{t_{ij} \text{ specified}} t_{ij}^{g_{ij}^1} > 1 \iff t_{23} t_{32} t_{44} > t_{24} t_{33} t_{42}$$

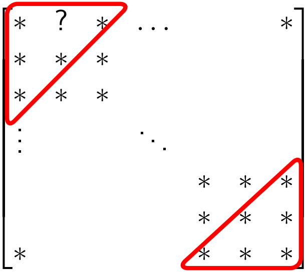
$$T = \begin{pmatrix} ? & t_{12} & t_{13} & t_{14} & ? & ? \\ t_{21} & ? & t_{23} & t_{24} & ? & t_{26} \\ ? & t_{32} & t_{33} & ? & t_{35} & ? \\ ? & t_{42} & ? & t_{44} & t_{45} & t_{46} \\ t_{51} & ? & ? & ? & t_{55} & ? \end{pmatrix}$$

- $t_{24} t_{46} > t_{26} t_{44}$
- $t_{32} t_{45} > t_{35} t_{42}$
- $t_{12} t_{33} > t_{13} t_{32}$
- $t_{13} t_{24} > t_{14} t_{23}$
- $t_{23} t_{32} t_{44} > t_{24} t_{33} t_{42}$
- $t_{23} t_{35} t_{46} > t_{26} t_{33} t_{45}$
- $t_{21} t_{33} t_{42} t_{55} > t_{23} t_{32} t_{45} t_{51}$
- $t_{ij} > 0$ for all specified entries t_{ij}.

TP-completion problem: one unspecified entry

Theorem (Fallat, Johnson, Smith 2000)

Let A be an m-by-n partial TP matrix with $4 \leq m \leq n$ and in which the only unspecified entry lies in the (s, t) position. Then A is completable iff $s + t \leq 4$ or $s + t \geq m + n - 2$.



TP-compeltion problem: one unspecified entry

Theorem (Akin, Johnson, Nasserasr 2014)

Let $A \in M_{n,n+2}$, and $k, j \in \{2, \ldots, n+1\}$. Suppose $\alpha = [n]$, and $\beta = [n+2] \setminus \{j, k\}$. If $j < k$, then

$$\det(A[\alpha, \{1,k\}^c]) \det(A[\alpha, \{j, n+2\}^c]) - \det(A[\alpha, \{1,j\}^c]) \det(A[\alpha, \{k, n+2\}^c
= \det(A[\alpha, \{j,k\}^c]) \det(A[\alpha, \{1, n+2\}^c]).$$

Theorem (Akin, Johnson, Nasserasr 2014)

For $m, n, k \geq 4$, an $m \times n$ pattern P with one unspecified entry in the (i, j) position is TP_k-completable if and only if $i + j \leq 4$ or $i + j \geq m + n - 2$.

TP-compeltion problem: $3 \times n$ case

Theorem (Carter, Johnson, 2024)

For $3 \times n$ cases, there are 13 distinct obstructions, listed below, accounting for symmetry.

$$\begin{bmatrix} \ast & ? & \ast \\ ? & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix} \begin{bmatrix} \ast & ? & \ast \\ ? & \ast & \ast \\ \ast & \ast & ? \end{bmatrix} \begin{bmatrix} \ast & \ast & \ast \\ ? & \ast & \ast \\ \ast & ? & \ast \end{bmatrix} \begin{bmatrix} \ast & ? & \ast \\ ? & \ast & \ast \\ \ast & \ast & ? \end{bmatrix} \begin{bmatrix} \ast & ? & ? & \ast \\ ? & \ast & \ast & ? \\ \ast & \ast & \ast & \ast \end{bmatrix}$$

$$\begin{bmatrix} \ast & ? & \ast & \ast \\ \ast & \ast & \ast & \ast \\ ? & \ast & \ast & \ast \end{bmatrix} \begin{bmatrix} \ast & ? & \ast & \ast \\ ? & \ast & \ast & ? \\ \ast & ? & \ast & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & ? & \ast \\ ? & \ast & \ast & \ast \\ \ast & \ast & \ast & ? \end{bmatrix} \begin{bmatrix} \ast & ? & \ast & \ast & ? \\ \ast & \ast & \ast & ? & \ast \\ \ast & ? & \ast & \ast & \ast \end{bmatrix}$$

$$\begin{bmatrix} \ast & ? & \ast & ? & \ast \\ \ast & \ast & \ast & \ast & \ast \\ ? & \ast & \ast & \ast & \ast \end{bmatrix} \begin{bmatrix} \ast & ? & \ast & \ast & ? \\ \ast & \ast & \ast & ? & \ast \\ \ast & ? & \ast & \ast & \ast \end{bmatrix} \begin{bmatrix} \ast & \ast & \ast & \ast & ? \\ ? & \ast & \ast & ? & \ast \\ \ast & \ast & \ast & \ast & \ast \end{bmatrix} \begin{bmatrix} \ast & ? & ? & \ast & \ast & ? \\ \ast & \ast & \ast & \ast & \ast & \ast \\ ? & \ast & \ast & ? & ? & \ast \end{bmatrix}$$

Relating minors

Lemma (Sylvester's Identity)

Let A be an n-by-n square matrix, with ordered subsets $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \subseteq \{1, \ldots, n\}$. If $\tilde{A}$ is the two-by-two matrix of determinants

$$\tilde{A} = \begin{bmatrix} \det A[\{\alpha_2\}^c, \{\beta_2\}^c] & \det A[\{\alpha_2\}^c, \{\beta_1\}^c] \\ \det A[\{\alpha_1\}^c, \{\beta_2\}^c] & \det A[\{\alpha_1\}^c, \{\beta_1\}^c] \end{bmatrix},$$

then $\det A \det A[\{\alpha_1, \alpha_2\}^c, \{\beta_1, \beta_2\}^c] = \det \tilde{A}$.

Theorem (Chen, Lu, Nasserasr, REU 2023)

For an $n \times n$ matrix A, if $i < j < k$ and $\ell < m < p$, then
$\det A[j^c, p^c] \det A[\{i, k\}^c, \{\ell, m\}^c] - \det A[k^c, m^c] \det A[\{i, j\}^c, \{\ell, p\}^c] =$
$\det A[i^c, p^c] \det A[\{j, k\}^c, \{\ell, m\}^c] - \det A[k^c, \ell^c] \det A[\{i, j\}^c, \{m, p\}^c]$

TP-completion problem: diagonal unspecified entries in symmetric TP matrices

The case where the partial matrix is symmetric with unspecified entries consecutively along the diagonal:

$$\begin{bmatrix} ? & * & * & * & \cdots & * \\ * & * & * & * & & \vdots \\ * & * & * & * & & \\ * & * & * & * & & \\ \vdots & & & & \ddots & \vdots \\ * & \cdots & & & \cdots & * \end{bmatrix}$$

1 unspecified entry: completable with large enough value

$$\begin{bmatrix} ? & * & * & * & \cdots & * \\ * & ? & * & * & & \vdots \\ * & * & * & * & & \\ * & * & * & * & & \\ \vdots & & & & \ddots & \vdots \\ * & \cdots & & & \cdots & * \end{bmatrix}$$

2 unspecified entries: completable (Fallat, Johnson, Smith 2000)

$$\begin{bmatrix} ? & * & * & * & \cdots & * \\ * & ? & * & * & & \vdots \\ * & * & ? & * & & \\ * & * & * & * & & \\ \vdots & & & & \ddots & \vdots \\ * & \cdots & & & \cdots & * \end{bmatrix}$$

3 unspecified entries: completable (Chen, Lu, N.)

TP-compeltion problem: diagonal unspecified entries in symmetric TP matrices

Theorem (Chen, Lu, Nasserasr, REU 2023)

Let A be an n-by-n symmetric partial TP matrix, $n \geq 7$ with the pattern below. Then, A is TP-completable if and only if the following three independent inequalities hold:

$$\begin{bmatrix} ? & * & * & * & * & \cdots & * \\ * & ? & * & * & * & & \vdots \\ * & * & ? & * & * & & \\ * & * & * & ? & * & & \\ * & * & * & * & * & & \\ \vdots & & & & & \ddots & \vdots \\ * & \cdots & & & & \cdots & * \end{bmatrix}$$

- $\downarrow A[\{1,2,3\}^c, \{1, n-1, n\}^c] <$ $\uparrow A[\{1,2\}^c, \{3, n\}^c]$
- $\downarrow A[\{1,2\}^c, \{1,3\}^c] <$ $\uparrow A[\{1,2,3\}^c, \{n-2, n-1, n\}^c]$
- $\downarrow A[\{1,2\}^c, \{1,3\}^c] <$ $\uparrow A[\{1, n\}^c, \{2,3\}^c]$

TP-compeltion problem: diagonal unspecified entries in symmetric TP matrices

$$\begin{bmatrix} ? & * & * & * & * & * & * \\ * & ? & * & * & * & * & * \\ * & * & ? & * & * & * & * \\ * & * & * & ? & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{bmatrix}$$

$\downarrow A[\{1,2,3\}^c, \{1,6,7\}^c] <$ $\uparrow A[\{1,2\}^c, \{3,7\}^c]$

$$\begin{bmatrix} ? & * & * & * & * & * & * \\ * & ? & * & * & * & * & * \\ * & * & ? & * & * & * & * \\ * & * & * & ? & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{bmatrix}$$

$\downarrow A[\{1,2\}^c, \{1,3\}^c] <$ $\uparrow A[\{1,2,3\}^c, \{5,6,7\}^c]$

$$\begin{bmatrix} ? & * & * & * & * & * & * \\ * & ? & * & * & * & * & * \\ * & * & ? & * & * & * & * \\ * & * & * & ? & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{bmatrix}$$

$\downarrow A[\{1,2\}^c, \{1,3\}^c] <$ $\uparrow A[\{1,7\}^c, \{2,3\}^c]$

TP-compeltion problem: diagonal unspecified entries in symmetric TP matrices

Theorem (Chen, Lu, Nasserasr, REU 2023)

Let A be a 6-by-6 symmetric partial TP matrix with the pattern below. Then, A is TP-completable if and only if seven inequalities hold.

$$\begin{bmatrix} ? & * & * & * & * & * \\ * & ? & * & * & * & * \\ * & * & ? & * & * & * \\ * & * & * & ? & * & * \\ * & * & * & * & ? & * \\ * & * & * & * & * & * \end{bmatrix}$$

Five Diagonal Case

The main obstacle is that we must 'fill in' the $(5,5)$ unspecified entry so that our inequalities for the $(4,4)$ entry hold.

For instance, we want

$$\downarrow A[\{3,4,5,6\},\{2,4,5,6\}] < \uparrow A[\{2,3,4,5\},\{1,4,5,6\}].$$

Equivalently,

$$\frac{-\det A[\{3,4,5,6\},\{2,4,5,6\}](0)}{\det A[\{3,5,6\},\{2,5,6\}]} < \frac{\det A[\{2,3,4,5\},\{1,4,5,6\}](0)}{\det A[\{2,3,5\},\{1,5,6\}]}$$

TP-compeltion problem: diagonal unspecified entries in symmetric TP matrices

Then, plugging in y for the $(5,5)$ unspecified entry, we get

$$\frac{-\det A[\{3,4,6\},\{2,4,6\}](0)y - \det A[\{3,4,5,6\},\{2,4,5,6\}](0,0)}{\det A[\{3,6\},\{2,6\}]y + \det A[\{3,5,6\},\{2,5,6\}](0)}$$

$$< \frac{-\det A[\{2,3,4\},\{1,4,6\}](0)y + \det A[\{2,3,4,5\},\{1,4,5,6\}](0,0)}{-\det A[\{2,3\},\{1,6\}]y + \det A[\{2,3,5\},\{1,5,6\}](0)},$$

which we may rewrite as $ay^2 + by + c < 0$.

This gives us an upper and lower bound, namely

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} < y < \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Table of Bounds for TP-compeltion problem with diagonal unspecified entries in symmetric TP matrices

k \ n	4	5	6	7	8	9+
3	0	0	0	0	0	0
4	0	1	2	3	3	3
5	n/a	1	7	≤ 45	≤ 60	≤ 66

Table: Number of necessary constraints for $n \times n$ symmetric matrix with k unspecified diagonal entries

International Linear Algebra Society (ILAS)

Please consider joining ILAS.

Thank You!