Total-positivity transforms in one and several variables

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The study of positivity preservers goes back to Schur (1911) and Pólya-Szegő (1925), who showed that every convergent power series, when applied entrywise to positive semidefinite matrices of all sizes, preserves positivity. The converse was shown by Schoenberg (1942) and Rudin (1959): there are no other such preservers. Many subsequent variants (via changing the domain) and extensions (multivariate case) exist; e.g. a recent work by Belton–Guillot–Khare–Putinar classified the multivariate transforms of matrices with prescribed negative inertias.

I will discuss the parallel problem of preserving totally nonnegative (TN) or totally positive (TP) kernels, on arbitrary totally ordered sets X, Y. We begin with the work of Belton–Guillot–Khare–Putinar (2023), where they classified the post-composition operators F that preserve TN/TP kernels of each specified order, and showed that such univariate preservers F are either constant or linear.

I will then explain how to extend this from preservers to transforms, and from one to several variables. Namely, we completely characterize the transforms $F[-]$ that send each tuple of TN/TP kernels of orders $k_1, \dots, k_p$ on $X \times Y$ to a TN/TP kernel of order l, for arbitrary prescribed positive integers (or infinite) $k_1, \dots, k_p, l$. The proofs use generalized Vandermonde kernels, (strictly totally positive) Pólya frequency functions, and a kernel that can be traced back to works of Schoenberg (1955) and Karlin (1964), and recent work by Jain (2010s). This is joint work with Apoorva Khare.