



Mathematical  
Institute

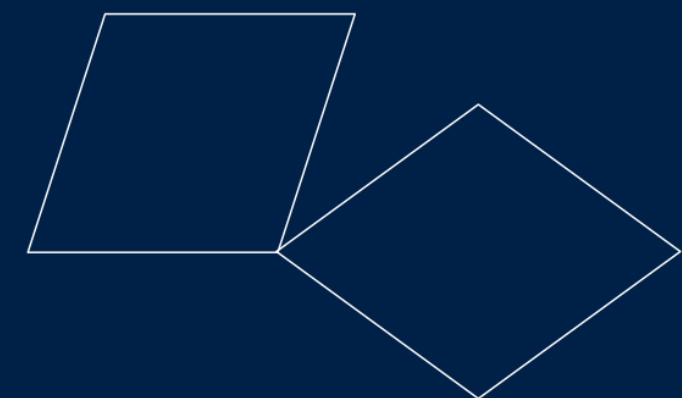
# Heterogeneity in soft matter

## *Does it matter?*

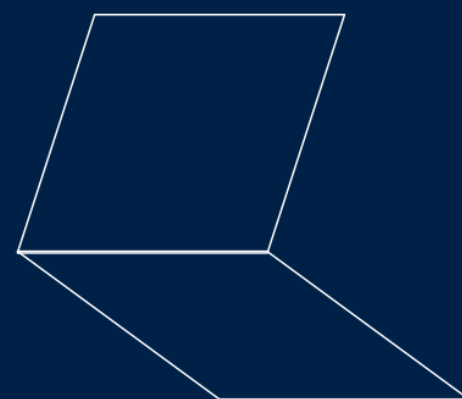
RWAM - 13th January 2025

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Mathematics



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Derek Moulton

# Motivation

## Tendons

Tissues connecting muscle to bone

Withstand large tensile loads

### **Tendinopathy**

Common, painful condition

**Overuse** injury / disease

**Altered** tissue **structure**, **composition** and **volume**

Model:

Non-linear poroelasticity with  
material heterogeneity

Gastrocnemius  
(lateral head)

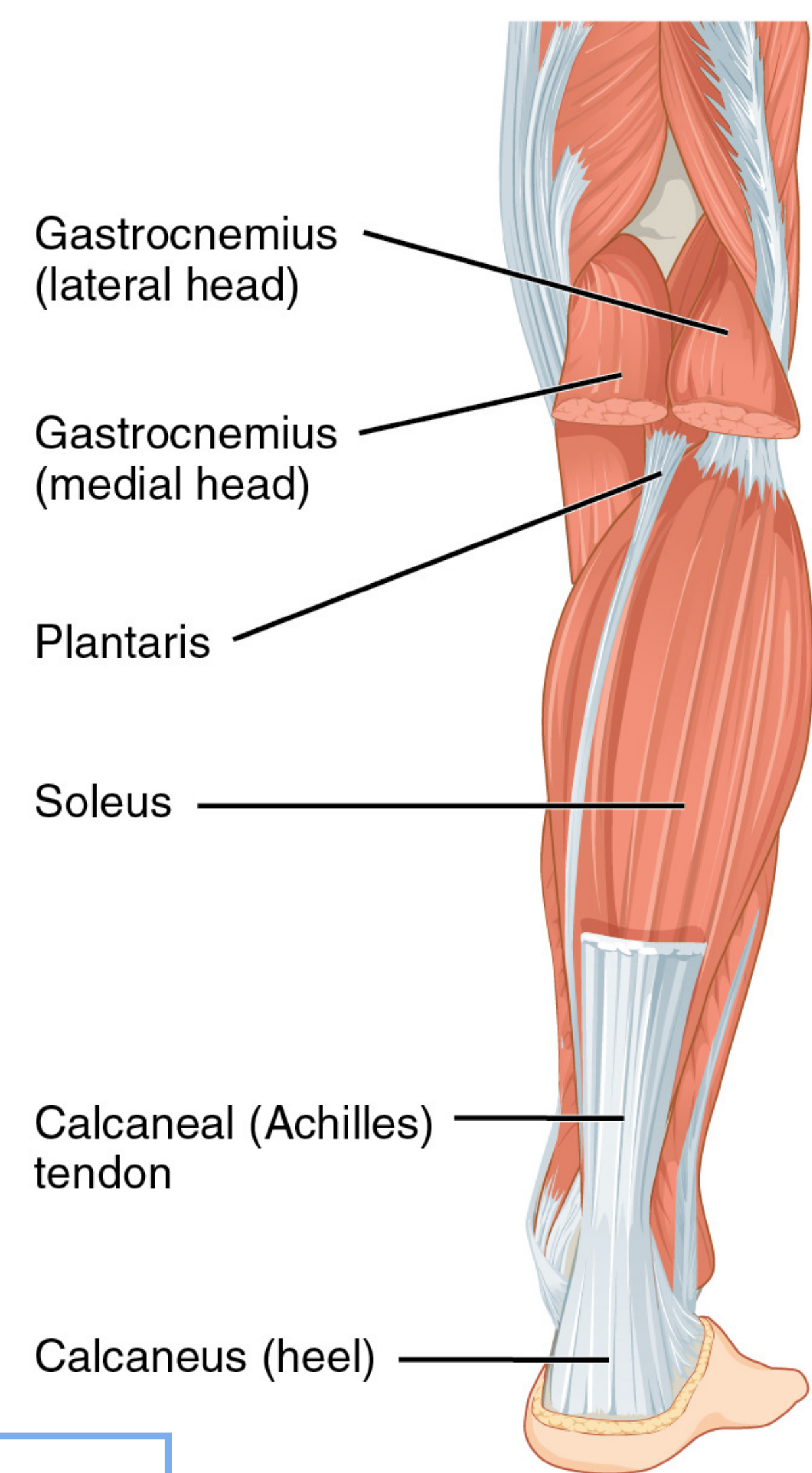
Gastrocnemius  
(medial head)

Plantaris

Soleus

Calcaneal (Achilles)  
tendon

Calcaneus (heel)



Wong M, Jardaly  
AH, Kiel J.  
Anatomy, Bony  
Pelvis and Lower  
Limb: Achilles  
Tendon. [Updated  
2023 Aug 8]. In:  
StatPearls  
[Internet]

How does material heterogeneity affect the response of a porous material to cyclical pulling?

# Non-linear poroelasticity

(in Lagrangian)

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial Z} \left[ \frac{k(\Phi)}{1 + \Phi - \Phi_0} \frac{\partial p}{\partial Z} \right],$$

$$\frac{\partial p}{\partial Z} = \frac{\partial s'}{\partial Z}$$

$$s = s' - p$$

total stress

Conservation of mass

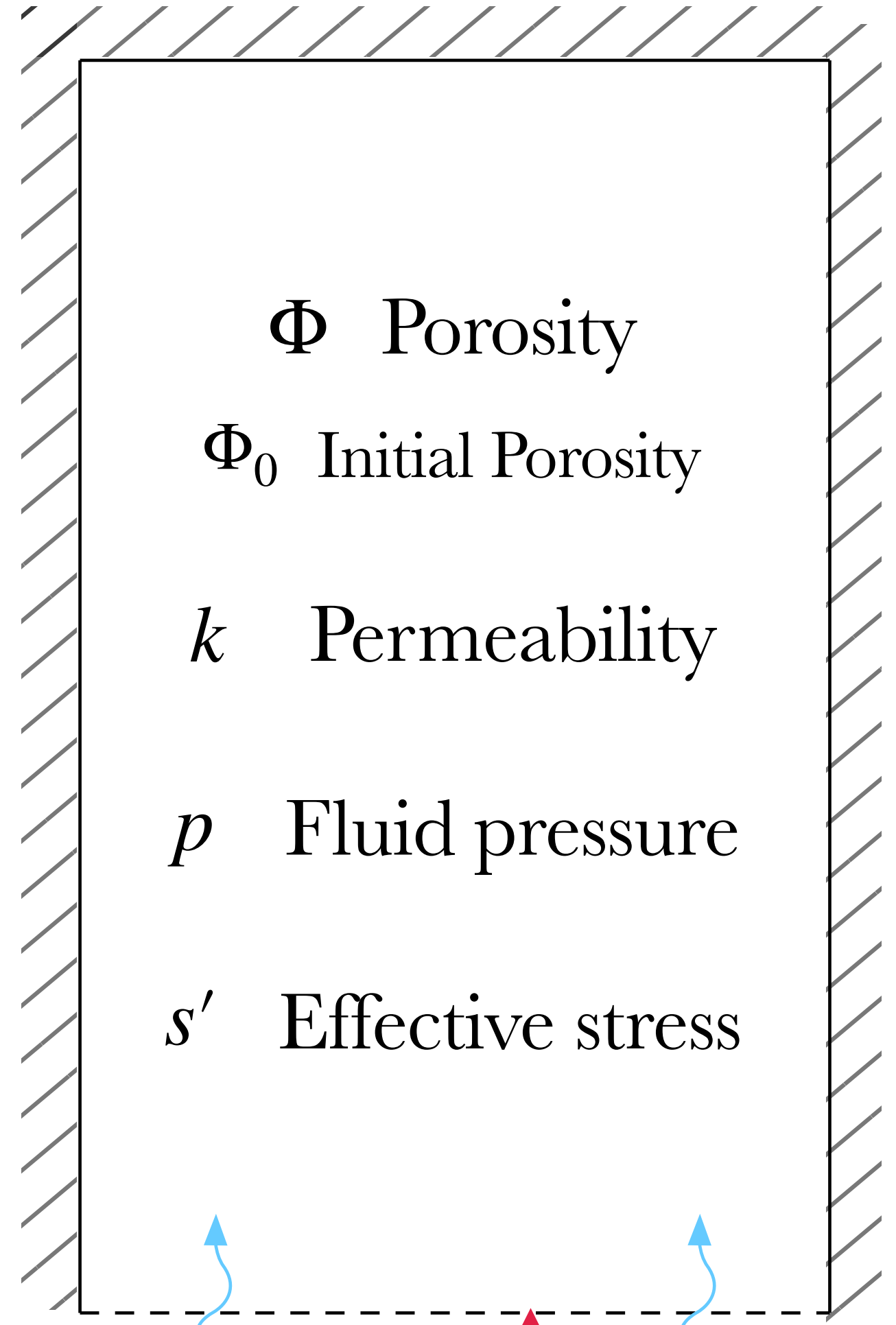
Darcy flow

Mechanical equilibrium

No flux, no displacement

*bone*  $Z = 1$

*muscle*  $Z = 0$



Applied Load or

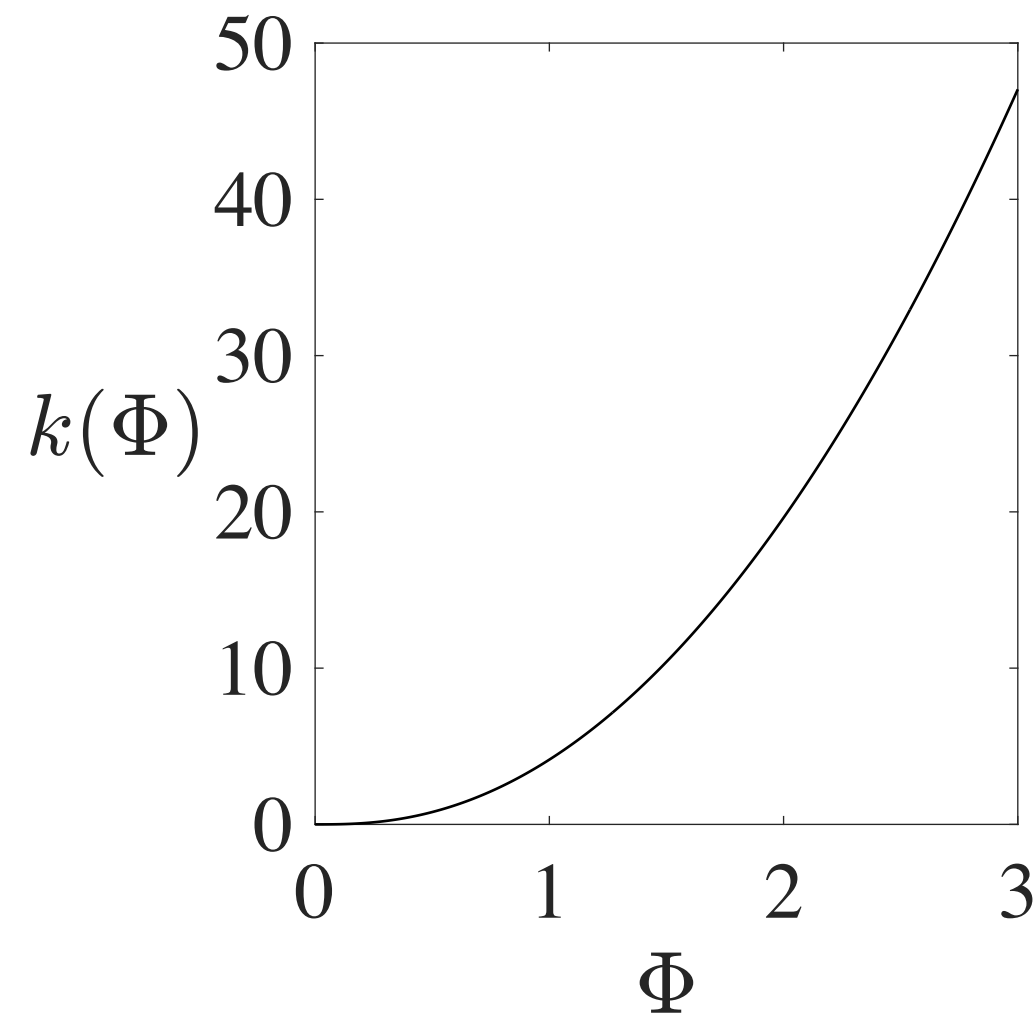
Applied Displacement

# Non-linear poroelasticity

(in Lagrangian)

Kozeny-Carman permeability

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial Z} \left[ \frac{k(\Phi)}{1 + \Phi - \Phi_0} \frac{\partial p}{\partial Z} \right], \quad \frac{\partial p}{\partial Z} = \frac{\partial s'}{\partial Z}$$

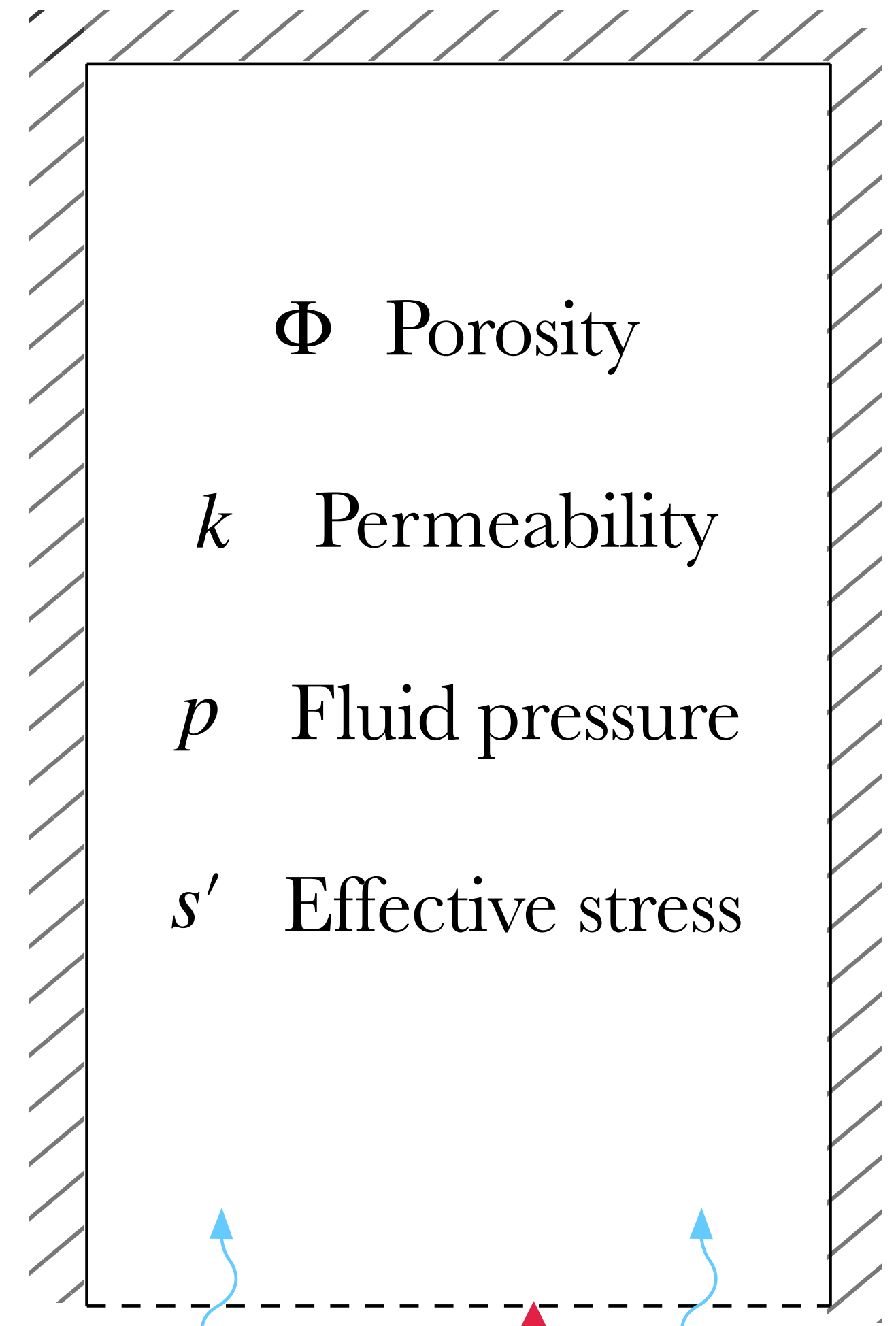


Neo-Hookean elasticity

$$s'(\Phi)$$

*bone*  $Z = 1$

No flux, no displacement



*muscle*  $Z = 0$

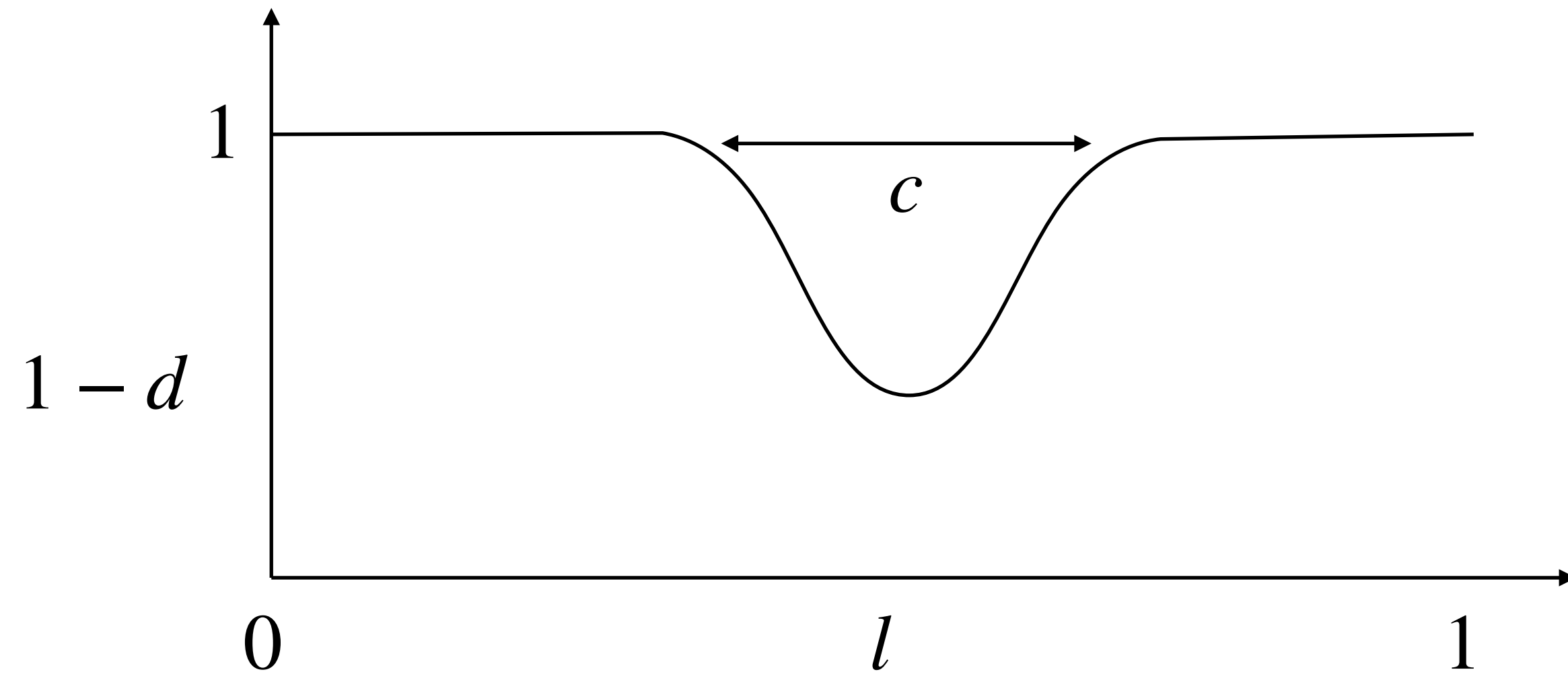
Applied Load or

Applied Displacement

**Non-linear advection diffusion equation for porosity**

# Heterogeneity

$$f(Z) = 1 - d \exp\left(-\frac{(Z-l)^2}{2c^2}\right)$$

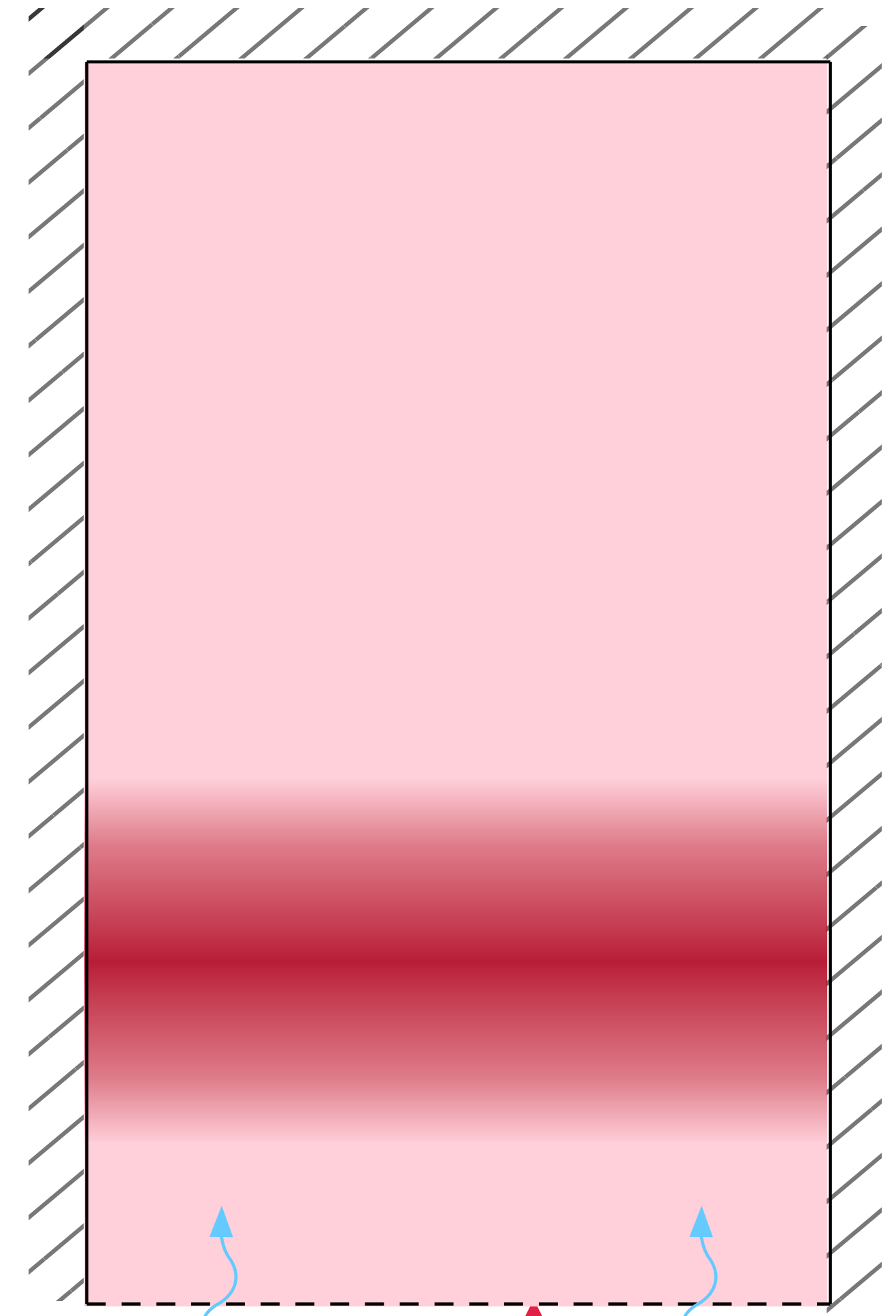


$$M_d(Z) = f(Z)$$

*bone*  $Z = 1$

*muscle*  $Z = 0$

No flux, no displacement



Applied Load or

Applied Displacement

$M$  Stiffness

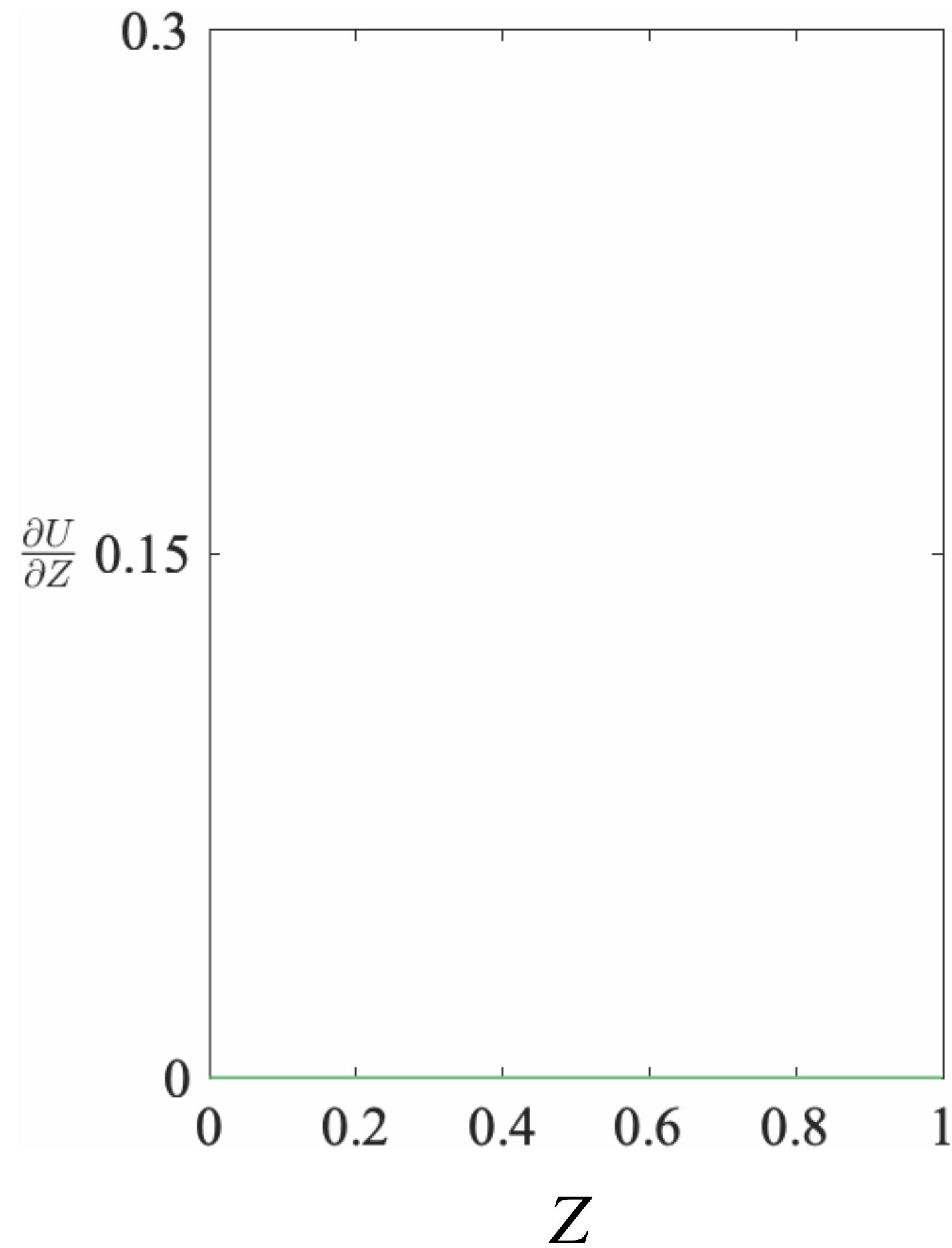
undamaged

damaged

# APPLIED LOAD - STRAIN / POROSITY RESPONSE (medium frequency)

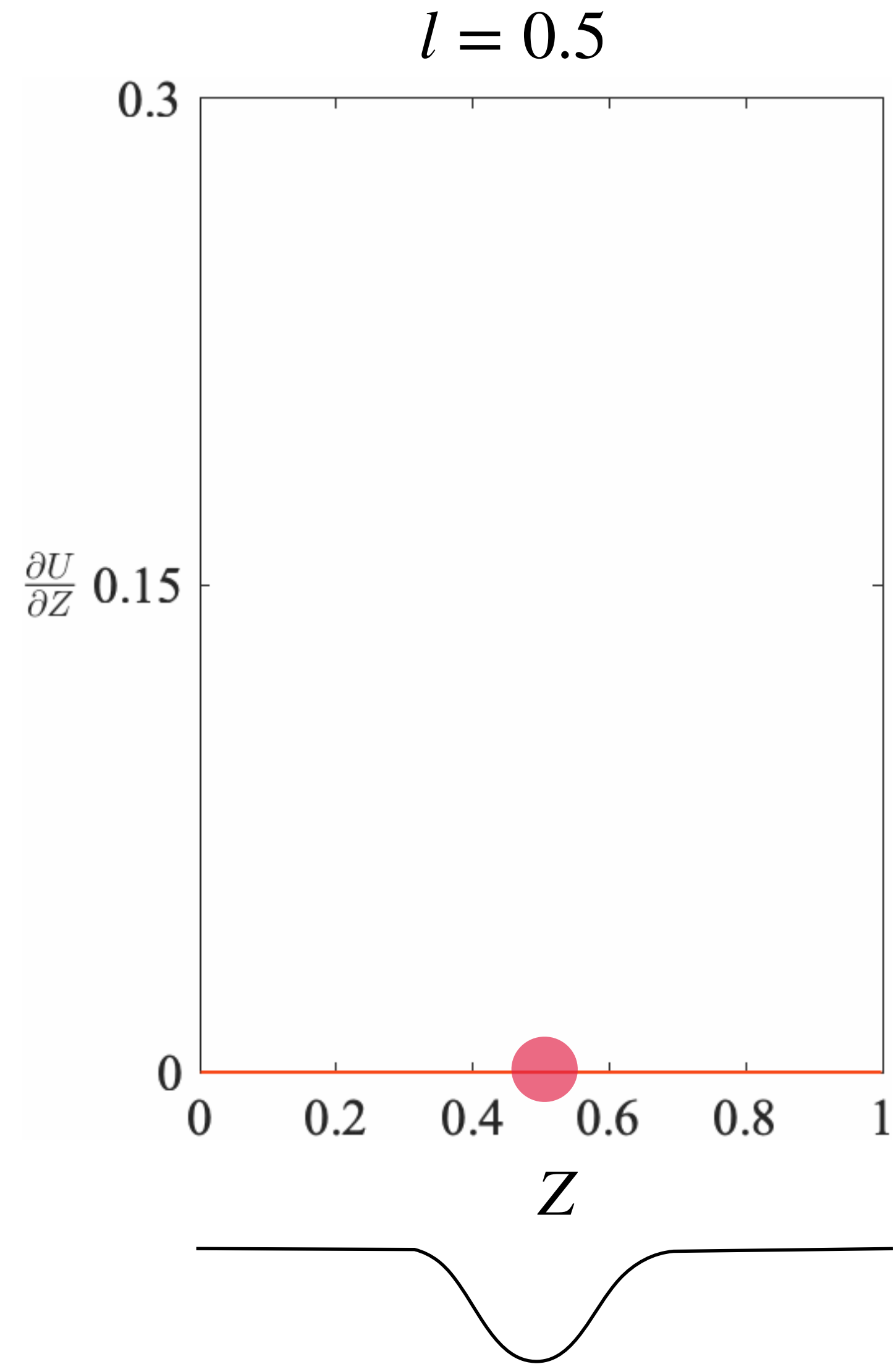
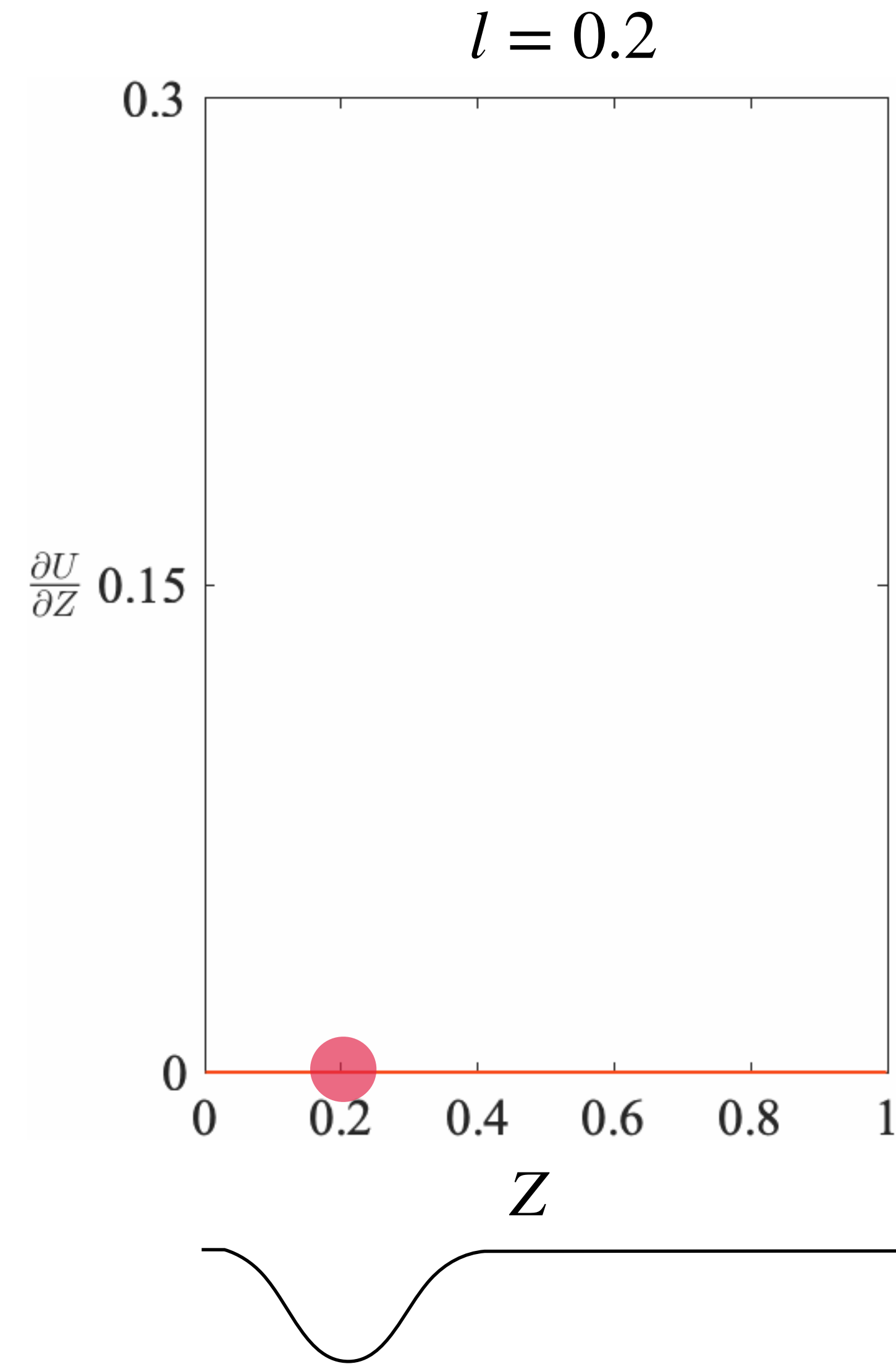
$$\frac{\partial U}{\partial Z} = \Phi - \Phi_0 \quad \text{Strain}$$

- Fluid enters when pulled, exits when let go
- Positive feedback loop from porosity-dependent permeability
- Strain / porosity maintained as  $Z \rightarrow 1$



APPLIED LOAD - STRAIN / POROSITY RESPONSE  
(medium frequency)

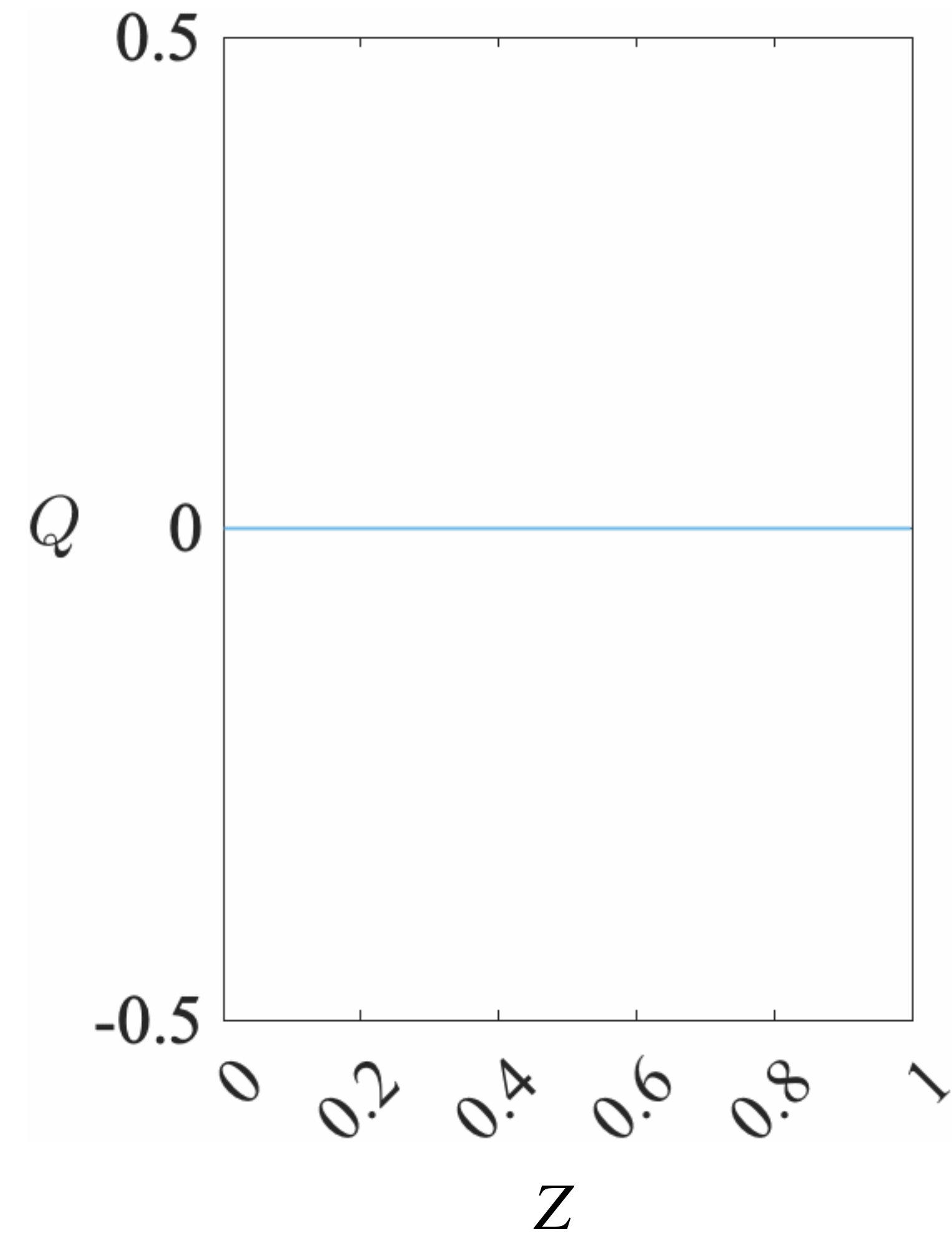
**DAMAGED STIFFNESS**



Strain increases around point of damage



# APPLIED DISPLACEMENT - FLUX RESPONSE (medium frequency)



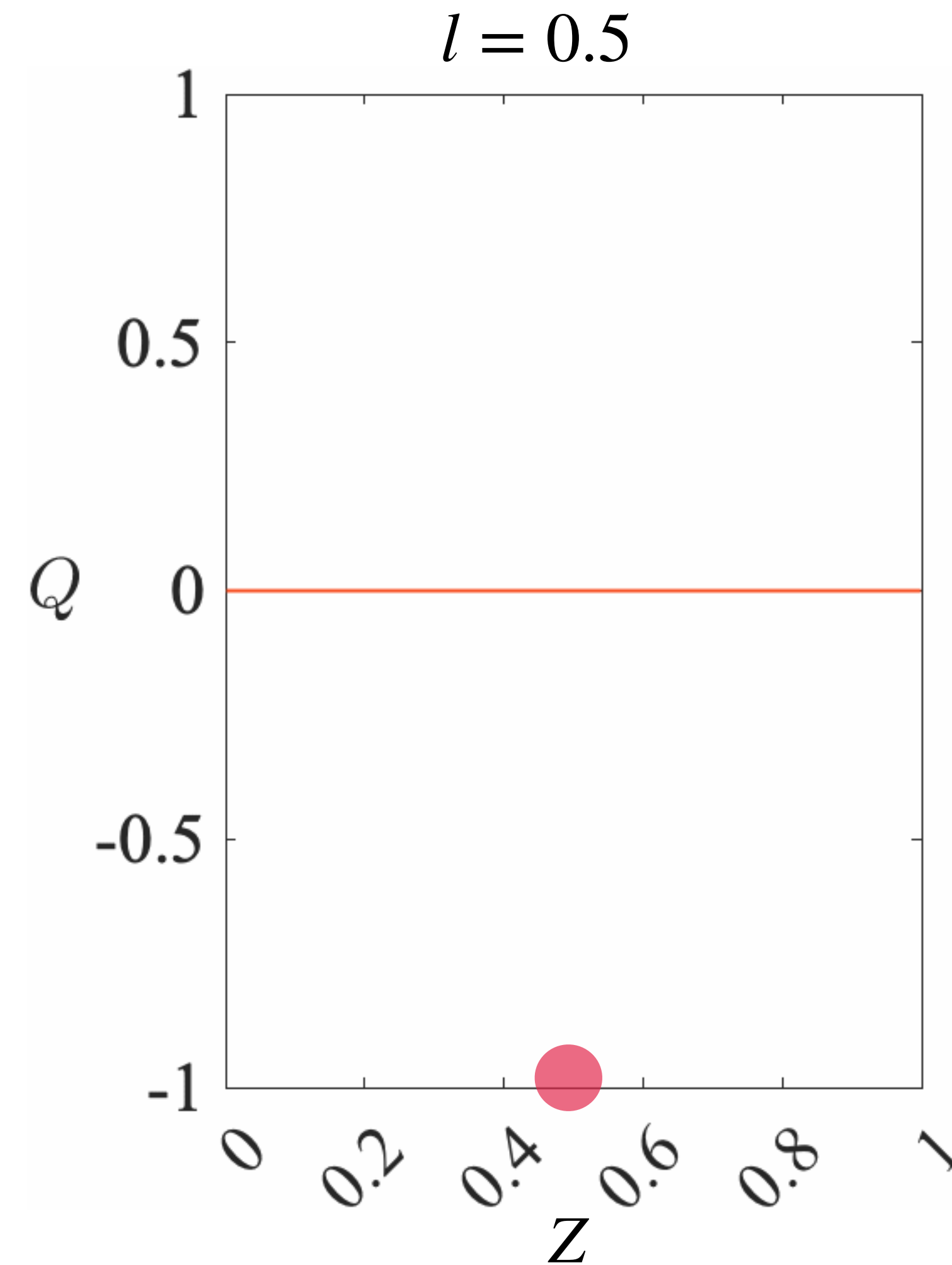
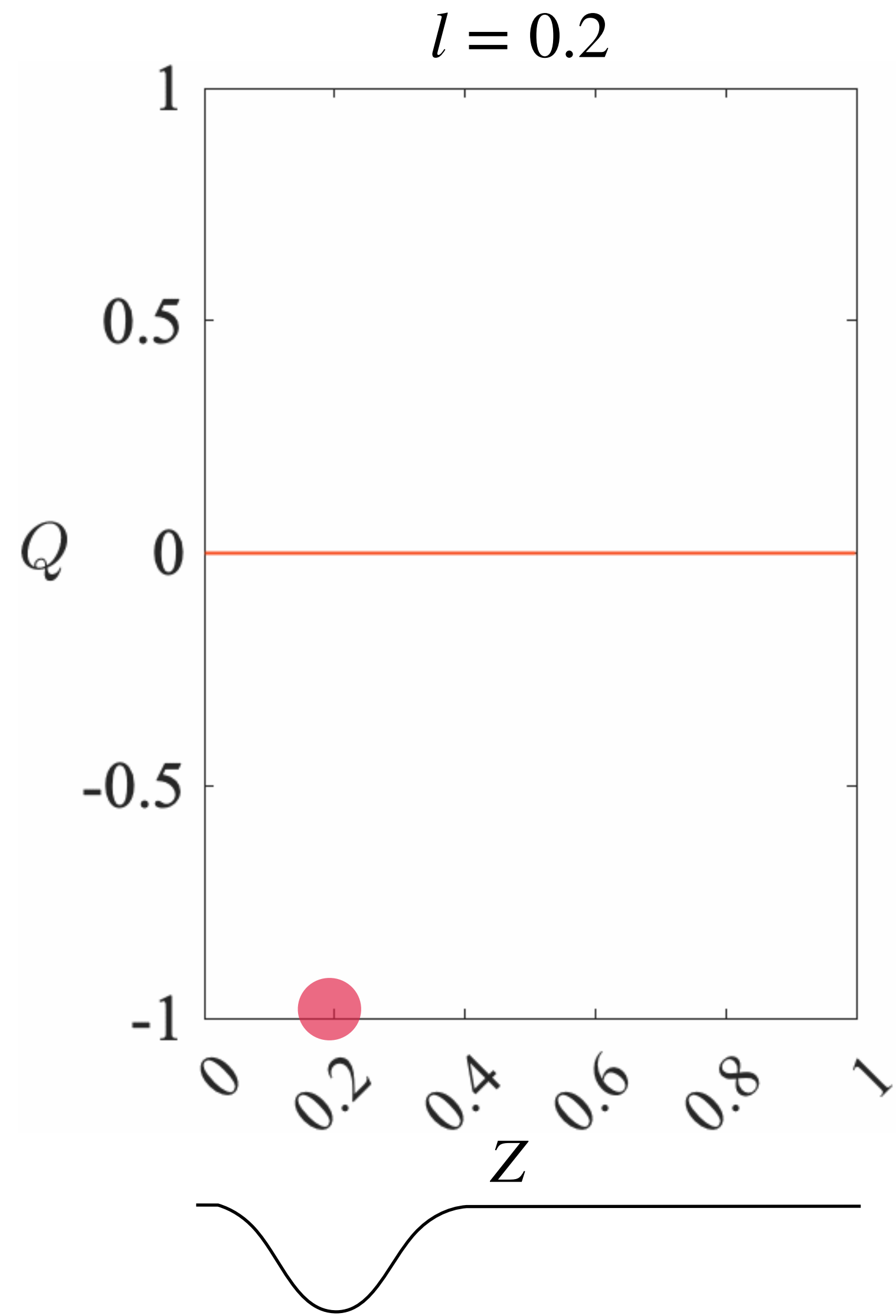
$Q$

Relative Flow

- Fluid enters when pulled, exits when pushed back
- No flux at  $Z = 0$

APPLIED DISPLACEMENT - FLUX RESPONSE  
(medium frequency)

**DAMAGED STIFFNESS**



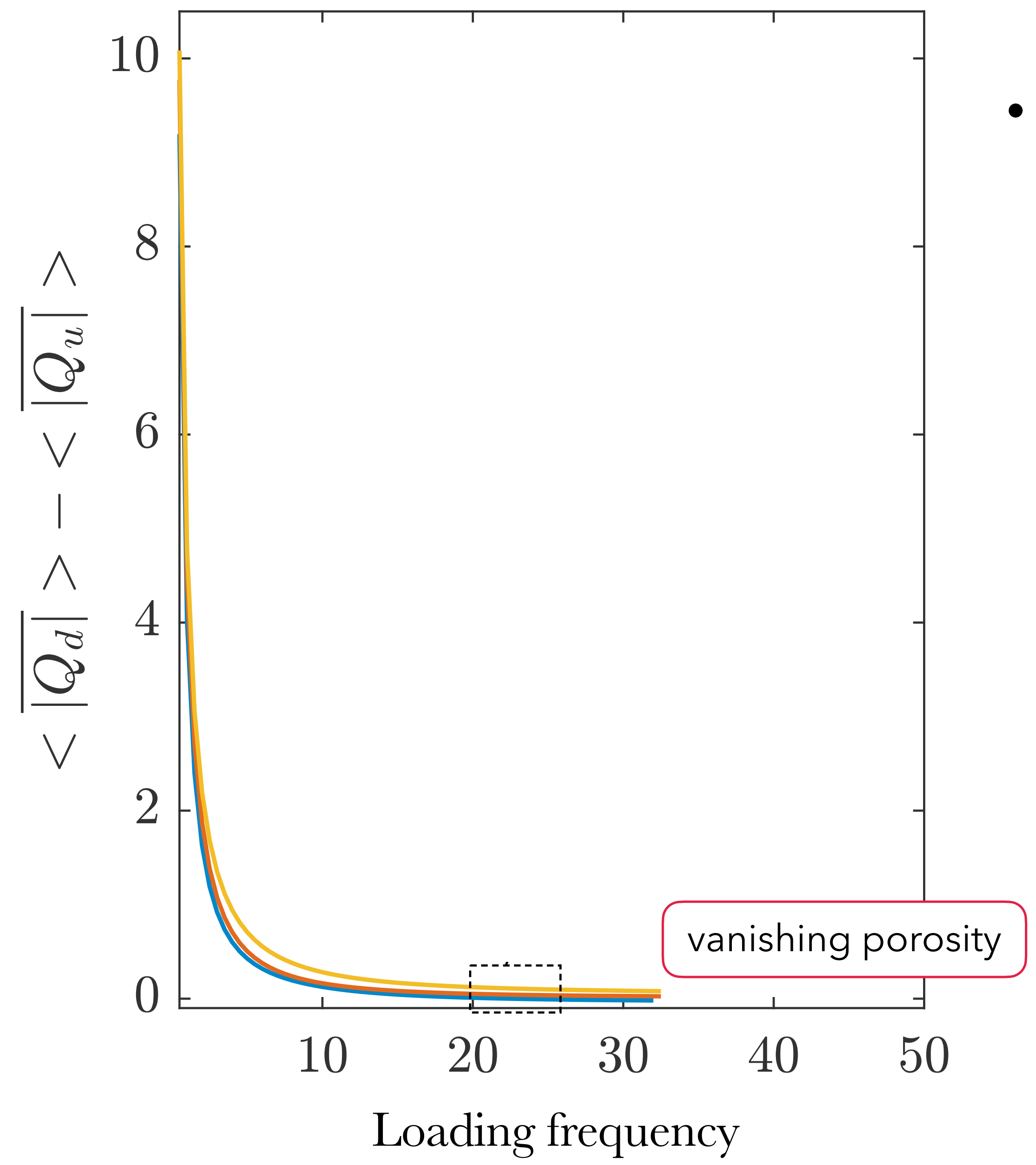
Generated flux into damaged region due to stress gradient (mostly)

How can we characterise how the response changes with frequency, damage magnitude and location?

The diagram illustrates the equivalence between a double integral and an average value notation. On the left, a double integral is shown:  $\int_0^1 \int_0^T |\cdot| dt dZ$ . The inner integral  $\int_0^T |\cdot| dt$  is enclosed in a yellow box and labeled "Time integral". The outer integral  $\int_0^1$  is enclosed in a teal box and labeled "Space integral". A purple bracket under the absolute value symbol  $|\cdot|$  is labeled "Absolute value". This is followed by an equivalence symbol  $\equiv$ . On the right, the average value notation  $\langle \overline{|\cdot|} \rangle$  is shown. The inner average  $\overline{|\cdot|}$  is enclosed in a yellow box and labeled "Time integral". The outer average  $\langle \cdot \rangle$  is enclosed in a teal box and labeled "Space integral". A purple bracket under the absolute value symbol  $|\cdot|$  is labeled "Absolute value".

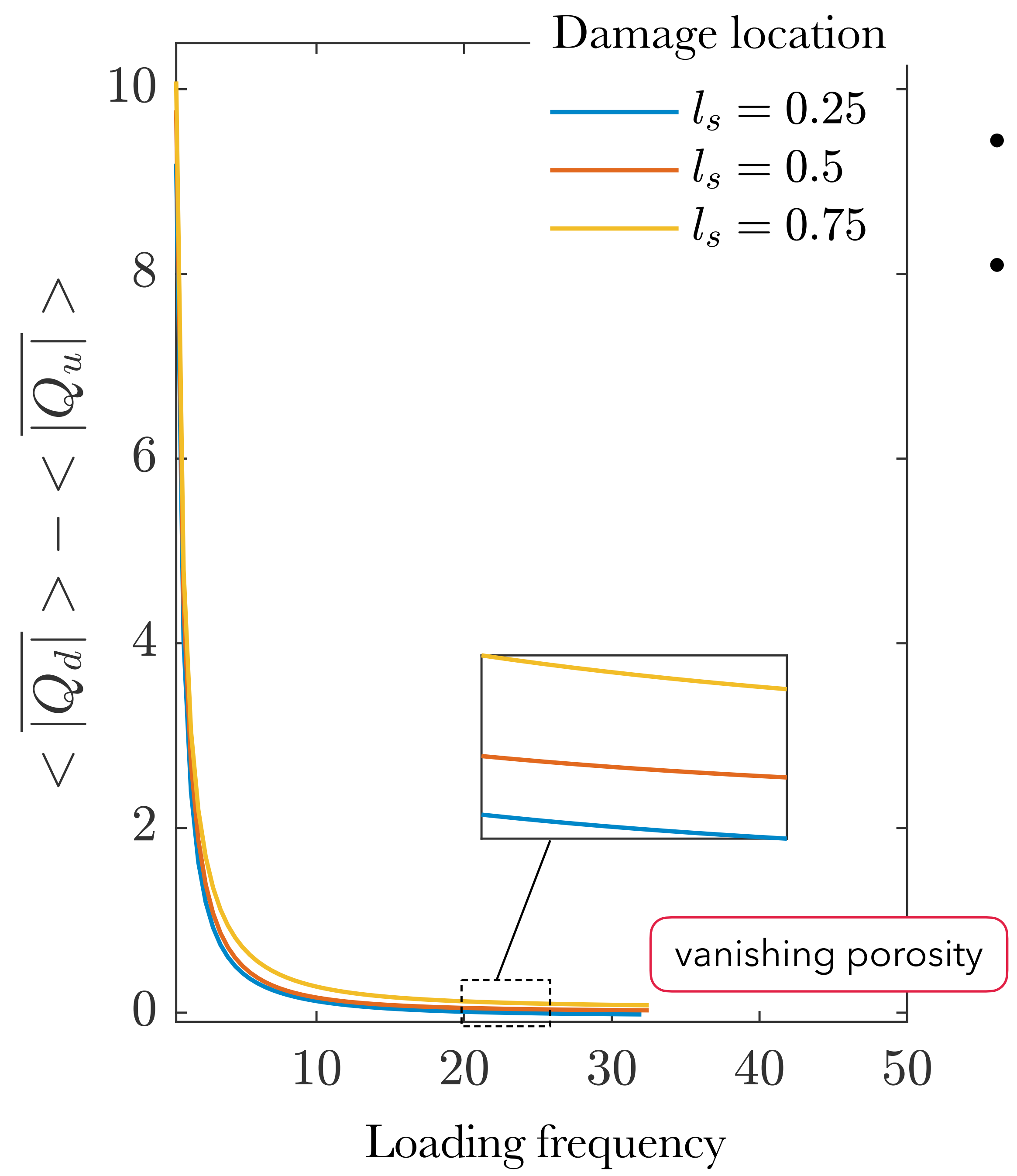
$$\int_0^1 \int_0^T |\cdot| dt dZ \equiv \langle \overline{|\cdot|} \rangle$$

# APPLIED DISPLACEMENT, varying loading frequency

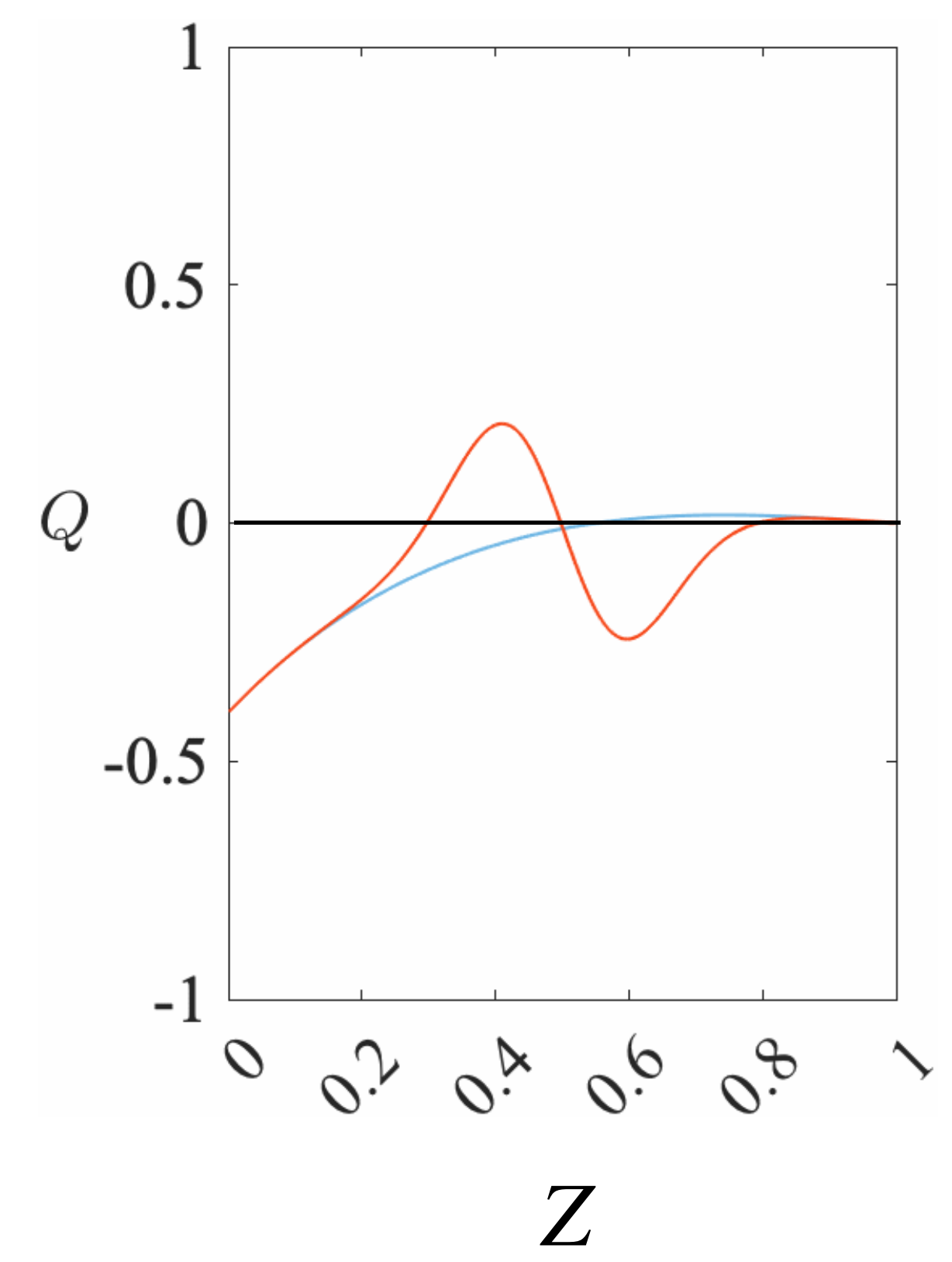
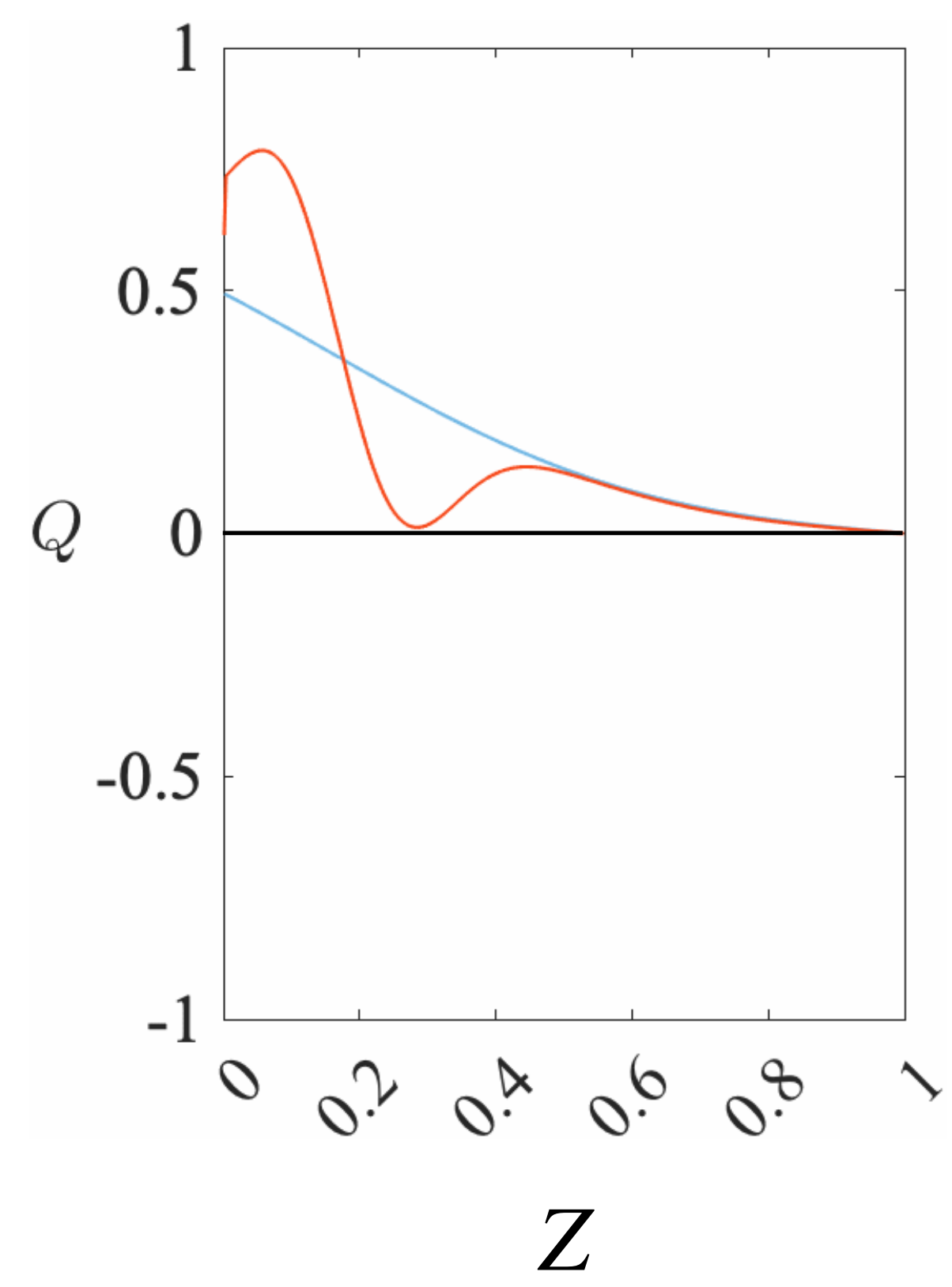


- Slower loading results in greater flux

# APPLIED DISPLACEMENT, varying loading frequency & location

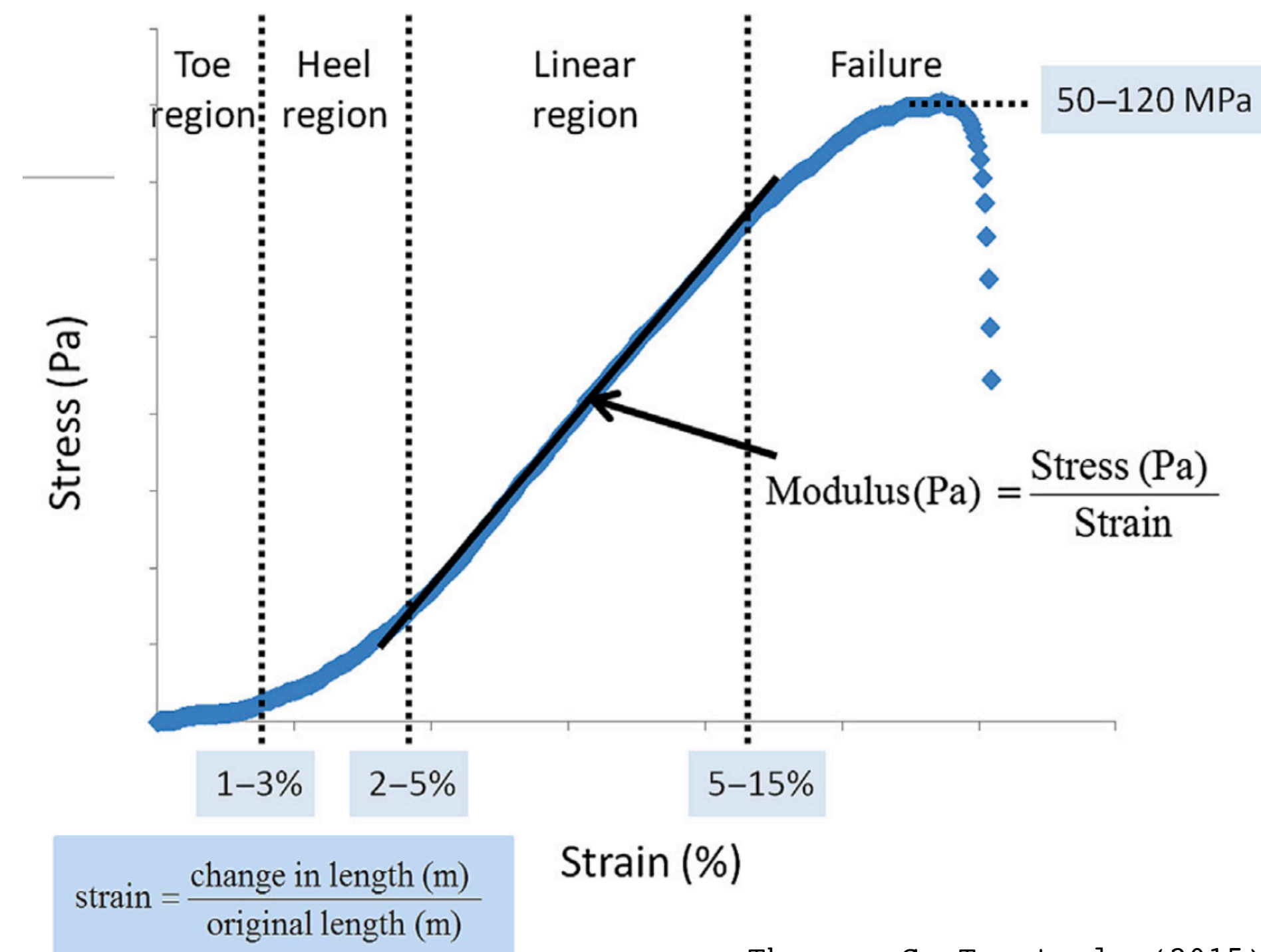


- Slower loading results in greater flux
- Damage further away from boundary results in greater flux



# Broader context and future directions

- Many parameters and variables to play with: which ones are important?
- Informed by context of problem
- Aim: build a more accurate model for tendon



Tendon stress-strain curve

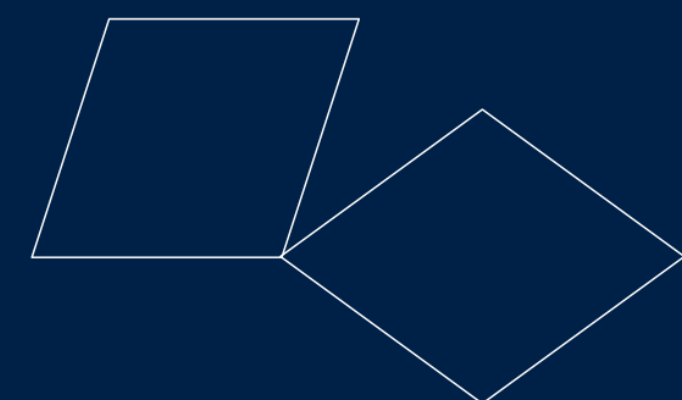


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# Thank you for listening!

Any questions?

contact: [zoe.godard@seh.ox.ac.uk](mailto:zoe.godard@seh.ox.ac.uk)



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Mathematics



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