The escaping set

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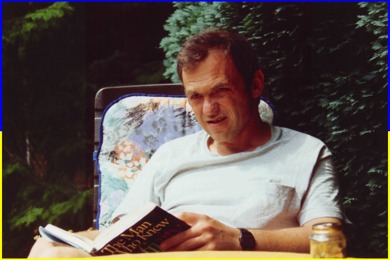

Classical Function Theory in Modern Mathematics

Celebrating Alexandre Eremenko's 70th birthday

Edinburgh, July 1-5, 2024

Our starting point is a paper which appeared 35 years ago, in the proceedings of a semester "Dynamical Systems and Ergodic Theory" at the Banach Center:

ON THE ITERATION OF ENTIRE FUNCTIONS

A. E. EREMENKO

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Kharkov, U.S.S.R.

This paper has had enormous impact on the further development of transcendental dynamics.

We will discuss some developments arising from this paper.

The paper is concerned with the escaping set $I(f)$:

Define

$$I(f) = \{z \in C \colon f^n z \to \infty,\ n \to \infty\}.$$

If f is a polynomial then $I(f)$ is a domain containing ∞. In this case we have

(1) $$J(f) = \partial I(f).$$

We shall study the set $I(f)$ for transcendental entire functions.

THEOREM 1. *For every entire function f the set $I(f)$ is non-empty.*

Here $J(f)$ is the Julia set, and Alex also showed in this paper that $J(f) = \partial I(f)$ for transcendental f as well.

I saw this paper first in February 1990 at a conference in Oberwolfach, where David Drasin showed it to me.

My talk had been on periodic points of entire functions, and one tool I had used was Wiman-Valiron theory.

David showed me that I was not the first to use this in dynamics.

THEOREM WV. *Let f be a transcendental entire function, $\alpha > \frac{1}{2}$. If*

(2) $$|z - w(r)| < r\left(N(r)\right)^{-\alpha},$$

then

(3) $$f(z) = \left(\frac{z}{w(r)}\right)^{N(r)} f\left(w(r)\right)(1+\varepsilon_1),$$

(4) $$f'(z) = N(r)\left(\frac{z}{w(r)}\right)^{N(r)} f\left(w(r)\right)\left(w(r)\right)^{-1}(1+\varepsilon_2)$$

where $\varepsilon_i = \varepsilon_i(r, z) \to 0$ uniformly with respect to z if $r \to \infty$, $r \notin E$. The exceptional set E depending of f and α has a finite logarithmic measure, i.e.

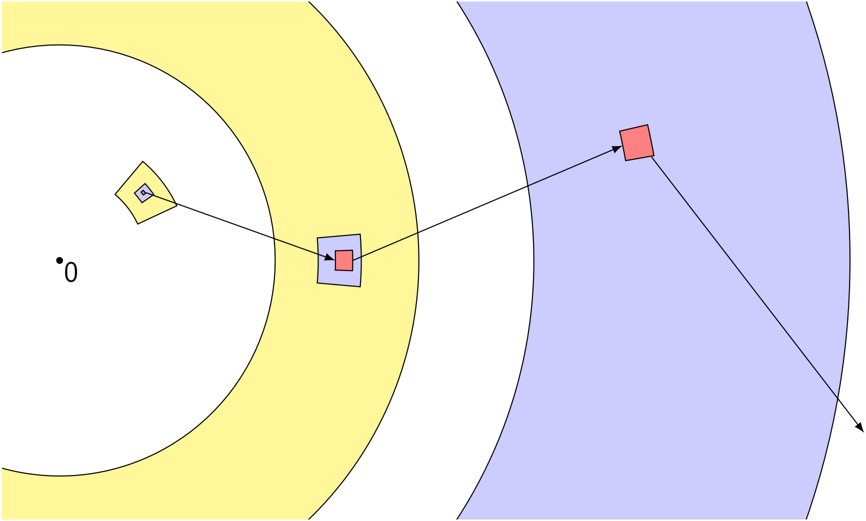
$$\operatorname{lm} E = \int_E \frac{dt}{t} < \infty.$$

Here $|w(r)| = r$ and $|f(w(r))| = \max_{|w|=r} |f(w)|$.

$N(r)$ central index: $f(z) = \sum_n a_n z^n$, $\max_n |a_n| r^n = |a_{N(r)}| r^{N(r)}$.

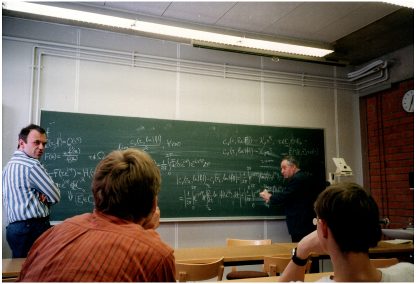
Essentially Wiman-Valiron theory says that near points of maximum modulus an entire function behaves like a monomial of large degree.

A small neighborhood of such a point is mapped to a large annulus.



The first time I met Alex in person was in March 1991, at a workshop in Joensuu organized by Ilpo Laine.

The workshop was very small – the only non-Finnish participants were Alex, his advisor Anatoly Gol'dberg, Jörg Winkler, and me.



Alex, Gol'dberg, and I shared an appartment.

We had no common language: With Gol'dberg I spoke German, with Alex English, and they spoke Russian.

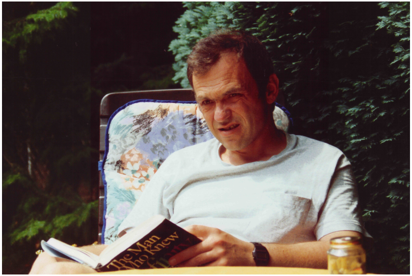
In 1993, there was some program for visiting lecturers in Aachen (where I was then). I could invite Alex.

He accepted, and I still remember his email I got some time later:

`I received my visa. It says: ‘‘Erwerbstätigkeit nicht gestattet’’. I only understand the word in the middle.`

We could overcome the obstacle: Alex visited for a couple of weeks.

My wife and I lived in Düren (a city between Aachen and Cologne) then, and Alex stayed with us there.



During this stay Alex and I wrote our first paper.

In the same year 1993 I also visited Alex in West Lafayette:

I moved to Kiel in 1996. Alex visited me there first in 1998.

In 2000 he obtained the Humboldt Prize – and came again.

Since then he has come to Kiel essentially every year – with only very few exceptions.

Back to Alex's paper from 1989:

This paper became famous not only because of the results proved there, but also because of a conjecture:

> It is plausible that the set $I(f)$ has no bounded connected components. We shall prove a weaker statement.
>
> THEOREM 3. *The closure $\overline{I(f)}$ of $I(f)$ has no bounded components.*

This conjecture became one of the central problems in transcendental dynamics and initiated a lot of research in the area.

A counterexample was recently constructed by Martí-Pete, Rempe, and Waterman — James will talk about this at 14:30 today.

I would like to convince you that this is not the end of the story:

Many interesting open questions about the escaping set remain – and the aim of this talk is to present some them.

Open questions about the escaping set

Question: Does Eremenko's conjecture hold in some "reasonable" classes of entire functions? E.g., does it hold for functions of finite order? Or in the Eremenko-Lyubich class $\mathcal{B}$? Or in the Speiser class $\mathcal{S}$?

Here f is said to be of finite order if there exists $\mu \in \mathbb{R}$ such that

$$|f(z)| \leq \exp |z|^{\mu} \quad \text{for large } |z|.$$

The infimum of these μ is the order of f and denoted by $\rho(f)$.

Let $\operatorname{sing}(f^{-1})$ be the set of singularities of the inverse of f.

The classes $\mathcal{B}$ and $\mathcal{S}$ are defined by

$\mathcal{B} = \{f\colon \operatorname{sing}(f^{-1}) \text{ is bounded}\}$ and $\mathcal{S} = \{f\colon \operatorname{sing}(f^{-1}) \text{ is finite}\}$.

Rottenfußer, Rückert, Rempe, Schleicher 2011, Barański 2007: Eremenko's conjecture holds (in a stronger form) for functions which are in $\mathcal{B}$ *and* of finite order.

Open questions about the escaping set

For points z constructed by Eremenko's method, $|f^n(z)|$ tends to infinity very fast.

Rippon, Stallard 2011: There exist points in $I(f)$ which escape arbitrarily slowly. That is, given a real sequence (a_n) tending to infinity, there exists $z \in I(f)$ such that $|f^n(z)| \leq a_n$ for large n.

Let f be entire and $R > 0$. A connected component of $\{z\colon |f(z)| > R\}$ is called a *tract* of f.

A modification of the Wiman-Valiron theory (by Phil Rippon, Gwyneth Stallard and me) shows that if D is a tract, then there exists $z \in I(f)$ such that $f^n(z) \in D$ for all $n \geq 0$.

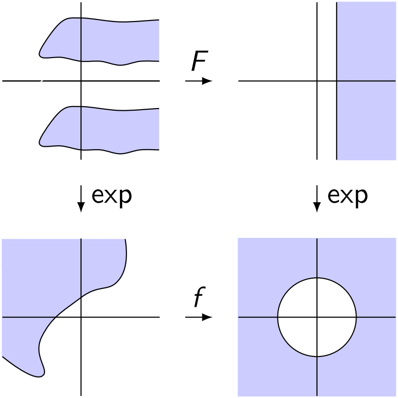
These points also escape fast.

Question: Given a tract D, do there exist points which escape arbitrarily slowly in D?

Waterman 2019: Yes, if $f \in \mathcal{B}$, or if the tract satisfies a certain geometric condition.

Open questions about the escaping set

Why is the hypothesis $f \in \mathcal{B}$ so helpful?



The *logarithmic transform* F is univalent on any component of its domain of definition – so we have good distortion estimates.

Eremenko and Lyubich introduced this tool to dynamics in 1992.

They used it to show that if $f \in \mathcal{B}$, then $I(f) \subset J(f)$.

Open questions about the escaping set

Various questions concern the size of $I(f)$.

Let $\dim_H X$ and $\dim_P X$ denote the Hausdorff and the packing dimension, and $\operatorname{meas} X$ the Lebesgue measure of a set X.

Question: Do we have $\dim_P I(f) = 2$ for every transcendental entire function f?

Rippon, Stallard 2005: Yes, if $f \in \mathcal{B}$.

B. 2012: Yes, if

$$\liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r} = \infty.$$

Here

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

For comparison: The order $\rho(f)$ already considered is given by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Open questions about the escaping set

McMullen 1987: $\dim_H I(\lambda e^z) = 2$ and meas $I(\lambda \cos z + \mu) > 0$.

One idea is that λe^z is large in one half of the plane, and $\lambda \cos z + \mu$ is large in the plane minus a strip around the real axis.

Quite generally, it is plausible that if $|f(z)|$ is large for a large set of z-values, then $I(f)$ is large.

The Theorem of Phragmén-Lindelöf says that if f is bounded outside a sector of opening angle $\alpha \in (0, 2\pi)$, then $\rho(f) \geq \pi/\alpha$.

The Theorem of Denjoy-Carleman-Ahlfors extends this to domains more general than sectors.

Schubert 2007, Barański 2008: If $f \in \mathcal{B}$ and $\rho(f) < \infty$, then $\dim_H I(f) = 2$.

B., Karpińska, Stallard 2009: If $f \in \mathcal{B}$, then $\dim_H I(f) \geq 1 + 1/q$, where

$$q = \limsup_{r \to \infty} \frac{\log \log \log M(r, f)}{\log \log r}.$$

Open questions about the escaping set

Question: Is the hypothesis $f \in \mathcal{B}$ necessary in the above results? In particular, do we have $\dim_H I(f) = 2$ whenever $\rho(f) < \infty$?

For small order perhaps even more could be true:

Question: Do we have $\operatorname{meas} I(f) > 0$ whenever $\rho(f) < \frac{1}{2}$?

If the answer to these questions is negative:

Question: Is there any growth rate which implies that $\operatorname{meas} I(f) > 0$, or at least that $\dim_H I(f) = 2$?

There are some results in this direction for functions which grow sufficiently regular.

Open questions about the escaping set

Periodic components of the Fatou set which are contained in $I(f)$ were named *Baker domains* by Eremenko and Lyubich.

As already mentioned, they showed that if $f \in \mathcal{B}$, then $I(f) \subset J(f)$. In particular, f has no Baker domains.

Bargmann 2001: If f has an invariant Baker domain, then there exists $K > 1$ such that for every large r the annulus $\{r \colon r < |z| < Kr\}$ intersects $\text{sing}(f^{-1})$.

Question: How sparse can the set of singularities of the inverse be for a function with an invariant Baker domain?

Fleischmann 2008: Example with invariant Baker domain, where the singularities of the inverse form a sequence (w_n) with $|w_n| \sim n^2$.

Another question about Baker domains (by Rippon and Stallard):

Question: Let U be a Baker domain. Do we have $\partial U \cap I(f) \neq \emptyset$?

For wandering domains in $I(f)$ they had shown that the answer is positive.

Open questions about the escaping set

Eremenko's conjecture was about the topology of the escaping set.

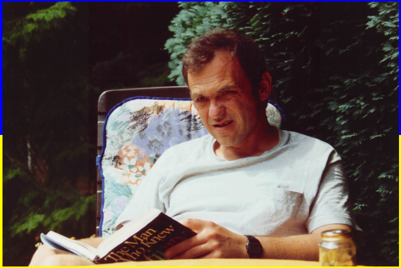
There are also topological questions that remain. The following ones are due to Rippon and Stallard:

Question: If $I(f)$ is disconnected, does it have uncountably many components?

Question: If $I(f)$ is connected, can $\mathbb{C} \setminus I(f)$ contain an unbounded continuum?

Many more open questions about the escaping set can be found (hopefully soon) in a forthcoming survey on the escaping set by Lasse Rempe and me.

Thank you very much



Happy Birthday, Alex!