When the circular dilatation at a point equals one

Classical Function Theory in Modern Mathematics
celebrating 70th birthday of Alexandre Eremenko

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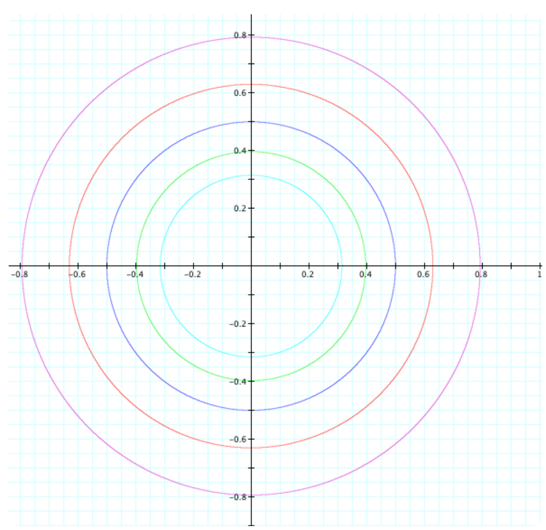
Abstract

We discuss geometric and analytic conditions implying certain local behavior of quasiconformal mappings at a point in the plane where the circular dilatation (a.k.a. linear distortion) equals one, e.g. conformality, $C^{1+\alpha}$ conformality, asymptotic homogeneity, weak conformality, or maximal stretch for the q.c. map at that point. Some results include extensions of the Teichmüller-Wittich–Belinskii theorem. Besides being of interest by themselves, they enjoy applications in Nevanlinna theory, modulus of continuity studies, complex dynamics, the theory of p-integrable Teichmüller spaces, some of which are highlighted.



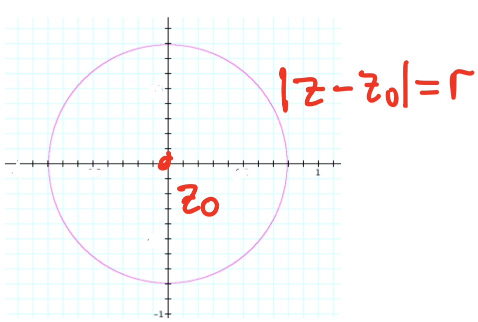
Circular Dilatation of a homeomorphism

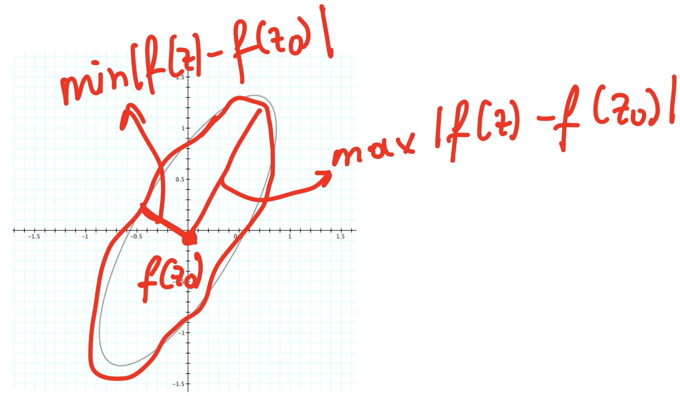
- Let Ω be a domain in $\mathbb{C}$.
- Let $f : \Omega \to \mathbb{C}$ be a homeomorphism.
- Let $z_0 \in \Omega$.
- Let $r > 0$. Then $D(z_0, r) = \{z : |z - z_0| < r\}, \mathbb{D} = D(0, 1)$

The **circular distortion** of the circle $|z - z_0| = r$ under f is

$$H_f(z_0, r) = \frac{\max\limits_{|z-z_0|=r} |f(z) - f(z_0)|}{\min\limits_{|z-z_0|=r} |f(\zeta) - f(z_0)|}$$

The **circular dilatation (a.k.a. linear distortion)** of f at z_0 is





$$H_f(z_0) = \limsup_{r \to 0} \frac{\max\limits_{|z-z_0|=r} |f(z) - f(z_0)|}{\min\limits_{|z-z_0|=r} |f(z) - f(z_0)|}.$$

Definition 1 *(Metric Definition) Let $f : \Omega \to \mathbb{C}$ be an colorblueorientation–preserving homeomorphism. Let $1 \leq K < \infty$. One says that f is K-quasiconformal in Ω if and only*

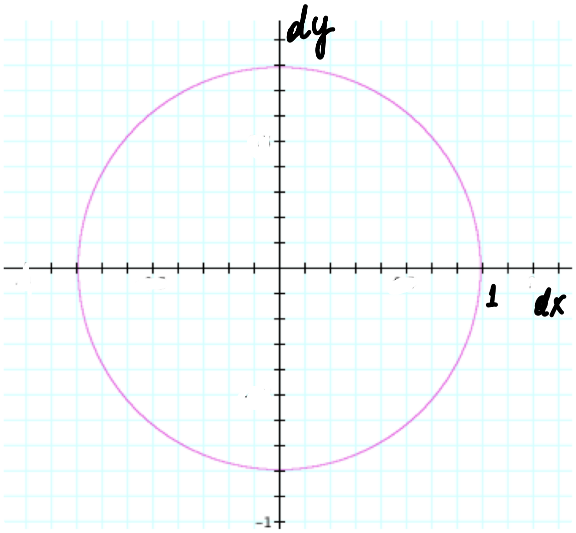
- $\sup\limits_{z \in \Omega} H_f(z) = M < \infty$.
- $H_f(z) \leq K$ *a.e. in* Ω.

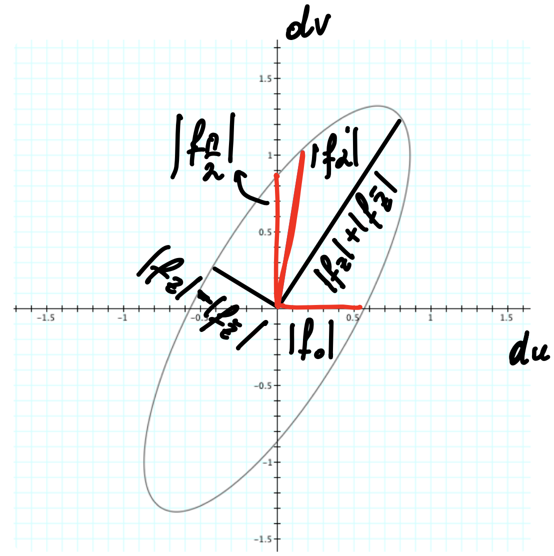
[O. Lehto, K. Virtanen, Springer Verlag, 1973],
[L. Ahlfors, Lectures on quasiconformal mappings, 2006]
[K. Astala, T. Iwaniec, G. Martin, PMS, 2009]
and others.

Qusiconformal Mappings–analytic properties

Let $\Omega \subset \mathbb{C}$ - domain, $f : \Omega \to \mathbb{C}$ K-quasiconformal

- $f_z = \frac{1}{2}(f_x - if_y), \quad f_{\bar z} = \frac{1}{2}(f_x + if_y)$
- f is $ACL \Rightarrow f$ is a.e. differ. $\Rightarrow \Delta f = f_z \Delta z + f_{\bar z}\Delta \bar z + o(|\Delta z|), \Delta z \to 0.$





- the Jacobian is $J_f = |f_z|^2 - |f_{\bar z}|^2 > 0$ a.e.
- the real dilatation is $D_f = \dfrac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \le K$ for a.e. $z \in \Omega$.
- the complex dilatation is $\mu_f = \dfrac{f_{\bar z}}{f_z}$, $\|\mu_f\|_\infty \le \dfrac{K-1}{K+1} < 1$.
- the directional derivative is

$$f_\alpha = f_z + f_{\bar z} e^{-2i\alpha}, \ \ \alpha \in [0, 2\pi)$$

- the directional dilatation is

$$\frac{\min\limits_\alpha |f_\alpha|}{\max\limits_\alpha |f_\alpha|} = \frac{1}{D_f} \le D_{f,\alpha} = \frac{|f_\alpha|^2}{J_f} = \frac{|1 + \mu_f e^{-2i\alpha}|^2}{1 - |\mu_f|^2} \le D_f = \frac{\max\limits_\alpha |f_\alpha|}{\min\limits_\alpha |f_\alpha|}.$$

$$D_f - 1 = \frac{|\mu_f|}{1-|\mu_f|} \sim \frac{|\mu_f|}{1-|\mu_f|^2}$$

$\boxed{z = re^{i\theta}}$

$$D_{f,(\theta+\alpha)} - 1 = \frac{|1+\mu_f e^{-2i(\alpha+\theta)}|^2}{1-|\mu_f|^2} = 2\,\frac{Re(\mu_f e^{-2i(\alpha+\theta)}) + |\mu_f|^2}{1-|\mu_f|^2}$$

- $f \in W^{1,2}_{loc}(\Omega)$.
- At a regular point $z \in \Omega$, $H_f(z) = D_f(z), \Rightarrow H_f = D_f$ a.e. in Ω.

$$D_{f,\alpha} - 1 = 2\frac{Re(\mu_f e^{-2i\alpha}) + |\mu_f|^2}{1-|\mu_f|^2}.$$

- The **Measurable Riemann mapping theorem.** Given a measurable $\mu \in \Omega$, s.t. $\|\mu\|_\infty = k < 1$, there exists a unique (up to normalization) K-quasiconformal mapping, $K = \dfrac{k+1}{k-1}$, satisfying the **Beltrami equation**

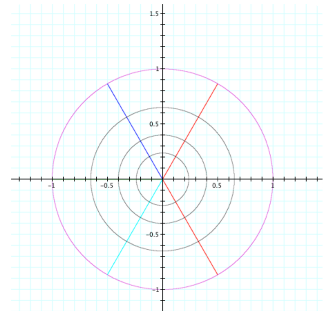
$$f_{\bar{z}} = \mu f_z \quad \text{a.e. in} \quad \Omega.$$

- Let $|\mu(z)| < 1$ be a measurable function in Ω. An ACL sense-preserving homeomorphism $f : \Omega \to \mathbb{C}$ with $J_f > 0$ a.e. that satisfies the Beltrami equation a.e. is called a **μ-homeomorphism.**

- If f is a K-quasiconformal mapping in Ω and $H_f = 1$ a.e. in Ω then f is conformal in Ω.

  Let $z_0 = 0,\ f(0) = 0, \Omega = \mathbb{D}$.

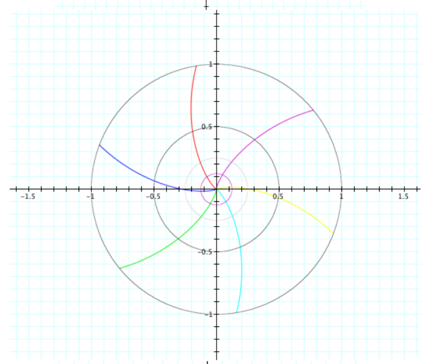
- If $\displaystyle\lim_{z\to 0} \frac{f(z)}{z} = A \neq 0$ then $H_f(0) = 1$.



- $f(z) = z(1 - \log|z|)$.

  $$\mu_f = \frac{1}{1 - 2\log|z|}.$$

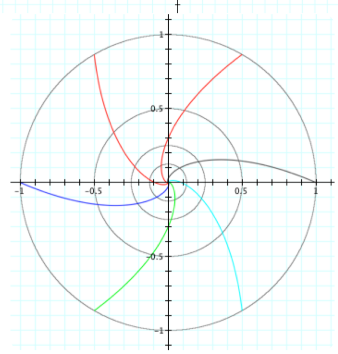
  $H_f(0) = 1$ and $\displaystyle\lim_{z\to 0} \mu_f = 0$.



- $f(z) = z \cdot e^{i \log\log(\frac{e}{|z|})}$.

  $$\mu_f = \frac{1}{1 + 2i\log(e/|z|)}\frac{z}{\bar{z}}.$$

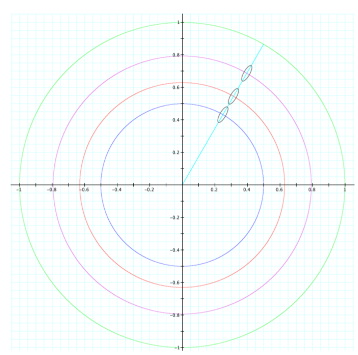
  $H_f(0) = 1$ and $\displaystyle\lim_{z\to 0} \mu_f = 0$.



- $f(z) = ze^{i(-\log|z|)}$.

  $$\mu_f = \frac{z}{\bar{z}(1 + 2i)}.$$

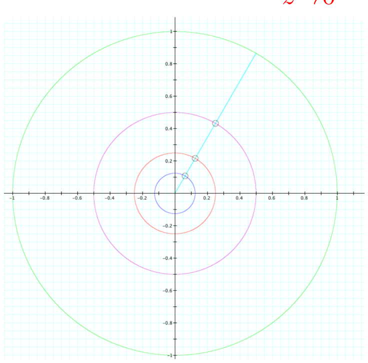
  $H_f(0) = 1$ and $\displaystyle\lim_{z\to 0} \mu_f \neq 0$.



- $K > 1,\ f(z) = z|z|^{\frac{1}{K}-1}$

  $\mu_f(z) = \frac{K-1}{K+1}\frac{z}{\bar{z}}$

  $H_f(0) = 1$ and $\displaystyle\lim_{z\to 0} \mu_f \neq 0$.



- (A. A. Gol'dberg.) Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ be the $n-$th partial sum of the harmonic series. Consider $f(z) = A_f(r)e^{i\theta}$, where

$$A_f(r) = \sqrt{r}e^{-\frac{3}{2}H_n} \qquad e^{-3H_n - \frac{2}{n+1}} \leq r \leq e^{-3H_n},$$

  and

$$A_f(r) = r^2 e^{3H_{n+1}} \qquad e^{-3H_{n+1}} \leq r \leq e^{-3H_n - \frac{2}{n+1}}.$$

  $f(re^{i\theta}) = re^{i\theta}$, when $r = e^{-3H_n}$, $n = 1, 2, ...$

  $\lim_{n\to\infty} \frac{e^{-3H_n}}{e^{-3H_{n+1}}} = 1$, the mapping f is conformal at $z = 0$.

  $D_f(re^{i\theta}) = 2$ *a.e.* and $\mu_f(re^{i\theta}) = -\frac{1}{3}e^{2i\theta}$ *a.e.*, $H_f(0) = 1$.

Teichmüller's Modulsatz

[Teichmüller '38, Deutsche Math.]]
[Alberge, MBT, Papadopoulos, Untersuchungen, translation, HTT, EMS, 2020]
[MBT, Weiss, Untersuchungen, translation, HTT, EMS, 2020]

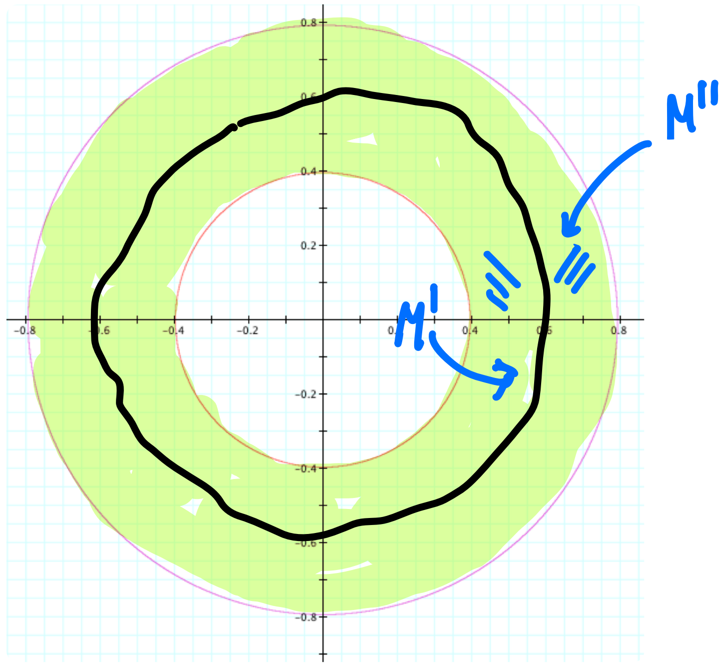
- $0 < r < R$, $\mathfrak{G} = \{z : r < |z| < R\}$ with conformal module $M = \frac{1}{2\pi} \log Rr$.
- $\mathfrak{G}'$ and $\mathfrak{G}''$, ring domains, separating 0 and ∞, inside G, $\mathfrak{G}$ separes 0 from $\mathfrak{G}''$ with conformal modules M' and M''
- $M' + M'' \leq M$.
- $\forall \varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ (independent of R, r $\mathfrak{G}', \mathfrak{G}''$) such that $\delta(\varepsilon) \sim \frac{\varepsilon^2}{\log \frac{1}{\varepsilon}}$, as $\varepsilon \to 0$ and such that

$$M' + M'' \geq M - \delta \implies$$

  the set of points that lie in $\mathfrak{G}$ between $\mathfrak{G}'$ and $\mathfrak{G}''$ belongs to the circular ring

$$\frac{1}{2\pi} \log r + M' - \varepsilon \leq \frac{1}{2\pi} \log |z| \leq \frac{1}{2\pi} \log R - M'' + \varepsilon.$$

....

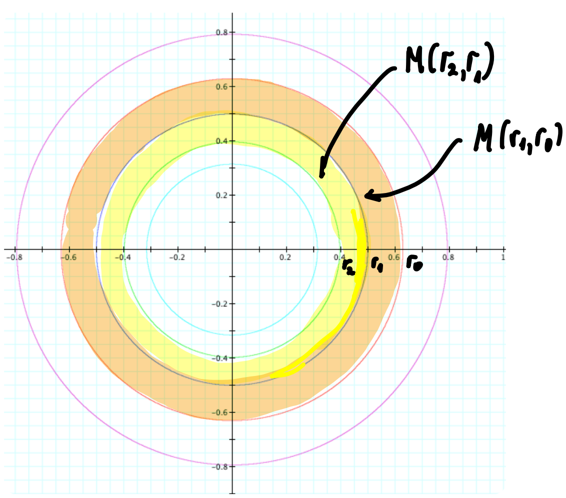


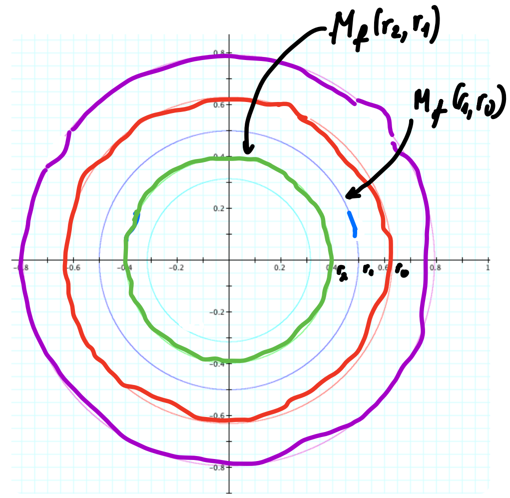
$$\implies \frac{1}{2\pi}\log r + M' - \varepsilon \leq \frac{1}{2\pi}\log|z| \leq \frac{1}{2\pi}\log r + M' + \delta + \varepsilon$$

Sufficient and necessary conditions for the circular dilatation to be equal to one.

Let $f : \mathbb{D} \to \mathbb{D},\ f(0) = 0$.

- Let $0 < r_2 < r_1 < r_0, A(r_2, r_1) = \{z : r_2 < |z| < r_1\}$.
- $M(r_2, r_1), M_f(r_2, r_1)$ are the conformal modules of $A(r_2, r_1)$ and $f(A(r_2, r_1))$, $M(r_2, r_1) = \dfrac{1}{2\pi} \log \dfrac{r_1}{r_2}$.





Theorem 2 *([Teichmüller '38, Deutsche Math.] Sufficient and necessary conditions for* $H_f(0) = 1$.*)*

- $H_f(0, r_1) \to 0,\ r_1 \to 0 \Rightarrow \lim\limits_{r_0 \to 0} \{M_f(r_2, r_0) - M_f(r_2, r_1) - M_f(r_1, r_0)\} = 0$.
- *If* $\forall \delta > 0, \exists X, \forall r_2 < r_1 < r_0 < \dfrac{1}{X},\ M_f(r_2, r_1) > X,\ M_f(r_1, r_0) > X$ *implies* $\{M_f(r_2, r_0) - M_f(r_2, r_1) - M_f(r_1, r_0)\} < \delta \Rightarrow H_f(0, r_1) \to 0,\ r_1 \to 0$.

- Let $C(|z|)$ be a continuous function such that $D_f(z) < C(|z|)$.

$$\frac{1}{2\pi}\int_{r_2}^{r_1}\frac{1}{C(r)}\frac{dr}{r} < M_f(r_2,r_1) < \frac{1}{2\pi}\int_{r_2}^{r_1}C(r)\frac{dr}{r}.$$

$$\implies |M_f(r_2,r_1) - \frac{1}{2\pi}\log\frac{r_1}{r_2}| \leq \frac{1}{2\pi}\int_{r_2}^{r_1}(C(r)-1)\frac{dr}{r}$$

- If $\lim_{r_1\to 0} M_f(r_2,r_1) - \frac{1}{2\pi}\log\frac{r_1}{r_2} = 0 \Rightarrow \exists\alpha\in\mathbb{R}\ \lim_{|z|\to 0}\log|f(z)| - \log|z| = \alpha.$
- If $w = f(z) : \mathbb{D}\to\mathbb{D}, f(0) = 0$ is K-q.c. and

$$\int_0^1 (C(r)-1)\frac{dr}{r}$$

converges $\Rightarrow\ \exists\gamma > 0$ such that $|f(z)| \sim \gamma|z|, z\to 0.$

- **Teichmüller-Wittich theorem**

If If $w = f(z) : \mathbb{D}\to\mathbb{D}, f(0) = 0$ is K-q.c., and

$$\int_{|z|<1}(D_f(z)-1)\frac{A_z}{|z|^2}$$

converges $\implies\ \exists\gamma > 0$ such that $|f(z)| \sim \gamma|z|, z\to 0.$

- $c > 0$, and $\int_0^r (C(r)-1) < const\cdot r^c$ then there exists $\alpha\in\mathbb{R}$

$$|\log|f(z)| - \log|z| - \alpha| \leq const\cdot\frac{|z|^{\frac{c}{c+2}}}{\sqrt[c+2]{\log|z|}}$$

Sufficient conditions for conformality at 0

[Belinskii, L'vov Gos. Univ. Uchen. Zap, 1954]
[O. Lehto, Ann. Acad. Sci. Fenn., 1960]

Theorem 3 *(T. W.-Belinskii-Lehto) Let f be a quasi-conformal mapping in $|z|<1, f(0)=0$. If*

$$\iint\limits_{|z|<1} (D_f(z)-1)\frac{A_z}{|z|^2}<\infty \quad \left(\iint\limits_{|z|<1} \frac{|\mu_f(z)|}{1-|\mu_f(z)|^2}\,\frac{dA_z}{|z|^2}<\infty\right)$$

then f is conformal at $z=0$, namely $\exists\gamma>0$,

$$\lim_{z\to 0}\frac{f(z)}{z}=\gamma.$$

Theorem 4 *([E.. Reich, H. Walczak, Trans. Amer. Math. Soc., 1995]) Suppose that f is K-quasiconformal in $|z|<1,\ f(0)=0, z=re^{i\theta}$. If*

$$\iint\limits_{|z|<1} |D_{f,\theta+\alpha}(z)-1|\,\frac{dA_z}{|z|^2}<\infty \tag{5}$$

for $\alpha=0,\pi/2$ then $\exists\gamma>0$ such that $\lim\limits_{z\to 0}|f(z)|/|z|\to\gamma$ and

$$\arg f(r^{i\theta})=o\left(\sqrt{-\log r}\right), \quad \text{as } r\to 0.$$

In addition, if (5) holds for $\alpha=0,\pi/2,\beta$, $\beta\neq n\cdot\pi/2, n=0,1,2,$ then f is conformal at $z=0$.

Conjecture 6 *(Reich and Walczak)*

Let $L(z):\mathbb{D}\to[1,\infty)$ be any measurable function $\operatorname{ess}\sup\limits_{\mathbb{D}} L(z)<\infty$.

Then there exists a q.c. map $f:\mathbb{D}\to f(\mathbb{D}),\ f(0)=0,$ conformal at 0 with $D_f(z)=L(z)$ a.e. in $\mathbb{D}$.

Theorem 7 *[MBT, Jenkins, Kodai Math. J., 1994]*

Let $f : \mathbb{D} \to f(\mathbb{D}), f(0) = 0$ be μ-homeomorphism, satisfying certain integrability conditions. If $\varphi = \arg z$ and

$$\iint\limits_{\mathbb{D}} \frac{|\mu_f(z)|^2 + |\Re e^{-2i\phi}\mu_f|}{1 - |\mu_f(z)|^2} \frac{dxdy}{|z|^2} < \infty$$

then $|f(z)| \sim A|z|, \; z \to 0, \; A > 0$ and f is a asymptotically rotation on circles, namely

$$\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) - (\theta_2 - \theta_1) = 0, \; r \to 0,$$

uniformly in θ for appropriate choice of the argument.

If $\lim\limits_{r\to 0} \arg f(re^{i\theta_0}) = a$, for some $a \in \mathbb{R}$ and particular value of θ_0, then f is conformal at 0.

Theorem 8 *([Gutlyanskiĭ and Martio, J. Anal. Math, 2003,])*

Let f be a quasi-conformal mapping in $\mathbb{D}$, $f(0) = 0$. If

$$\iint\limits_{\mathbb{D}} \frac{|\mu_f(z)|^2}{|z|^2} dxdy < \infty \tag{9}$$

$$\text{and } \lim_{r\to 0} \iint\limits_{r<|z|<1} \frac{\mu_f(z)}{z^2} dxdy \text{ exists and is finite.}$$

then f is conformal at $z = 0$.

Theorem 10 *([Gutlyanskiĭ and Martio, J. Anal. Math, 2003,]) Let $\mathbb{D}$ be the unit disk. For each measurable function μ satisfying (9) in $\mathbb{D}$, with $\|\mu\|_\infty < 1$, there exists a quasiconformal mapping $F : \mathbb{D} \to \mathbb{D}$ conformal at $z = 0$ and*

$$|\mu_F(z)| = |\mu(z)| \;\; a.e.$$

Theorem 11 *[MBT, Complex Variables and Elliptic Equations, 2010] Let* f *be a* μ*-homeomorphism in* $\mathbb{D}$, $f(0) = 0$. *Let* $0 < r_2 < r_1 < 0, 0 < r < 1$,

- $\lim\limits_{|z| \to 0} |f(z)|/|z| = \gamma \neq 0$ *if and only if*

  $$M(A_f(r_2, r_1)) - M(A(r_2, r_1)) = o(1), \text{ as } r_1 \to 0, \tag{12}$$

  where $\log \dfrac{r_1}{r_2} > K_0$ *for some preliminary fixed* K_0.

- *If* f *satisfies (12) and for* $t > 1$

  $$\lim_{r \to 0} M(Q_f(r, tr, \theta_1, \theta_2)) = \frac{\theta_2 - \theta_1}{\log t}, \tag{13}$$

  uniformly in $\theta_1, \theta_2 \in [0, 2\pi)$ *then* $\exists \gamma > 0$ $\lim\limits_{|z| \to 0} |f(z)|/|z| = \gamma$ *and* f *is asymptotically a rotation on circles.*

- *Let* $\beta > 0$. *If* f *satisfies (12), (13) and*

  $$M(S_f^\beta(r_2, r_1)) - M(S^\beta(r_2, r_1)) \to 0 \text{ as } r_1 \to 0,$$

  then f *is conformal at* $z = 0$.
  Here $S^\beta(r_2, r_1) = \bigcup\limits_{\theta \in [0, 2\pi)} \{z = re^{i(-\beta \log r + \theta)}, r_2 < r < r_1\}$, *and* $S_f^\beta(r_2, r_1)$ *is the image of the family of curves* $S^\beta(r_2, r_1)$ *under* f.

- *If for some fixed* θ, $\lim\limits_{r \to 0} \arg f(re^{i\theta}) - \theta = const$, *then* f *is conformal at the origin if and only if (12), (13) hold.*

Theorem 14 *Let f be a μ-homeomorphism in $\mathbb{D}$, $f(0)=0$. If for $\alpha=0$ and $\alpha=\frac{\pi}{2}$ the limits*

$$\lim_{r\to 0}\iint\limits_{r<|z|<1}\left(D_{f,\theta+\alpha}-1\right)\frac{dA_z}{|z|^2} \tag{15}$$

are finite, then $\exists\gamma>0$ such that $\lim\limits_{|z|\to 0}|f(z)|/|z|=\gamma$ and f is asymptotically a rotation on circles. If (15) exists for $\alpha=0$, $\alpha=\pi/2$ and $\alpha_0\neq\pi/2,\pi,3\pi/2,\alpha_0\in(0,2\pi)$, then f is conformal at $z=0$.

$$\lim_{r\to 0}\iint_{r<|z|<1}\left(D_{f,\theta+\alpha}-1\right)\frac{dA_z}{|z|^2} \iff$$

$$\iint\limits_{\mathbb{D}}\frac{|\mu_f(z)|^2}{|z|^2}dxdy<\infty \text{ and } \lim_{r\to 0}\iint\limits_{r<|z|<1}\frac{\mu_f(z)}{z^2}dxdy \text{ exists and is finite.}$$

Peano derivatives, $C^{1+\alpha}$ conformality and other smoothness results

Theorem 16 *(Shishikura)([Shishikura , Ann. Acad. Sci.. Fenn. Math, 2018])*
Let $f:\mathbb{C}\to\mathbb{C},\ f(0)=0$, *be a quasiconformal mapping and suppose*

$$I(r)=\iint\limits_{|z|<r}\frac{|\mu_f(z)|}{1-|\mu_f(z)|^2}\frac{dA_z}{|z|^2}$$

is finite and has order $O(r^\beta), r\to 0$, *for some* $\beta>0$.

Then for any $0<\alpha<\dfrac{\beta}{2+\beta}$, f *is* $C^{1+\alpha}$ *-conformal at* 0 *in the sense that*

$$f(z)=f'(0)z+O(|z|^{1+\alpha}),\ \text{as } z\to 0.$$

Conjecture 17 *Let* $f:\mathbb{D}\to\mathbb{D},\ f(0)=0$, *be a quasiconformal mapping and suppose for some* $C>0$, $I(r)=\iint\limits_{|z|<r}\dfrac{|\mu_f|}{1-|\mu_f(z)|^2}\dfrac{dA_z}{|z|^2}<Cr^\beta$. *Then*

$$f(z)=f'(0)z+O\left(\frac{|z|^{1+\frac{\beta}{2+\beta}}}{\sqrt[\beta+2]{\log|z|}}\right), z\to 0.$$

and the result is sharp.

Theorem 18 *([I. Nikolaev and S. Shefel, Siberian Math. J, 1986])*
Let $R>0, f$ *be quasiconformal* $D(0,R)$, $|\mu_f(z)|\le C|z|^\beta, \beta>0$ *for almost every* z *in* $\mathbb{D}$.

Then there exists a polynomial P_{n+1} *of degree at most* $n+1$, $0<\beta-n\le 1$, *such that*

$$|f(z)-P_{n+1}(z)|\le\tilde{C}|z|^{\beta+1},$$

where $\tilde{C}$ *depends on* C,β,R *and the diameter of* $f(D(0,R))$.

Approximate continuity

- $\mu \in L^\infty(\Omega)$, $\|\mu(z)\|_\infty < 1$. Let $z_0 \in \Omega$.
- If $\exists E$–measurable $\lim_{\varepsilon \to 0} \dfrac{|E \cap D(z_0, \varepsilon)|}{|D(z_0, \varepsilon)|} = 1$, and $\lim_{z \to z_0, z \in E} \mu(z) = \mu_0$, then $app \lim_{z \to z_0} \mu(z) = \mu_0$.
- $app \lim_{z \to z_0} \mu(z) = \mu_0 \iff \lim_{r \to 0} \frac{1}{r^2} \int_{|z - z_0| < r} |\mu(z) - \mu_0| dA_z = 0.$

A q.c. mapping $f(z) : \Omega \to \mathbb{C}$ such that $f(0) = 0$ is **asymptotically homogeneous at** 0 if for all $\zeta \in \mathbb{C}$ $\lim_{z \to 0} \dfrac{f(\zeta z)}{f(z)} = \zeta$.

1. $\iff$ $\exists A(x) : \mathbb{R} \to \mathbb{C}$, so that $f(z) = A(|z|)(z + o(|z|))$, $z \to 0$, where $\lim_{x \to 0} \dfrac{A(tx)}{A(x)} = 1$, for all $t > 0$.
2. $\implies$ $H_f(0) = 1$.
3. $\implies$ $\lim_{z \to 0} |f(\zeta z)| / |f(z)| = \zeta, \zeta \in \mathbb{C} \setminus 0$.
4. $\implies$ $\lim_{z \to 0} [\arg f(\zeta z) - \arg f(z)] = \arg \zeta$.
5. $\implies$ $\lim_{z \to 0} \log |f(z)| / \log |z| = 1$.

Theorem 19 *([Gutlyanskiĭ and Ryazanov, Izv. Math., 1995])*
Let $f : \mathbb{C} \to \mathbb{C}$, $f(0) = 0$, be a quasiconformal mapping

$$app \lim_{z \to 0} \mu_f(z) = 0,$$

f is asymptotically homogeneous at $z = 0$, and therefore satisfies 1.-5. f is weakly conformal at z=0 (if it satisfies 1. and 3.)

- $app \lim_{z \to 0} \mu_f(z) = 0 \iff \lim_{r \to 0} \dfrac{1}{r^2} \int_{|z| < r} |\mu_f(z)| dA_z = 0.$
- If $\lim_{z \to 0} \mu_f(z) = 0$ (f is asymptotically conformal at 0) then f is asymptotically homogeneous at 0.
- Observation: Let $p > 0$. If $\iint_{|z| < 1} \dfrac{|\mu_f(z)|^p}{|z|^2} dA_z < \infty$ then $app \lim_{z \to 0} \mu_f(z) = 0$.

Theorem 20 *[MBT, 1988]*

Let $f(z) : \mathbb{D} \to \mathbb{D}$ be a K q.c. mapping. The following conditions are equivalent:

1. $H_f(0) = 1$
2. *For a fixed $r_0 < 1$ and $0 < r_2 < r_1 < r_0$,*
$$M(r_2, r_0) - M(r_2, r_1) - M(r_1, r_0) = o(1)$$
, as $r_1 \to 0$.
3. *For a fixed $r_0 > 0$, $r < r_0$, and for some constant $A = A(r_0) > 0$, $\theta \in [0, 2\pi)$*
$$|f(re^{i\theta})| \sim A \exp\{-2\pi M(r, r_0)\}, \ r \to 0.$$

$$\int_{r_2}^{r_1} \frac{dr}{r \int_0^{2\pi} D_{f,\theta+\frac{\pi}{2}} d\theta} \le M_f(r_2, r_1) \le \frac{1}{(2\pi)^2} \iint_{A(r_2,r_1)} D_{f,\theta} \frac{dA_z}{|z|^2}.$$

Theorem 21 *([MBT, Lecture Notes in Math, 1988], L'vov) Let $f : \mathbb{D} \to f(\mathbb{D})$, be a K-q.c. mapping, $f(0) = 0$,*

$$\iint_{|z|<1} \frac{|\mu_f(z)|^2}{|z|^2} dA_z < \infty.$$

Then $H_f(0) = 1$ and for any $0 < r_0 < 1$, $\exists A = A(r_0) > 0$

$$|f(z)| \sim A \exp\left(-\frac{1}{2\pi} \int_{|z|}^{r_0} \int_0^{2\pi} D_{f,\pi/2}(z) \frac{dA_z}{|z|^2} \right), \ |z| \to 0.$$

1. $f(z) = z(1 - \log|z|)$ satisfies $\iint_{|z|<r} \frac{|\mu_f(z)|^2}{|z|^2} dA_z < \infty$ and
2. $\exp\left(-\frac{1}{2\pi} \int_{|z|}^{r_0} \int_0^{2\pi} D_{f,\pi/2}(z) \frac{dA_z}{|z|^2} \right) = A(r_0)|z|(1 - \log|z|)$.
3. $\iint_{|z|<r} \frac{|\mu_f(z)|}{|z|^2} dA_z < \infty \Rightarrow \exp\left(-\frac{1}{2\pi} \int_r^{r_0} \int_0^{2\pi} D_{f,\pi/2} \frac{dA_z}{|z|^2} \right) \sim A|z|, \ |z| \to 0$ for some constant $A > 0$.

$$\frac{4|\mu_f|^2}{1 - |\mu_f|^2} = \frac{|1 + e^{-2i\theta}\mu_f|^2}{1 - |\mu_f|^2} + \frac{|1 - e^{-2i\theta}\mu_f|^2}{1 - |\mu_f|^2} - 2.$$

Applications to the type problem and Nevanlinna theory

- [Drasin and A. Weitsman, Advances inMath., 1975]
- [Drasin, Acta Math., 1977]
- [Drasin, Results in Math., 1986]

Theorem 22 *([Drasin, Results in Math., 1986]) Let $f(z)$ be a quasiconformal mapping in $\mathbb{C}$, $f(\infty) = \infty$, with complex dilatation μ_f such that*

$$\int\limits_{2^n<|z|<2^{n+1}} \frac{|\mu_f|}{|z|^2} dxdy = o(1), \quad n \to \infty. \tag{23}$$

Then as $r \to \infty$

$$\sup_{\frac{1}{2}r<|z|<2r} \left| \frac{f(z)}{f(r)} - \frac{z}{r} \right| = o(1).$$

The proof of this theorem provided in [Drasin, Results in Math., 1986] is remarkably short and concise. In [Drasin and A. Weitsman, Advances inMath., 1975] one can find a more elaborate proof.

An immediate consequence of this theorem is that the circular dilatation is one at infinity.

- ”The theorem seems to be more natural and convenient for the study of deficient values than the original one of Teichmüller–Wittich. It enables us to work with less precise estimates on the dilatation and still retain sufficient precision for computing those quantities relevant to Nevanlinna's theory.
- This generalization of Teichmüller-Wittich theorem has played a key role in [Drasin and A. Weitsman, Advances inMath., 1975] and in the solution to the inverse problem for meromorphic functions [Drasin, Acta Math., 1977] in using quasiconformal deformation.

Theorem 24 *Let $f(z)$ be a q.c. map $\mathbb{C}$, $f(0)=0$, with dilatation μ_f.*

$$\int\limits_{2^{-(n+1)}<|z|<2^{-n}} \frac{|\mu_f|}{|z|^2} dxdy = o(1), \quad n \to \infty \tag{25}$$

if and only if app $\lim\limits_{z\to 0} \mu(z) = 0$.

Proof. Let $t_n = 2^{-n}$.

$$\left(\int\limits_{t_{n+1}<|z|<t_n} \frac{|\mu_f|}{|z|^2} dxdy \right)^2 \leq \left(\int\limits_{t_{n+1}<|z|<t_n} \frac{1}{|z|^4} dxdy \right) \cdot \left(\int\limits_{t_{n+1}<|z|<t_n} |\mu_f|^2 dxdy \right)$$

$$\leq 2\pi \int\limits_{t_{n+1}}^{t_n} \frac{1}{r^3} dr \cdot \int\limits_{t_{n+1}<|z|<t_n} |\mu_f| dxdy$$

$$\leq \pi(2^{2(n+1)} - 2^{2n}) \cdot \int\limits_{t_{n+1}<|z|<t_n} |\mu_f| dxdy \leq \frac{3\pi}{t_n^2} \int\limits_{|z|<t_n} |\mu_f| dxdy.$$

$$\lim_{n\to\infty} \frac{1}{t_n^2} \int\limits_{|z|<t_n} |\mu(z)| dA_z = 0 \implies \lim_{n\to\infty} \int\limits_{t_{n+1}<|z|<t_n} \frac{|\mu_f|}{|z|^2} dxdy = 0. \tag{26}$$

Let $\varepsilon > 0$. $\exists n > 0$ such that $\int\limits_{t_{n+k+1}<|z|<t_{n+k}} \frac{|\mu_f|}{|z|^2} dxdy < \varepsilon, k = 0, 1, 2, \cdots$

$$\sum_{k=0}^{\infty} \int\limits_{t_{n+k+1}<|z|<t_{n+k}} |\mu_f| dxdy < \sum_{k=0}^{\infty} \varepsilon t_{n+k}^2 = \varepsilon \sum_{k=0}^{\infty} \frac{1}{2^{2(n+k)}} = \frac{4\varepsilon}{3 \cdot 2^{2n}}$$

$$\frac{1}{t_n^2} \int\limits_{|z|<t_n} |\mu(z)| dA_z < \frac{4}{3}\varepsilon,$$

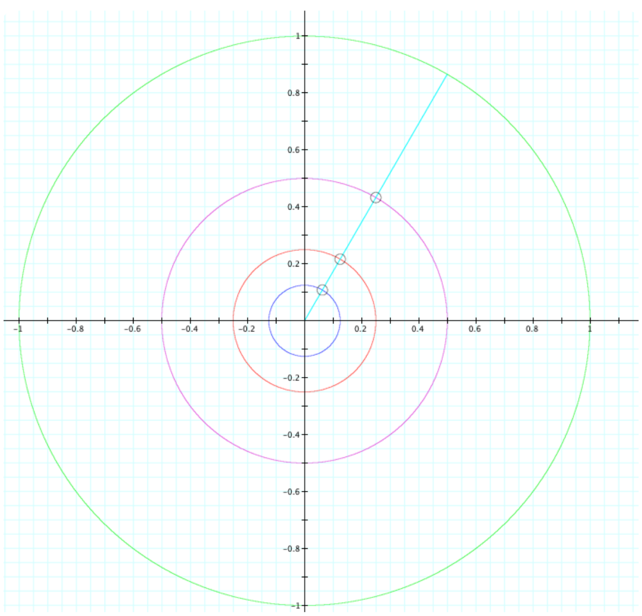
$$\lim_{r\to 0} \frac{1}{r^2} \int\limits_{|z|<r} |\mu(z)| dA_z = 0 \iff \lim_{n\to\infty} \int\limits_{t_{n+1}<|z|<t_n} \frac{|\mu_f|}{|z|^2} dxdy = 0. \tag{27}$$

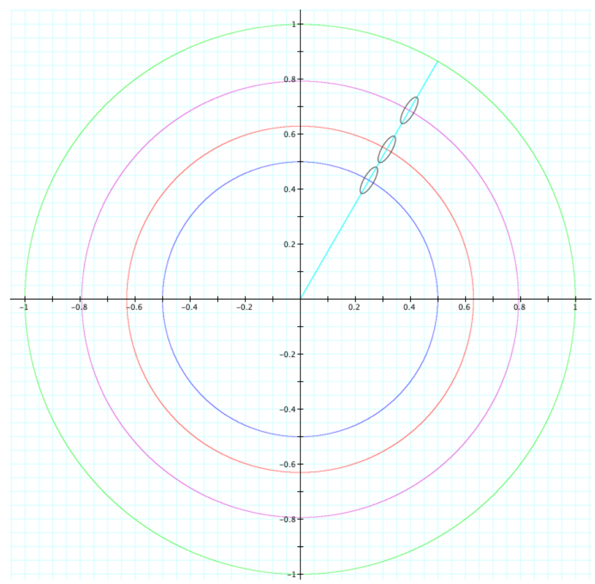
Maximal modulus of continuity and $F(z) = Az|z|^{1/K-1}, A \neq 0$

A K-q.c/quasiregular mapping $f : \Omega \to \mathbb{C}$, where $\Omega \in \mathbb{C}$ is locally Hölder continuous with exponent $\frac{1}{K}$. The exponent $\frac{1}{K}$ is the best possible, since

$$f_K(z) = z|z|^{\frac{1}{K}-1}$$

is $\dfrac{1}{K}$ Hölder continuous at $z = 0$.





- $H_{f_K}(0) = 1$ and $D_{f_K}(0)$ does not exist.
- $z_0 \neq 0, D_{f_K}(z_0) = K,\ H_{f,K}(z_0) = K.$

Theorem 28 *[MBT, Jenkins, POMI, 1997] Let $f : \mathbb{D} \to f(\mathbb{D})$ be an ACL homeomorphism, satisfying additional integrability conditions. Let $\varphi = \arg z$. If there exists $K > 0$, such that for $\theta = \arg z$*

$$\iint_{\mathbb{D}} \left| \frac{1 + |\mu_f|^2}{1 - |\mu_f(z)|^2} - \left(K + \frac{1}{K} \right) \right| \frac{dxdy}{|z|^2} < \infty, \quad \iint_{\mathbb{D}} \left| \frac{2e^{-2i\theta}\mu_f}{1 - |\mu_f|^2} - \left(K - \frac{1}{K} \right) \right| \frac{dxdy}{|z|^2} < \infty,$$

then

$$|f(z)| \sim A|z|^{\frac{1}{K}-1} z, z \to 0,$$

uniformly in $\arg z$.

Let f ne a quasiregular/quasiconformal mapping in Ω, $z_0 \in \Omega$. The **local modulus of continuity** for $z_0 \in \Omega$ and small $\delta > 0$

$$\omega_{f,\delta}(z_0) = \max\{|f(z) - f(z_0)| : z \in \Omega, |z - z_0| \leq \delta, \}$$

Since $\omega_{f,\delta}(z_0) \leq A\delta^{1/K}, A > 0$, we have $\omega_f(z_0) = \limsup\limits_{\delta \to 0} \dfrac{\omega_{f,\delta}(z_0)}{\delta^{\frac{1}{K}}} < \infty$.

If $\omega_f(z_0) > 0$ then z_0 is called *a point of maximal stretch.*

Theorem 29 *([Kovalev, Ann.Acad. Sci. Fenn., 2004] Let $f : \Omega \to \mathbb{C}$ be a K-quasiregular mapping, $z_0 \in \Omega$. If $\omega_f(z_0) > 0$ then f is injective in a neighborhood of z_0 and there exist a continuous function $\theta : (0, 1) \to \mathbb{R}$ such that*

$$\lim_{z \to z_0} \left\{ \frac{f(z) - f(z_0)}{|z - z_0|^{1/K-1}(z - z_0)} - \omega_f(z_0)e^{i\theta(|z-z_0|)} \right\} = 0. \tag{30}$$

Theorem 31 *([Kovalev, Ann.Acad. Sci. Fenn., 2004]) Let $f : \Omega \to \mathbb{C}$ be a K-quasiconformal mapping. For any $\Omega_1 \subset \Omega$, there exist a constant $C = C(\Omega_1, \Omega_1)$ such that for $z_1, z_2 \in \Omega_1$*

$$|f(z_1) - f(z_2)| \geq C\sqrt{\omega_f(z_1)\omega_f(z_2)}|z_1 - z_2|^{\frac{1}{K}}.$$

Boundary correspondence

A quasiconformal mapping of a Jordan domain onto a Jordan domain can be extended to the closure as a homeomorphism. To study boundary behavior of q.c. maps one often considers quasiconformal authomorphisms of the upper half plane

- $\mathbb{H} = \{z = x + iy : y > 0\}$, $\mathbb{D} = \{z = x + iy : |z| < 1\}$
- $f : \mathbb{H} \to \mathbb{H}$ is a K-quasiconformal automorphism of $\mathbb{H}$, $f(\infty) = \infty$.
- The induced boundary correspondence is also denoted by f and so is its extension by reflection to the lower half-plane $\mathbb{H}^*$.
- For $x, y \in \mathbb{R},\ t > 0$, the interval distortion $h_f(x,t) = \dfrac{f(x+t) - f(x)}{f(x) - f(x-t)}$, of the interval $|x - y| \le t$ can be viewed as the trace on $\mathbb{R}$ of the circular distortion $H_f(x,t), x \in \mathbb{R}$.

Theorem 32 *There exists a number M such that on $\mathbb{R}$*

$$\frac{1}{M} \le \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \le M \tag{33}$$

Moreover, if $f : \mathbb{R} \to \mathbb{R}, f(\infty) = \infty$ is a homeomorphism that satisfies the above inequality, then its extention to $f : \mathbb{H} \to \mathbb{H}$

$$f(x+iy) = \frac{1}{2}\int_0^1 (f(x+ty) + f(x-ty))\,dt + i\int_0^1 (f(x+ty) - f(x-ty))\,dt.$$

is a K-q.c. such that $K = K(M)$ and $K \to 1$, as $M \to 1$.

The second part of the theorem is due to [Beurling, Ahlfors, Acta Math., 1956]. They showed that quasisymmetric maps could be purely singular.

In [Carleson, J. Analyse Math., 1967], was the first to obtain sufficient conditions on the growth of the real dilatation of a quasiconformal map defined on the upper-half plane near the real line that imply differentiability of its boundary extension or absolute continuity of the boundary extension so that its generalized derivatives are in $L^2_{loc}(\mathbb{R})$.

Further improvements or related results:

[J. Anderson, J. Becker, M. D. Lesley, '88, J. London Math soc.],
[Becker, Complex. Var. 1987],
[J. Anderson, A. Hinkkanen, '94, Bull. Lond. Math. Soc.],
[Gutlyanskiĭ and Ryazanov, Ann.Acad. Sci. Fenn, 1996],
[MBT, J. Jenkins, Ann. Acad. Sci. Fenn, 2002]
and many others.

Quasisymmetric and symmetric maps on the real line and the unit circle.

A homeomorphism $f : \mathbb{R} \to \mathbb{R}, f(\infty) = \infty$ that satisfies

$$\frac{1}{M} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq M \tag{34}$$

$\forall x \in \mathbb{R}, t > 0$, is called **quasisymmetric** on $\mathbb{R}$.

A homeomorphism $f : \mathbb{R} \to \mathbb{R}, f(\infty) = \infty$ that satisfies.

$$\lim_{t \to 0} \frac{f(x+t) - f(x)}{f(x) - f(x-t)} = 1, \tag{35}$$

uniformly in x is called **symmetric** on $\mathbb{R}$.

$G(z) = \exp(2\pi i z)$ or $H(z) = i\dfrac{z-1}{z+1}$ map $\mathbb{H}$ to $\mathbb{D}$.

Let $f : \mathbb{S} \to \mathbb{S}$ be orientation–preserving homeomorphism and $\mathbb{S} = \{z : |z| = 1.\}$

$f : \mathbb{S} \to \mathbb{S}$ is **quasisymmetric** if $\exists M > 0$ such that $\forall \theta \in \mathbb{R}, \forall t > 0$:

$$\frac{1}{M} \leq \left| \frac{f(e^{i(\theta+t)}) - f(e^{i\theta})}{f(e^{i\theta}) - f(e^{i(\theta-t)})} \right| \leq M.$$

$f : \mathbb{S}^1 \to \mathbb{S}^1$ is **symmetric** on $\mathbb{S}$ if

$$\frac{f(e^{i(\theta+t)}) - f(e^{i\theta})}{f(e^{i\theta}) - f(e^{i(\theta-t)})} \underset{t \to 0^+}{\longrightarrow} 1,$$

uniformly on $\mathbb{R}$.

The universal Teichmüller space $\mathcal{T}$ and some of its subspaces.

The **universal Teichmüller space** $\mathcal{T}(\mathbb{D})$ is the space of *quasisymmetric* homeomorphisms of the unit circle $\mathbb{S}$ fixing 1, i, and -1.

Two quasiconformal automorphisms f, g of $\mathbb{D}$ are said to be equivalent iff $f|_{\mathbb{S}} = g|_{\mathbb{S}}$. Denote by $[f]$ the equivalent class of f.

$$\mathcal{T}(\mathbb{D}) = \{[f]\,, f \text{ is a quasiconformal automorphism of } \mathbb{D}\}.$$

The Teichmüller metric $d_{\mathcal{T}(\mathbb{D})}\left([f]), [g]\right) = \dfrac{1}{2} \inf\limits_{f\in[f], g\in[g]} \log K(g \circ f^{-1})$.

Here $K(g \circ f^{-1})$ is the maximal dilatation.

One can also define the Kobayashi metric on $\mathcal{T}.(\mathbb{D})$

The space $\mathcal{T}(\mathbb{D})$ was introduced by Bers [Bers, Anal. Meth.s in Math. Phys., 1965], via the Bers embedding. It embeds bi-holomorphically in a bounded subspace of the space of holomorphic quadratic differential of $\mathbb{D}^* = \{z : |z| > 1.\}$ equipped with the Bers norm and has Bers complex manifold structure.

- Let $g_i \in [f]\,, i = 1, 2$. Let μ_{g_i} be its complex dilatations in $\mathbb{D}$.
- Let G_i be the solution to the Beltrami equation in $\mathbb{C}$ so that $\mu_{G_i} = \mu_{g_i}$ on $\mathbb{D}$ and $\mu_{G_i} = 0$ in $\mathbb{D}^*$ so that G_i fixes ∞ and its derivative is 1 at ∞.
- Then $G_1|_{\mathbb{D}^*} = G_2|_{\mathbb{D}^*}$.
- Let S_{G_i} be the Schwarzian derivative of $G_i, i = 1, 2$
- The Bers embedding is $\beta([f]) = S_{G_i}$ since for $g_i \in [f]\,, i = 1, 2, \quad S_{G_1} = S_{G_2}$. Here S_{G_i} is a holomorphic function in a Banach space of holomorphic quadratic differentials Φ in $\mathbb{D}^*$, with the Bers norm

$$\|\Phi\|_\beta = \sup_{z\in\mathbb{D}^*} |\Phi(z)|(|z|^2 - 1)^2.$$

The *symmetric Teichmüller space* $\mathcal{T}_s(\mathbb{D})$,

[Gardiner, Sullivan, Amer. J. Math., 1992] is the space of normalized symmetric maps.

- A q.c. map $f : \mathbb{D} \to \mathbb{D}$ is said to be asymptotically conformal if for every $\epsilon > 0$, there exists a compact subset K_ϵ of $\mathbb{D}$ such that for any $z \in \mathbb{D} \setminus K_\epsilon$, $|\mu_F(z)| < \epsilon$.
- The space $\mathcal{T}_0(\mathbb{D})$ of normalized quasisymmetric maps which have an asymptotically conformal extension is the little Teichmüller space $\mathcal{T}_o$.
- Gardiner and Sullivan showed that $\mathcal{T}_s(\mathbb{D}) = \mathcal{T}_o(\mathbb{D})$, and that $\mathcal{T}_s(\mathbb{D})$ has complex Banach manifold structure.
- Later it was shown that the Kobayashi and Teichmüller metric coincide on $\mathcal{T}_s$.
- One can introduce in a similar way $\mathcal{T}_0(\mathbb{R})$ and $\mathcal{T}_s(\mathbb{R})$, however a special care here has to be taken with regard to the point at ∞ and if one does not include it into consideration, one has $\mathcal{T}_0(\mathcal{R}) \subsetneq \mathcal{T}_s(\mathcal{R})$.
- The space $C^{1+H} = \bigcup_{0<\alpha\leq 1} C^{1+\alpha} studiedin$[Jiang, Acta Math. Sinica, 2020], has complex Banach manifold structure and moreover that the Kobayashi and Teichmüller mertics coincide on it.
- For further references about the universal Teichmüller space and its many subspaces, we refer to

  [O. Lehto, Univ. functions and Teichm. spaces, Springer, 87],

  [Fletcher, Markovic, Q. Maps and Teichm. theory, Oxford U. Press, 07] and references therein.

The p-integrable Teichmüller spaces, $p > 0$.

$\mathcal{T}^p(\mathbb{D}) = \left\{ f \in \mathcal{T} \mid \exists F : \mathbb{D} \to \mathbb{D}, \text{ q.c. such that } F_{|\mathbb{S}} = f \text{ and } |\mu_F| \in \mathrm{L}^p(\mathbb{D}, \sigma) \right\}$,

$d\sigma(z) = (1 - |z|^2)^{-2} dxdy$

$$\iint_{\mathbb{D}} \frac{|\mu_F(z)|^p}{(1-|z|^2)^2} dxdy < \infty.$$

- If $0 < p < q$, then $\mathcal{T}^p(\mathbb{D}) \subset \mathcal{T}^q(\mathbb{D})$.
- $\mathcal{T}^p(\mathbb{D})$, $p \geq 2$, was introduced in [Guo, Sci. China, 2000] through the equivalent description of holomorphic functions. The complex Banach manifold structure for $\mathcal{T}^p(\mathbb{D})$ was provided by Yanagishita including the case with the Fuchsian group action.
- $\mathcal{T}^2(\mathbb{D})$ was introduced in [Cui '00, Sci. China, 2000], who gave a few important characterizations of the elements of $\mathcal{T}^2(\mathbb{D})$.

  In particular, he proved that the Beltrami coefficient associated with the *Douady–Earle extension* (see [Douady, Earle, Acta Math.1986]) of any element of $\mathcal{T}^2(\mathbb{D})$ belongs to $\mathrm{L}^2(\mathbb{D}, \sigma)$ and that $\mathcal{T}^2(\mathbb{D})$ is a symmetric space.
- [L. Takhtajan, L.P. Teo, Memoirs of the Amer. Math. Soc., 2006] introduced a Hilbert manifold structure on $\mathcal{T}(\mathbb{D})$ which makes $\mathcal{T}^2(\mathbb{D})$, the connected component of the identity map. $\mathcal{T}^2(\mathbb{D})$, is now known as the *Weil–Petersson Teichmüller space*.
- [Bishop, Curves, beta-numbers, and minimal surfaces, 2020] introduced (more than 25) equivalent definitions of the Weil-Petersson Teichmüller space, showing connections between and applications in fields such as geometric function theory, SLE, harmonic analysis and geometric measure theory, knot energies, bi-Lipschitz involutions, Menger curvature, minimal surfaces, while taking on an innovative geometric viewpoint to many of them.
- For further studies of $\mathcal{T}^p, p > 1$, we refer to also to [Y. Shen, Amer. J. Math., 2018] and [Wei, Matzusaki '23, Proc. Japan Acad].

Intrinsic characterization of the $\mathcal{T}^p$-spaces, $p > 0$.

It was proposed by Tahtajan and Teo that the Weil–Petersson class could be characterized in an intrinsic way, without using its extensions or its embedding.

Theorem 36 *(Shen, [Y. Shen, Amer. J. Math., 2018]) A sense-preserving homeomorphism h on the unit circle belongs to the Weil–Petersson class if and only if h is absolutely continuous (with respect to the arc-length measure) and such that $\log h'$ belongs to the Sobolev class $H^{1/2}$.*

Later this characterization was presented in the language of Besov spaces.

One says that a locally integrable, complex-valued function u on $\mathbb{S}$ belongs to the p-Besov spave $B_p(\mathbb{S})$ if, for $p > 1$

$$\|u\|_{B_p} = \left(\int\limits_{\mathbb{S}} \int\limits_{\mathbb{S}} \frac{|u(s) - u(t)|^p}{|s - t|^2} |ds||dt| \right)^{\frac{1}{p}} < \infty.$$

Theorem 37 *([X. Liu, Y. Shen, Math. Zeit., 1922]) A sense-preserving homeomorphism on the unit circle belongs to the class $T^p(\mathbb{D}), p \geq 2$ if and only if it is absolutely continuous (with respect to the arc-length measure) and $\log h'$ belongs to the Besov class $B^p(\mathbb{S})$.*

In [Wei, Matzusaki '23, Proc. Japan Acad] the authors show, using different methods, that the result extends to $\mathcal{T}^p, p > 1$.

$\mathcal{T}^p$ on $\mathbb{R}$

Let $f : \mathbb{H} \to \mathbb{H}$ be an K-q.c. automorphism of $\mathbb{H}$, fixing the boundary points $0, 1, \infty$. We say that the Beltrami coefficient μ_f is p-integrable if

$$\iint\limits_{\mathbb{H}} \frac{|\mu_f(z)|^p}{y^2} dxdy < \infty$$

Let $\mathcal{T}(\mathbb{R})$ denote the set of all quasisymmetric maps on $\mathbb{R}$ fixing $0, 1, \infty$.

Let $\mathcal{T}_s(\mathbb{R})$ denote the set of all symmetric maps on $\mathbb{R}$. Such maps have a q.c. extension F such that $\mu_F(z) \to 0, y \to 0$, uniformly in x.

Let $\mathcal{T}_0(\mathbb{R})$ be the elements of $\mathcal{T}$ with extension F in $\mathbb{H}$ such that $\mu_F \to 0$ as $z \to \infty$.

Define $\mathcal{T}^p(\mathbb{R}) = \{f \in QS(\mathbb{R}) : f$ has a p-integrable q.c. extension in $\mathbb{H}.\}$

Lemma 38 *Let $f \in T^p(\mathbb{R})$ and let f be its extension to $\mathbb{H}$ with $p-$integrable complex dilatation. Denote again by f the extension of f to the lower half-plane by reflexion. If uniformly in $\xi \in \mathbb{R}, z \in \mathbb{C}$, $H_f(\xi) = 1$, namely*

$$\lim_{r\to 0} \frac{\max\limits_{|z-x|=r} |f(z) - f(\xi)|}{\min\limits_{|z-x|=r} |f(z) - f(\xi)|} = 1.$$

then $f : \mathbb{R} \to \mathbb{R}$ is symmetric.

Let $t > 0$.

$$\frac{\min\limits_{|z-\xi|=t} |f(z) - f(\xi)|}{\max\limits_{|z-x|=r} |f(z) - f(\xi)|} \leq \frac{f(\xi + t) - f(\xi)}{f(\xi) - f(\xi - t)} \leq \frac{\max\limits_{|z-\xi|=t} |f(z) - f(\xi)|}{\min\limits_{|z-\xi|=t} |f(z) - f(\xi)|}.$$

Theorem 39 *[MBT, Workshop on Grothendieck–Teichmüller Theories (2016), 2022] We have $\mathcal{T}^2(\mathbb{R}) \subset \mathcal{T}_s(\mathbb{R})$ and $\mathcal{T}^2(\mathbb{R}) \subset \mathcal{T}_0(\mathbb{R})$.*

In addition, $\mathcal{T}^2(\mathbb{D}) \subset \mathcal{T}_s(\mathbb{D})$.

The proof is based on conformal module techniques and *Teichmüller's Modulsatz.*

Theorem 40 *[MBT, Anal. Math. Phys., 2018] Let $p>0$. We have $\mathcal{T}^p(\mathbb{R}) \subset \mathcal{T}_0(\mathbb{R})$ and $\mathcal{T}^p(\mathbb{R}) \subset \mathcal{T}_s(\mathbb{R})$. In addition $\mathcal{T}^p(\mathbb{S}) \subset \mathcal{T}_0(\mathbb{S})$.*

- **Proposition 41** *([B. Bojarski et. al. Infinitesimal Geometry, EMS 2013]) Let $\mathcal{M} \in \mathbb{C}$ be a set, $f:\mathbb{C}\to\mathbb{C}$ is such that its complex dilatation μ_f satisfies*
$$\lim_{r\to 0} \frac{1}{r^2} \iint_{|z|<r} |\mu_f(z+\xi)|dxdy = 0$$
uniformly in $\xi \in \mathcal{M}$. Then f is uniformly weakly conformal in $\mathcal{M}$.
- Let $f \in \mathcal{T}^p(\mathbb{R})$. Denote by f the extension of f with p-integrable μ_f in $\mathbb{H}$ and its reflection to the lower half-plane.
- Let $\varepsilon > 0$.
- $\iint\limits_{\mathbb{H}} \frac{|\mu_f(z)|^p}{y^2} dxdy < \infty \implies \exists$ compact K such that
$$\iint\limits_{\mathbb{H}\setminus K} \frac{|\mu_F(z)|^p}{y^2} dxdy < \left(\frac{\varepsilon}{C_q}\right)^p, \text{ where } C_q = \left(\frac{\pi}{q}\right)^{\frac{1}{q}}.$$
- Let $r_\varepsilon = d(K,\mathbb{R}), \xi \in \mathbb{R}$ and $r < r_\varepsilon$.
$$\frac{1}{r^2}\iint_{|z|<r} |\mu_f(\xi+z)|dxdy \leq$$
$$\frac{1}{r^2}\left(\iint_{|z|<r} |z|^{2q/p} dxdy\right)^{\frac{1}{q}} \left(\iint_{|z|<r} \frac{|\mu_f(\xi+z)|^p}{|z|^2} dxdy\right)^{\frac{1}{p}}$$
$$\leq \frac{1}{r^2}\left(\frac{\pi}{q}\right)^{\frac{1}{q}} \cdot r^2 \left(\iint_{D(\xi,r)} \frac{|\mu_f(z)|^p}{|z-\xi|^2} dxdy\right)^{\frac{1}{p}}$$
$$\leq \left(\frac{\pi}{q}\right)^{\frac{1}{q}} \left(\iint_{\mathbb{H}\setminus K} \frac{|\mu_f(z)|^p}{y^2} dxdy\right)^{\frac{1}{p}} < \varepsilon.$$
- Uniformly in $\zeta \in \mathbb{R}$, $\lim_{r\to 0} \frac{\max_{|z-\xi|=r} |f(z)-f(\xi)|}{\min_{|z-\xi|=r} |f(z)-f(\xi)|} = 1.$
- $f|_{\mathbb{R}}$ is symmetric.

Smoothness of $\mathcal{T}^p(\mathbb{S})$

Theorem 42 *([V. Alberge, MBT., 2021]) Let* $0 < p \leq 1$. *Every element of* $\mathcal{T}^p(\mathbb{D})$ *is a* $\mathcal{C}^1$*-diffeomorphism.*

- Let $f \in \mathcal{T}^1(\mathbb{D})$
- $\exists$ an extension of f to $\mathbb{D}$ such that

  $$\iint_{\mathbb{D}} \frac{|\mu_f|}{(1-|z|^2)^2} dxdy$$

  is convergent. Using reflection over $\mathbb{S}$ we extend further f to the outside of the disk.
- Let $0 < r < 1$ be sufficiently small.
- $\iint_{D(\xi_0,r)} \frac{|\mu_f(z)|}{|z-\xi_0|^2} dxdy$ is convergent, uniformly in $\xi_0 \in \mathbb{S}$.
- By T.W. Belinskii-Lehto theorem $f'(\xi_0)$ exists and is not equal to 0.
- One uses the reduced module to show that $f'(\xi)$ is continuous on $\mathbb{D}$.

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