Classical Function Theory in Modern Mathematics
In Honor of Alexandre Eremenko's 70th Birthday

Brody holomorphic curves on the degree six Fermat surface

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Two-term Fermat functional equations

One of the most famous problems in number theory is the Fermat's Last Theorem which says that there is no natural numbers x, y and z such

$$x^n + y^n = z^n \quad (1)$$

for any natural number n greater than 2.

- The problem was eventually solved by Andrew Wiles (1995).
- The corresponding problem in one complex variable function theory is whether the equation (1) has entire function solutions (x, y, z).

Two-term Fermat functional equations

This is equivalent to asking if the following functional equation has non-constant meromorphic solutions f and g on the complex plane $\mathbb{C}$:

$$f^n + g^n = 1 \tag{2}$$

- It was proved by Iyer (1939) that (2) has no non-constant entire solutions when $n > 2$ and when $n = 2$, all entire solutions are of the form $f(z) = \cos(\alpha(z))$ and $g = \sin(\alpha(z))$, where α is a non-constant entire function.

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- Gross (1966) showed that (2) has no non-constant meromorphic solutions when $n > 3$ and when $n = 2$, all the meromorphic solutions are of the form

$$f(z) = \frac{2\beta(z)}{1+\beta(z)^2}, \quad g(z) = \frac{1-\beta(z)^2}{1+\beta(z)^2},$$

where β is a meromorphic function.

$f^n + g^n = 1$

For $n = 3$, Baker (1966) showed that all meromorphic solutions are of the form

$$f(z) = F(\alpha(z)) \quad \text{and} g = cG(\alpha(z))$$

where α is an entire function, F and G are the elliptic functions

$$\frac{1 + 3^{-1/2}\wp'(z)}{2\wp(z)} \quad \text{and} \quad \frac{1 - 3^{-1/2}\wp'(z)}{2\wp(z)}$$

respectively.

Here c is a cubic root of unity and $\wp$ is the Weierstrass $\wp$ function.

Three-term Fermat functional equations

We consider the three-term Fermat functional equations,

$$f^n + g^n + h^n = 1, \tag{3}$$

where n is an integer and f, g, h are functions on $\mathbb{C}$. For each integer n, one can ask whether there are **non-constant** solutions (**non-trivial solution**) to (1) that are

(a) meromorphic;

(b) rational;

(c) entire; or

(d) polynomial.

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(a) meromorphic;
(b) rational;
(c) entire; or
(d) polynomial.

Trivial solutions: Solutions of the form $(f(t), \omega_1 f(t), \omega_2)$ or by permutation of the indices, where $\omega_1, \omega_2 \in \mathbb{C}$ such that $\omega_1^n = -1$ and $\omega_2^n = 1$.

$f^n + g^n + h^n = 1$

The results to date regarding non-constant solutions to three-term Fermat functional equations:

Requirement	Exist	Don't exist	Unknown
Meromorphic	$n \leq 6$	$n \geq 9$	$n = 7, 8$
Rational	$n \leq 5$	$n \geq 8$	$n = 6, 7$
Entire	$n \leq 5$	$n \geq 7$	$n = 6$
Polynomial	$n \leq 3$	$n \geq 6$	$n = 4, 5$

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- Non-existence proofs regarding meromorphic or rational solutions employ H. Cartan's version of Nevanlinna theory (1933) by Hayman in 1985.
- Actually, Toda (1971) also proved a more general results for the entire case and Fujimoto (1974) proved the meromorphic case for meromorphic maps on $\mathbb{C}^k$.
- Examples of transcendental meromorphic and entire solutions were constructed by Green, Gross, Gundersen, Reznick, Tohge, etc.

$f^n + g^n + h^n = 1$

Examples of entire solutions also exist for $n \leq 5$. They are given as follows where α is a non-constant entire function:

Case n = 1. f, g non-constant entire, $h = -f - g + 1$.

Case n = 2. $f = \dfrac{\alpha^2 - 2}{\sqrt{3}}$, $g = \dfrac{(\alpha^2 + 1)i}{\sqrt{3}}$, $h = \sqrt{2}\alpha$

Case n = 3. Lehmer (1956): $f = 9\alpha^4$, $g = -9\alpha^4 + 3\alpha$, $h = -9\alpha^3 + 1$

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Case n = 4. Gross (1966):

$$f = 2^{1/4}(\sin^2\alpha - \cos^2\alpha + i\sin\alpha\cos\alpha)$$
$$g = (-1)^{1/4}(2i\sin\alpha\cos\alpha + \sin^2\alpha)$$
$$h = (-1)^{1/4}(2i\sin\alpha\cos\alpha - \cos^2\alpha)$$

or Green (1975):

$$f = 8^{-1/4}(e^{3\alpha} + e^{-\alpha}),\ g = (-8)^{-1/4}(e^{3\alpha} - e^{-\alpha}),\ h = (-1)^{1/4}e^{2\alpha}.$$

$f^n + g^n + h^n = 1$

Case n = 5. Gundersen and Tohge (2004):

$$f = \frac{1}{3}[(2-\sqrt{6})e^{\alpha} + 1 + (2+\sqrt{6})e^{-\alpha}]$$

$$g = \frac{1}{6}[\{\sqrt{6} - 2 + (3\sqrt{2} - 2\sqrt{3})i\}e^{\alpha} + 2 - \{\sqrt{6} + 2 - (3\sqrt{2} + 2\sqrt{3})i\}e^{-\alpha}]$$

$$h = \frac{1}{6}[\{\sqrt{6} - 2 + (2\sqrt{3} - 3\sqrt{2})i\}e^{\alpha} + 2 - \{\sqrt{6} + 2 + (3\sqrt{2} + 2\sqrt{3})i\}e^{-\alpha}]$$

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$$g = \frac{1}{6}[\{\sqrt{6}-2+(3\sqrt{2}-2\sqrt{3})i\}e^{\alpha} + 2 - \{\sqrt{6}+2-(3\sqrt{2}+2\sqrt{3})i\}e^{-\alpha}]$$
$$h = \frac{1}{6}[\{\sqrt{6}-2+(2\sqrt{3}-3\sqrt{2})i\}e^{\alpha} + 2 - \{\sqrt{6}+2+(3\sqrt{2}+2\sqrt{3})i\}e^{-\alpha}]$$

- In 1998, Gundersen constructed meromorphic (elliptic) solutions for $n = 6$ by expressing certain binary form as sum of powers of linear form (see also Tohge (2011) for a detailed explanation of Gundersen's construction).
- Then in 2001, Gundersen again constructed meromorphic solutions for the case $n = 5$ using a result on the unique range sets of meromorphic functions.

Remaining open problems for $f^n + g^n + h^n = 1$

Problem A: Whether there exist non-trivial **entire** solutions when $n = 6$?

Problem B: Whether there exist non-trivial **meromorphic (non-entire)** solutions when $n = 7$?

Problem C: Whether there exist non-trivial **meromorphic (non-entire)** solutions when $n = 8$?

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- The above three problems were asked by Hayman in many occasions.
- These problems are also mentioned in the work of Ishizaki (2002), Gundersen(2003), Gundersen and Tohge (2004).
- In 2015, Gundersen (2015) proposed to study these problems again in his problem list (see Question 3.1–3.4 of this list).

Remaining open problems for $f^n + g^n + h^n = 1$

Problem A: Whether there exist non-trivial **entire** solutions when $n = 6$?

- A result of Toda (1971) implies that if (f, g, h) is non-trivial, then each f, g and h must have at least one zeros.
- Y.H. Li and M. Su (2009) proved that if one of f, g, h has order strictly less than 1, then (f, g, h) is trivial.

Fermat surface

We will study Brody curves on the Fermat surface M_n defined by

$$X^n + Y^n + Z^n = W^n \tag{4}$$

on the complex projective space $\mathbb{P}^3 = \{[W : X : Y : Z]\}$.

On the affine part of $\mathbb{P}^3$ ($W \neq 0$), the equation is given by

$$x^n + y^n + z^n = 1 \tag{5}$$

where $x := \frac{X}{W}$, $y := \frac{Y}{W}$ and $z := \frac{Z}{W}$.

So any entire solution (f, g, h) will give a holomorphic curve $F : \mathbb{C} \to \mathbb{P}^3$ defined by $[1 : f : g : h]$.

Brody curve solution

Definition

Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve. Let $f = [f_0 : ... : f_n]$ be a reduced representation of f where $f_0, ..., f_n$ are entire functions in $\mathbb{C}$.
Let $||f||^2 = \sum_{j=0}^{n} |f_j|^2$ and $\|df\|_s$ be the Fubini–Study derivative of f which is given by

$$\|df\|_s^2 = ||f||^{-4} \sum_{i,j=0}^{n} |f_i f_j' - f_j f_i'|^2.$$

A holomorphic curve is called a *Brody curve* if its Fubini–Study derivative is bounded.

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Theorem (N. and Yeung)

There exists no non-trivial Brody curve on a Fermat surface of degree 6 of the form $[1 : f : g : h]$ given by entire holomorphic functions f, g, h.

Key ingredients of the proof

1. A version of Wiman-Valiron theory for vector-valued entire functions developed by Jank and Volkmann in 1986.

Proposition 1

There is no non-trivial entire solution to $f^6 + g^6 + h^6 = 1$ if one of f, g, h has order strictly less than $119/117$.

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There is no non-trivial entire solution to $f^6 + g^6 + h^6 = 1$ if one of f, g, h has order strictly less than $119/117$.

2. A potential theoretical result in Eremenko's work on Brody curves omitting hyperplanes in 2010.

Proposition 2

Assume that $F = [1 : f_1 : f_2 : f_3]$ is a Brody curve on a Fermat surface of degree 6 and all the f_j are entire. Let ρ be the order of f_i for $j = 1, 2, 3$. Then $\rho \leq 1$ unless F is trivial.

Wiman-Valiron theory for vector-valued entire functions

For one variable entire functions g_i, let

$$\mathbf{g}(z) = \begin{pmatrix} g_1(z) \\ g_2(z) \\ \vdots \\ g_n(z) \end{pmatrix} = \sum_{k=0}^{\infty} \mathbf{a}_k z^k, \mathbf{a}_k \in \mathbb{C}^n. \tag{6}$$

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The *maximum term* and the *central index* of $\mathbf{g}$ are defined by

$$\mu(r) = \mu(r, \mathbf{g}) = \max_{k \geq 0} ||\mathbf{a}_k|| r^k$$

$$\nu(r) = \nu(r, \mathbf{g}) = \max\{m| \, ||\mathbf{a}_m|| r^m = \mu(r, g)\}$$

Here, we always use the maximum norm $|| \cdot ||$ in $\mathbb{C}^n$ and we let $M(r) = M(r, \mathbf{g}) = \max_{|z|=r} ||\mathbf{g}(z)||$ for $r > 0$.

Wiman-Valiron theory for vector-valued entire functions

The function $\mathbf{g}$ is called transcendental if at least one of the components g_j in (6) is transcendental. Then we have

Theorem A (Jank and Volkmann, 1986) Let $\mathbf{g}(z) = (g_1(z), \ldots, g_n(z))$ be a vector-valued transcendental entire function, $0 < \delta < \frac{1}{4}$ and suppose z with $|z| = r$ satisfies

$$||\mathbf{g}(z)|| > M(r, \mathbf{g})[\nu(r, \mathbf{g})]^{-\frac{1}{4}+\delta}. \tag{7}$$

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Then, for $m \in \mathbb{N}$

$$\left|\left|\frac{1}{||\mathbf{g}(z)||}\left[\left(\frac{z}{\nu(r)}\right)^m \mathbf{g}^{(m)}(z) - \mathbf{g}(z)\right]\right|\right| \leq \nu(r)^{-\delta/2} = o(1), \tag{8}$$

and hence

$$\mathbf{g}^{(m)}(z) = \left(\frac{\nu(r)}{z}\right)^m (I + o(1))\mathbf{g}(z), r \notin E,$$

where I is the $n \times n$ identity matrix, $o(1)$ is a matrix which goes to 0 as $|z| \to \infty$ for $z \notin E$ and E is a set with finite logarithmic measure.

Wiman-Valiron theory for vector-valued entire functions

It follows from (8) that for each i,

$$\left(\frac{z}{\nu(r)}\right)^m g_i^{(m)}(z) = g_i(z) + R_{im}(z) \tag{9}$$

where $|R_{im}(z)| \leq \nu(r)^{-\delta/2}||\mathbf{g}(z)||$ as $r = |z| \to \infty$ for $r \notin E$.

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Proposition 1

There is no non-trivial entire solution to $f^6 + g^6 + h^6 = 1$ if one of f, g, h has order strictly less than $119/117$.

Remark. Y.H. Li and M. Su (2009) proved a similar result when one of f, g, h has order strictly less than 1.

Proof of Proposition 1

We will first construct 2-jet differentials from the following:

$$1 = x^{n-1}x + yy^{n-1} + zz^{n-1} \qquad (3)$$

By taking derivatives of equation (3), we obtain

$$\begin{aligned} 0 &= x^{n-1}dx + y^{n-1}dy + z^{n-1}dz & (10) \\ 0 &= x^{n-1}D^2x + y^{n-1}D^2y + z^{n-1}D^2z & (11) \end{aligned}$$

where $D^2F = d^2F + \frac{n-1}{F}(dF)^2$ for a function F.

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where $D^2F = d^2F + \frac{n-1}{F}(dF)^2$ for a function F. From equations (3), (10) and (11) and apply Crammer's rule, it follows that

$$\Phi := \frac{\begin{vmatrix} dy & dz \\ D^2y & D^2z \end{vmatrix}}{x^{n-1}} = \frac{\begin{vmatrix} dz & dx \\ D^2z & D^2x \end{vmatrix}}{y^{n-1}} = \frac{\begin{vmatrix} dx & dy \\ D^2x & D^2y \end{vmatrix}}{z^{n-1}} \qquad (12)$$

Proof of Proposition 1

We shall need the following properties of the 2-jet differential Φ:

$$\Phi = \begin{vmatrix} x & y & z \\ dx & dy & dz \\ D^2x & D^2y & D^2z \end{vmatrix} = (xyz)M_{xyz}, \tag{13}$$

where

$$M_{xyz} = \begin{vmatrix} 1 & 1 & 1 \\ \frac{dx}{x} & \frac{dy}{y} & \frac{dz}{z} \\ \frac{D^2x}{x} & \frac{D^2y}{y} & \frac{D^2z}{z} \end{vmatrix}.$$

Hence for

$$M_{yz} = \begin{vmatrix} \frac{dy}{y} & \frac{dz}{z} \\ \frac{D^2y}{y} & \frac{D^2z}{z} \end{vmatrix}, \ M_{zx} = \begin{vmatrix} \frac{dz}{z} & \frac{dx}{x} \\ \frac{D^2z}{z} & \frac{D^2x}{x} \end{vmatrix}, \ M_{xy} = \begin{vmatrix} \frac{dx}{x} & \frac{dy}{y} \\ \frac{D^2x}{x} & \frac{D^2y}{y} \end{vmatrix},$$

$$\Phi = \frac{(yz)M_{yz}}{x^{n-1}} = \frac{(zx)M_{zx}}{y^{n-1}} = \frac{(xy)M_{xy}}{z^{n-1}} = (xyz)M_{xyz}. \tag{14}$$

Proof of Proposition 1

Lemma (1)

Let $F := [1 : f : g : h]$*. If* $p := F^*(xyz\Phi) = 0$*, then* (f, g, h) *is a trivial solution.*

Proof of Proposition 1

Lemma (1)

Let $F := [1 : f : g : h]$. If $p := F^*(xyz\Phi) = 0$, then (f, g, h) is a trivial solution.

Suppose $F^*(xyz\Phi) = 0$. Then from

$$\Phi = \frac{(yz)M_{yz}}{x^{n-1}} = \frac{(zx)M_{zx}}{y^{n-1}} = \frac{(xy)M_{xy}}{z^{n-1}} = \frac{(xy)}{z^{n-1}} \begin{vmatrix} \frac{dx}{x} & \frac{dy}{y} \\ \frac{D^2x}{x} & \frac{D^2y}{y} \end{vmatrix},$$

unless $F(\mathbb{C})$ lies in a coordinate plane, we may assume that $F^*M_{xy} = 0$, since the former case can be handled easily.

Proof of Proposition 1

$F^*M_{xy} = 0$ gives

$$
\begin{aligned}
& f'(g'' + \frac{n-1}{g}(g')^2) - g'(f'' + \frac{n-1}{f}(f')^2) = 0 \\
\Rightarrow \quad & f'g'' - f'g'' = -(n-1)f'g'((\ln g)' - (\ln f)') \\
\Rightarrow \quad & f'g'' - g'f'' = -(n-1)f'g'(\ln(\frac{g}{f}))' \\
\Rightarrow \quad & (\frac{g'}{f'})' = -(n-1)\frac{g'}{f'}(\ln(\frac{g}{f}))' \\
\Rightarrow \quad & \ln(\frac{g'}{f'}) = -(n-1)\ln(\frac{g}{f}) + c \\
\Rightarrow \quad & \frac{g'}{f'} = k_1(\frac{g}{f})^{-(n-1)} \\
\Rightarrow \quad & g^n = k_1 f^n + k_2, \qquad (15)
\end{aligned}
$$

where k_1 and k_2 are constants.

Proof of Proposition 1

If both k_1 and k_2 are non-zero, the genus of the algebraic curve $y^n = k_1 x^n + k_2$ is greater than one and hence the curve is hyperbolic and f or g must be a constant.

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If both k_1 and k_2 are non-zero, the genus of the algebraic curve $y^n = k_1 x^n + k_2$ is greater than one and hence the curve is hyperbolic and f or g must be a constant.

Therefore, $k_1 = 0$ or $k_2 = 0$ and in either case, we conclude that the image of F lies in a rational curve of the form $(f(t), \omega_1 f(t), \omega_2, 1)$ or by permutation of the indices, where $\omega_1, \omega_2 \in \mathbb{C}$ such that $\omega_1^n = -1$ and $\omega_2^n = 1$.

Proof of Proposition 1

Consider the mapping $F = [1 : f : g : h] : \mathbb{C} \to \mathbb{P}^3$ where $f^6 + g^6 + h^6 = 1$. It suffices to show that,

$$p := F^*(xyz\Phi) = 0$$

where

$$\Phi = \begin{vmatrix} x & y & z \\ dx & dy & dz \\ D^2x & D^2y & D^2z \end{vmatrix} \tag{16}$$

and $D^2G = d^2G + \frac{n-1}{G}(dG)^2$ for a function G.

Proof of Proposition 1

Lemma (Ishizaki 2003)

Let (f, g, h) *be an entire solution of (3) when* $n = 6$ *such that* $F = [1 : f : g : h] : \mathbb{C} \to M_6 \subset \mathbb{P}^3$.

Suppose $F^* M_{xyz} \not\equiv 0$. *Then we have*
(a). $T(r, f) + S(r) = T(r, g) + S(r) = T(r, h) + S(r)$.
(b). $p := F^*(xyz\Phi)$ *is a polynomial whenever one of* f, g *and* h *is of finite order.*

Proof of Proposition 1

Let $g_1 = f^6, g_2 = g^6$ and $g_3 = h^6$ and we will apply Theorem A to $\mathbf{g}(z) = (g_1(z), g_2(z), g_3(z))$.

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We may assume that there is some uncountable $S \subset (0, +\infty) \backslash E$ such that for any $r \in S$, there is some $w = w(r) \in \mathbb{C}$ such that

$$M(r, g_1) = |g_1(w)| \geq |g_2(w)| \geq |g_3(w)|. \tag{17}$$

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In addition, from $f^6(w) + g^6(w) + h^6(w) = g_1 + g_2 + g_3 = 1$, we have

$$M(r, g_1) \leq |g_2(w)| + |g_3(w)| + 1 \leq 2|g_2(w)| + 1$$

and hence

$$|g_2(w)| \geq \frac{1}{3} M(r, g_1), \tag{18}$$

for $r \in S$.

Proof of Proposition 1

By Theorem A, for each $r \in S$ and the corresponding $w = w(r)$ satisfying $M(r, g) = |g(w)|$, we have

$$g_i^{(m)}(w) = \left(\frac{\nu(r)}{w}\right)^m (g_i(w) + R_{im}(w))$$

where $m = 1, 2$ and $|R_{im}(w)| \leq \frac{1}{\nu(r)^{\delta/2}} ||\mathbf{g}(w)|| = \frac{1}{\nu(r)^{\delta/2}} |g_1(w)|$.

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By Theorem A, for each $r \in S$ and the corresponding $w = w(r)$ satisfying $M(r, g) = |g(w)|$, we have

$$g_i^{(m)}(w) = \left(\frac{\nu(r)}{w}\right)^m (g_i(w) + R_{im}(w))$$

where $m = 1, 2$ and $|R_{im}(w)| \leq \frac{1}{\nu(r)^{\delta/2}} ||\mathbf{g}(w)|| = \frac{1}{\nu(r)^{\delta/2}} |g_1(w)|$. Let $r_{im}(w) = \frac{R_{im}(w)}{g_i(w)}$, then $|r_{1m}| \leq \frac{1}{\nu(r)^{\delta/2}}$ and $|r_{2m}| \leq \frac{3}{\nu(r)^{\delta/2}}$.
Therefore, we have for $i = 1, 2$,

$$g_i^{(m)}(w) = \left(\frac{\nu(r)}{w}\right)^m g_i(w)(1 + r_{im}(w)), \tag{19}$$

$|r_{im}(w)| \leq O(\frac{1}{\nu(r)^{\delta/2}})$.

Proof of Proposition 1

One can prove that

$$(g_1'g_2'' - g_2'g_1'')^3 = 36^3 p^3 g_3^2 g_1^2 g_2^2. \qquad (20)$$

Apply (19) to (20), we have

$$\left\{ \left(\frac{\nu(r)}{w}\right)^3 g_1(w)g_2(w)R(w) \right\}^3 = 6^6 p^3(w)(g_3(w))^2 g_1^2(w) g_2^2(w),$$

where $|R(w)| \leq O(\frac{1}{\nu(r)^{\delta/2}})$.

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Eventually, we have for $r_n \in S$ and $r_n = |w_n| = w(r_n)| \to \infty$,

$$|g_3^2(w_n)| \leq \frac{1}{6^6}|g_1(w_n)|\ |g_2(w_n)| \frac{\nu(r_n)^{9-\frac{3\delta}{2}}}{r_n^9}\ \frac{1}{|p^3(w_n)|}$$

(21)

Proof of Proposition 1

By interchanging the role of g_2 and g_3 and g_1 and g_3 in (20), we also have

$$(g_1'g_3'' - g_3'g_1'')^3 = 36^3(-p^3)g_2^2g_1^2g_2^3. \tag{22}$$

$$(g_3'g_2'' - g_2'g_3'')^3 = 36^3(-p^3)g_2^2g_1^2g_3^2. \tag{23}$$

We can obtain inequalities similar to (21) by the Logarithmic Derivative Lemma and finally to get

$$|p(w_n)| \leq O(r_n^c)$$

where $c < 0$ if one of the order of $f, g, h < \frac{119}{117}$.

In this case, $|p(w_n)| \to 0$ as $r_n = |w_n| \to \infty$.
As p is a polynomial, $p \equiv 0$ and we are done.

Proof of Proposition 2

Proposition 2

Assume that $F = [1 : f_1 : f_2 : f_3]$ is a Brody curve on a Fermat surface of degree 6 and all the f_j are entire. Let ρ be the order of f_i for $j = 1, 2, 3$. Then $\rho \leq 1$ unless F is trivial.

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First let us make some observations.

Lemma (2)

(a). Suppose $F = [1 : f_1 : f_2 : f_3]$ is a Brody curve to the Fermat surface of degree 6 in $\mathbb{P}^3$. The projection of F to each coordinate plane $\mathbb{P}^2$ is a Brody curve. Hence $F_{ij} = [1 : f_i : f_j]$ is a Brody curve for $i \neq j$.

(b). Assume that the spherical derivative $\|dF\|_s \leqslant c\|z\|^\ell$, so is the projection to each coordinate, that is $\|dF_{ij}\|_s \leqslant c'\|z\|^\ell$ for each i, j.

Proof of Proposition 2

Lemma (3)

Suppose $F = [1 : f_1 : f_2 : f_3]$ *is a Brody curve to the Fermat surface of degree* 6 *in* $\mathbb{P}^3$. *Then there exists some* $c > 0$ *such that whenever* $w \in \mathbb{C}$ *satisfying* $|f_1(w)| = 1$, *we have*

$$\|df_j(w)\|_s := \frac{|f_j'(w)|}{1 + |f_j(w)|^2} \leqslant c$$

for $j = 2, 3$.

Proof of Proposition 2

Theorem B (Eremenko 2010)

For entire functions $f_0, ..., f_n$ in $\mathbb{C}$, let $f = [f_0 : \cdots : f_n]$ and $u_i = \log|f_i|$ for $i = 0, ..., n$. Suppose f_0 has a zero a. Let $z \in$ such that $|z| > |a|$ and $u_0(z) \geqslant \max_{1 \leqslant j \leqslant n} u_j(z)$.

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Let $B(z; R)$ be the largest ball of radius R so that $u_0(s) \geqslant \max_{1 \leqslant j \leqslant n} u_j(s)$ for all $s \in B(z; R)$. Hence there exists $z_1 \in B(z; R)$ with $u_0(z_1) = \max_{1 \leqslant j \leqslant n} u_j(z_1)$.

Then

$$u_0(z) \leqslant \max_{1 \leqslant j \leqslant n} u_j(z) + 4(n+1)R\|df\|_s(z_1) \tag{24}$$

where $R \leqslant 2|z|$

Brody curves omitting hyperplanes

Theorem C (Eremenko 2010)

Brody curves $f : \mathbb{C} \to \mathbb{P}^n$ omitting n hyperplanes in general position satisfy

$$T(r,f) := \int_0^r \frac{dt}{t}\left(\frac{1}{\pi}\int_{\Delta(t)} \|df\|_s^2(z)dm_z\right) = O(r)$$

Remark. This extends Clunie and Hayman(1966)'s result when $n = 1$.

Proof of Proposition 2

We are going to prove the Proposition 2 in three steps:

- *Step 1*, to prove that either $\rho < \frac{5}{3}$ or that the entire holomorphic curve is a trivial one, on the assumption that it is a Brody curve, and

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- *Step 3*, to prove that $\rho \leqslant 1$ or that it is trivial, on the assumption that it has order $\rho < \frac{4}{3}$.

Remark. If $F = [1 : f_1 : f_2 : f_3]$ is a Brody curve in M_6, then $\rho_{f_i} = \rho_F \leq 2$ for $i = 1, 2, 3$.

Proof of Proposition 2 (Step 1)

Denote $u_i = \log|f_i|$. It suffices for us to show that for some positive ϵ, $u_j(z) \leqslant O(|z|^{\frac{5}{3}-\epsilon})$ for all $|z|$ sufficiently large as it is the same as showing that f_i has order less than $< \frac{5}{3} - \epsilon$ for each $i = 1, 2, 3$.

Let ϵ be a sufficiently small positive number.

Let $A := \{z : \max_{1 \leqslant j \leqslant 3} u_j(z) > |z|^{\frac{5}{3}-\epsilon}\}$.

We will assume A is an unbounded set and try to deduce that F is trivial.

Proof of Proposition 2 (Step 1)

For $1 \leqslant i \neq j \leqslant 3$, let $F_{ij} = [1 : f_i : f_j]$. Then (24) implies that for $|z|$ sufficiently large,

$$u_i(z) \leqslant \max(u_j(z), 0) + 24|z| \sup \|dF_{ij}\|_s \leqslant \max(u_j(z), 0) + 24C_0|z| \quad (25)$$

and $C_0 := \max\{\sup \|dF_{12}\|_s, \sup \|dF_{13}\|_s, \sup \|dF_{23}\|_s\}$, note that $C_0 < \infty$ follows from the fact that each F_{ij} is a Brody curve.

Proof of Proposition 2 (Step 1)

Hence, for any $1 \leqslant i, j \leqslant 3$ and $|z|$ sufficiently large,

$$u_i(z) \leqslant \max\{u_j(z), 0\} + 24C_0|z| \tag{26}$$

Let $r > 0$ be a fixed sufficiently large number so that (26) holds for $|z| > r$.

Now let $z \in A \backslash B(0; r)(\neq \phi)$ be a fixed number in $\mathbb{C}$.

Proof of Proposition 2 (Step 1)

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Now let $z \in A \backslash B(0; r)(\neq \phi)$ be a fixed number in $\mathbb{C}$.

We first show that $u_i(z) > 0$ for each $1 \leq i \leq 3$.

Without loss of generality assume at z that $u_1(z) \leqslant u_2(z) \leqslant u_3(z)$.

If $u_1(z) < 0$, then from (26), we have $u_3(z) \leqslant 0 + 24C_0|z|$ which is impossible as $z \in A$ and r can be arbitrarily large.

Therefore, $u_1(z) > 0$ and so are $u_2(z)$ and $u_3(z)$.

Proof of Proposition 2 (Step 1)

As each f_i has at least one zero b_i so that $u_i(b_i) < 0$, we can consider the largest radius $R_i > 0$ such that $u_i(w) > 0$ for all $w \in B(z; R_i)$.

Clearly for each i, there exists some $s_i \in \partial B(z; R_i)$ with $u_i(s_i) = 0$.

Let $R = \max_{i=1}^{3} R_i$ and we may assume $R_1 \leq R_2 \leq R_3 = R$.

Consider now two subcases,

(1a) $R \leqslant |z|^{\frac{5}{3}-2\epsilon}$, and

(1b) $R > |z|^{\frac{5}{3}-2\epsilon}$.

Proof of Proposition 2 (subcase 1a)

Consider first subcase (1a). Applying Theorem B to the mapping $G_j = [f_j : 1] :\to \mathbb{P}^1$ for $j = 1, 2, 3$ we get

$$u_j(z) \leqslant 8R_j\|dG_j'(z_j)\|_s. \tag{27}$$

for some point $z_j \in \partial B(z; R_j)$ with $u_j(z_j) = 0$. In fact, z_i is a candidate of s_i above.

Proof of Proposition 2 (subcase 1a)

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for some point $z_j \in \partial B(z; R_j)$ with $u_j(z_j) = 0$. In fact, z_i is a candidate of s_i above.

Let j_{z_1} be the index satisfying $|f_{j_{z_1}}(z_1)| = \max_{j=1}^{3} |f_j(z_1)|$. In our setting, j_{z_1} can be taken to be either 2 or 3 as $u_1(z_1) = 0$ and $R_2, R_3 \geq R_1$.

Applying $w = z_1$ in the statement of Lemma 3, noting that $|f_1(z_1)| = 1$ as $u_1(z_1) = 0$, we conclude that $\|dG_{j_{z_1}}(z_1)\|_s \leqslant c$ and hence the above estimate (27) implies that

$$|u_{j_{z_1}}(z)| \leqslant 8cR.$$

Proof of Proposition 2 (subcase 1a)

Apply now Theorem B to the map $[1 : f_1 : f_{j_{z_1}}] :\to \mathbb{P}^2$, we conclude from (24) or (26) that

$$u_1(z) \leq \max\{u_{j_{z_1}}(z), 0\} + 24C_0|z| \leqslant C_1|z|^{\frac{5}{3}-2\epsilon} + 24C_0|z| \leqslant C|z|^{\frac{5}{3}-2\epsilon} \quad (28)$$

for ϵ sufficiently small which is impossible as $z \in A \backslash B(0; r)$.

Hence subcase 1a leads to a contradiction if A is unbounded.

Proof of Proposition 2 (subcase 1b)

Subcase (1b), $R > |z|^{\frac{5}{3}-2\epsilon}$.

Use the notation that $F_1 = f_1^6, F_2 = f_2^6, F_3 = f_3^6$ and $u_i = \log|F_i|$. At a point z, we assume without loss of generality that $|F_3(z)| \geqslant |F_2(z)| \geqslant |F_1(z)|$, otherwise we may just permute the coordinates in the following argument, where p is independent of the choice.

Proof of Proposition 2 (subcase 1b)

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We rewrite $p := F^*(xyz\Phi)$ in the following way.

$$p \;=\; (\frac{F_1}{F_3}\frac{F_2}{F_3})^{1/3}\{(u_1'(\log\frac{F_2}{F_1})'' + (\log\frac{F_2}{F_1})'u_1'' + u_1'u_2'(\log\frac{F_2}{F_1})')\}$$

From our choices, $|(\frac{F_1}{F_3}\frac{F_2}{F_3})^{1/3}| \leqslant 1$.

Proof of Proposition 2 (Subcase 1b)

Lemma (4)

Let f_1, f_2 *and* f_3 *be entire functions of order* ρ *satisfying* $f_1^6 + f_2^6 + f_3^6 = 1$ *and we denote* f_i^6 *by* F_i*.*

Let $\rho' > \rho$ *and* $|z| = r$ *be sufficiently large so that* $\log |F_i(z)| = O(r^{\rho'})$ *for all* i*. Assume that* $\log |F_i| > 0$ *on* $B(z; R)$ *for* $i = 1, 2, 3$*.*

Then the following holds, where we denote $\frac{\partial h}{\partial z}$ *by* h'*.*

(a). $|(\log F_i)'(z)| \leqslant c\frac{r^{\rho'}}{R}$.

(b). $|(\log F_i)''(z)| \leqslant c\frac{r^{\rho'}}{R^2}$.

(c). $|\frac{F_i''}{F_i}(z)| \leqslant c\frac{r^{2\rho'}}{R^2}$.

(d). $|(\log \frac{F_i}{F_j})(z)| \leqslant c(\log(\frac{r^{9\rho'}}{R^9}) + 3\rho' \log r)$ *for* $i \neq j$*.*

(e). $|(\log \frac{F_i}{F_j})'(z)| \leqslant c\frac{\log r}{R}$ *for* $i \neq j$ *if* $R \geqslant 1$*.*

(f). $|(\log \frac{F_i}{F_j})''(z)| \leqslant c\frac{\log r}{R^2}$ *for* $i \neq j$*.*

Proof of Proposition 2 (subcase 1b)

Since $R > r^{\frac{5}{3}-2\epsilon}$, where $r = |z|$. Then from Lemma 4,

i) $|u_i'(z)| \leqslant cr^{\rho' - \frac{5}{3}+\epsilon}$,
ii) $|u_i''(z)| \leqslant cr^{\rho' - \frac{10}{3}+2\epsilon}$,
iii) $|(\log \frac{F_2}{F_1})'(z)| \leqslant c\frac{\log r}{r^{\frac{5}{3}-2\epsilon}}$,
iv) $|(\log \frac{F_2}{F_1})''(z)| \leqslant c\frac{\log r}{r^{\frac{10}{3}-4\epsilon}}$.

$$p = (\frac{F_1}{F_3}\frac{F_2}{F_3})^{1/3}\{(u_1'(\log\frac{F_2}{F_1})'' + (\log\frac{F_2}{F_1})'u_1'' + u_1'u_2'(\log\frac{F_2}{F_1})')\}$$

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$$p \;=\; (\frac{F_1}{F_3}\frac{F_2}{F_3})^{1/3}\{(u_1'(\log \frac{F_2}{F_1})'' + (\log \frac{F_2}{F_1})'u_1'' + u_1'u_2'(\log \frac{F_2}{F_1})')\}$$

The dominating term above is $u_1'u_2'(\log \frac{F_2}{F_1})'$, which is of order $\frac{r^{2\rho'}\log r}{R^3}$. In this case, can take $R = r^{\frac{5}{3}-2\epsilon}$. Hence

$$|p| \leqslant cr^{2\rho'-5+6\epsilon}\log r.$$

Hence $p = 0$ if $\rho \leqslant 2$, after taking r sufficiently large. From earlier discussions, $p = 0$ implies that f is trivial.

Proof of Proposition 2 (subcase 1b)

We conclude that for subcase 1b, f is trivial if A is unbounded.

In conclusion, if A is unbounded, f has to be trivial.

So if f is non-trivial Brody curve in M_6, then $\rho_f < 5/3$.

This completes the proof for *Step 1*.

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This completes the proof for *Step 1*.

Similarly, we can prove

- *Step 2*, to prove that $\rho < \frac{4}{3}$ or that it is trivial, on the assumption that it has order $\rho < \frac{5}{3}$,
- *Step 3*, to prove that $\rho \leqslant 1$ or that it is trivial, on the assumption that it has order $\rho < \frac{4}{3}$.

Green-Griffiths'conjecture (compact case)

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If D is a nonsingular hypersurface in the projective space $\mathbb{P}^n$ of degree $d \geq n+2$, then the image of any holomorphic mapping $f : \mathbb{C}^p \to D$ lies in some proper algebraic subvariety of D

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In the case where D is a Fermat type hypersurface, Green obtained the following results

Theorem D (Green 1975)

Let $p \geq 1$,$n \geq 2$ be positive integers. Let M be the Fermat hypersurface of degree d in $\mathbb{P}^n$. If $d > n^2 - 1$, then the image of every holomorphic map $f : \mathbb{C}^p \to M$ lies in a linear subspace of dimension at most $[\frac{n-1}{2}]$.

For $p = 1$, the unsettled cases for Green-Griffiths'conjecture (Fermat surface) are $d = 6$ and 5,

Nevanlinna current

Let α be a smooth $(1,1)$-form on a complex projectuve variety X of dimension two.

Let ω be a fixed Kähler form on X and $T(r, G, \omega)$ be the characteristic function of an entire holomorphic curve $G : \mathbb{C} \to M$ defined by

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Define the family of positive currents of bounded mass $R_{\omega,r}[G]$ by

$$R_{\omega,r}[G](\alpha) = \langle R_{\omega,r}[G],\alpha\rangle := \frac{1}{T(r,G,\omega)}\int_0^r \frac{dt}{t}\int_{\Delta(t)} G^*\alpha \tag{29}$$

Nevanlinna current

McQuillan (1998) showed that there exist infinitely many sequences $\{r_k\}$ converging to ∞ such that the sequence of currents $\{R_{\omega,r_k}[G]\}$ converges in weak topology to a closed positive $(1,1)$ current given by

$$R_\omega[G] := \lim_{k\to\infty} R_{\omega,r_k}[G], \tag{30}$$

where $r_k \to \infty$ as $k \to \infty$.

Such limit currents are called *Nevanlinna currents* for G. Nevanlinna currents can be considered as the logarithmic average analogs of Ahlfors currents.

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Lemma (McQuillan 2012)

Let T be a Nevanlinna current in a projective manifold X of dimension two. Assume that T charges a compact subset K of X. Then there exists a Brody curve intersecting K with non-zero area.

Duval (2008) first established a similar result for Ahlfors currents.

Nevanlinna current

From Siu's decomposition of positive currents (1974), we have

$$R_\omega[G] = \sum_i \beta_i [D_i] + A,$$

where the sum is a possibly countably infinite sum, D_i are distinct irreducible algebraic curves, $[D_i]$ are the currents of integration over D_i, β_i are the generic Lelong numbers of $R_\omega[G]$ along D_i, and A is a positive closed $(1,1)$ current which has positive Lelong number only on a set of dimension 0.

Nevanlinna current

Theorem (N. and Yeung)

Let $C_1, \ldots, C_l$ be trivial rational curves on M_6. Let $G : \mathbb{C} \to M_6$ be holomorphic.

Then $R_\omega[G]$ can be represented as

$$R_\omega[G] = \sum_{i=1}^{l} \beta_i [C_i] + A \tag{31}$$

in Siu's decomposition of positive current, where $\beta_i \geqslant 0$, and A is a countable set of points supported on $\cup_i [C_i]$. Furthermore,

$$\sum_i \beta_i \leqslant 1. \tag{32}$$