Some conjectures concerning the zeros of the deformed exponential function

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https://webspace.maths.qmul.ac.uk/p.j.cameron/csgnotes/sokal/

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Question: What can we say about the roots $x_k(y)$?

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- Can also use theta function in Rouché (Eremenko)

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Theorem (Eremenko 2009, unpublished)

No root can escape to infinity for y in the open unit disc $\mathbb{D}$.

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- **Conjecture** that roots cannot escape to infinity even in the *closed* unit disc except at $y = 1$

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Consequence of Big Conjecture #2. The roots are actually separated in modulus by a factor at least $|y|$, i.e.

$$|x_k(y)| < |y|\,|x_{k+1}(y)| \qquad \text{for all } k \geq 0$$

PROOF. Apply the Schwarz lemma to $x_k(y)/x_{k+1}(y)$.

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- $\Longrightarrow$ If $F(\,\cdot\,, y)$ has a *unique zero of minimum modulus* for all $y \in \mathbb{D}$ (a weakened form of **Big Conjecture #2**), then the zeros of $\overline{C}_n(y)$ do not accumulate anywhere in the open unit disc.

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- $\Longrightarrow$ the limit points of zeros of $\overline{C}_n$ are the values y for which the zero of minimum modulus of $F(\,\cdot\,, y)$ is *nonunique*.
- $\Longrightarrow$ If $F(\,\cdot\,, y)$ has a *unique zero of minimum modulus* for all $y \in \mathbb{D}$ (a weakened form of **Big Conjecture #2**), then the zeros of $\overline{C}_n(y)$ do not accumulate anywhere in the open unit disc.

But I actually conjecture more (based on computations up to $n \approx 80$):

Consequences for the zeros of $\overline{C}_n(y)$

- Recall that $\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n}$
- Use a variant of the Beraha–Kahane–Weiss theorem
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- $\Longrightarrow$ the limit points of zeros of $\overline{C}_n$ are the values y for which the zero of minimum modulus of $F(\,\cdot\,, y)$ is *nonunique*.
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But I actually conjecture more (based on computations up to $n \approx 80$):

Big Conjecture #3. For each n, $\overline{C}_n(y)$ has no zeros with $|y| < 1$.

[and, I suspect, no zeros with $|y| = 1$ except the point $y = 1$]

What is the evidence for these conjectures?

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- **Evidence #2:** From numerical computation of the series $x_k(y)$ . . .

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3. Use $\overline{C}_n(y) \;=\; -(n-1)! \sum_k x_k(y)^{-n}$ together with recurrence

$$\overline{C}_n(y) \;=\; y^{n(n-1)/2} - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \overline{C}_j(y)\, y^{(n-j)(n-j-1)/2}$$

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4. Use Dandelin–Lobachevskii–Graeffe iteration (repeated squaring)

$\longrightarrow$ I went to $n = 65535$ for $k = 0$

Results of numerical computations

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& + \tfrac{493}{1152}y^7 + \tfrac{163}{384}y^8 + \tfrac{323}{768}y^9 + \tfrac{1603}{3840}y^{10} + \tfrac{57283}{138240}y^{11} \\
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and all the coefficients (so far) are nonnegative!

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NEED TO DO: Extended computations for $k = 1, 2, \ldots$ and for symbolic k.

But more is true . . .

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\begin{aligned}
-\frac{1}{x_0(y)} \;=\; & 1 - \tfrac{1}{2}y - \tfrac{1}{4}y^2 - \tfrac{1}{12}y^3 - \tfrac{1}{16}y^4 - \tfrac{1}{48}y^5 - \tfrac{7}{288}y^6 \\
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and all the coefficients (so far) beyond the constant term are nonpositive!

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Big Conjecture #5. For each k, the series $-(k+1)y^{-k}/x_k(y)$ has all nonpositive coefficients after the constant term 1.

[This implies Big Conjecture #4, but is stronger.]

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- Relative simplicity of the coefficients of $-1/x_0(y)$ compared to those of $-x_0(y)$ $\longrightarrow$ simpler combinatorial interpretation?

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- Note that $x_k(y) \to -\infty$ as $y \uparrow 1$ (this is fairly easy to prove). So $1/x_k(y) \to 0$. Therefore:

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Consequence of Big Conjecture #5. For each k, the coefficients in the series $1 + (k+1)y^{-k}/x_k(y)$ are the *probabilities* for a positive-integer-valued random variable.

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What might such a random variable be???

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- Define $D_n(y) = \dfrac{\overline{C}_n(y)}{(-1)^{n-1}(n-1)!}$ [it has constant term 1]

  and recall that $-x_0(y) = \lim_{n\to\infty} D_n(y)^{-1/n}$

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- Since $D_n(y) > 0$ for $0 \le y < 1$, Vivanti–Pringsheim shows that **Big Conjecture #6a** implies **Big Conjecture #3**:

  For each n, $\overline{C}_n(y)$ *has no zeros with* $|y| < 1$.

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- **Big Conjecture #6b** $\Longrightarrow$ for each n, the coefficients in the series $1 - D_n(y)^{1/n}$ are the *probabilities* for a positive-integer-valued random variable.

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- Define $D_n(y) = \dfrac{\overline{C}_n(y)}{(-1)^{n-1}(n-1)!}$ [it has constant term 1]

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- **Big Conjecture #6b** $\Longrightarrow$ for each n, the coefficients in the series $1 - D_n(y)^{1/n}$ are the *probabilities* for a positive-integer-valued random variable.
- Such a random variable would generalize the one for $1/x_0(y)$ in roughly the same way that the binomial generalizes the Poisson.

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- Presumably has something to do with random graphs on n vertices
- Maybe some structure built on top of a random graph
  (some kind of tree? Markov chain?)

What might such a random variable be?

- Probability generating function $P_n(y) = 1 - D_n(y)^{1/n}$
- Try to understand the first two cases:

$$\begin{aligned} P_2(y) &= 1-(1-y)^{1/2} \\ &= \tfrac{1}{2}y + \tfrac{1}{8}y^2 + \tfrac{1}{16}y^3 + \tfrac{5}{128}y^4 + \tfrac{7}{256}y^5 + \tfrac{21}{1024}y^6 \\ &\qquad + \tfrac{33}{2048}y^7 + \tfrac{429}{32768}y^8 + \tfrac{715}{65536}y^9 + \tfrac{2431}{262144}y^{10} + \ldots \\ &\sim \text{Sibuya}(\tfrac{1}{2}) \text{ random variable} \end{aligned}$$

$$\begin{aligned} P_3(y) &= 1-(1-\tfrac{3}{2}y+\tfrac{1}{2}y^3)^{1/3} \\ &= \tfrac{1}{2}y + \tfrac{1}{4}y^2 + \tfrac{1}{24}y^3 + \tfrac{1}{24}y^4 + \tfrac{1}{48}y^5 + \tfrac{5}{288}y^6 \\ &\qquad + \tfrac{7}{576}y^7 + \tfrac{23}{2304}y^8 + \tfrac{329}{41472}y^9 + \tfrac{553}{82944}y^{10} + \ldots \end{aligned}$$

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- How are these related to random graphs on 2 or 3 vertices??
- I have an analytic proof that $P_3(y) \succeq 0$, but it doesn't shed any light on the possible probabilistic interpretation.

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&\quad + \tfrac{7}{576}y^7 + \tfrac{23}{2304}y^8 + \tfrac{329}{41472}y^9 + \tfrac{553}{82944}y^{10} + \ldots
\end{aligned}$$

- How are these related to random graphs on 2 or 3 vertices??
- I have an analytic proof that $P_3(y) \succeq 0$, but it doesn't shed any light on the possible probabilistic interpretation.
- Jim Fill has a probabilistic interpretation for $n = 2, 3$ in terms of birth-and-death chains, but it doesn't seem to generalize to $n \geq 4$.

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- Unfortunately, the series
$$\frac{x_1(y)}{x_2(y)} = \frac{2}{3}y + \frac{1}{18}y^2 + \frac{17}{216}y^3 + \frac{23}{810}y^4 + \frac{343}{17280}y^5 + \ldots$$
has a negative coefficient at order y^{13}. This doesn't contradict the conjecture that $|x_1(y)/x_2(y)| < 1$ in the unit disc, but it does rule out the simplest method of proof.

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Anyone knowledgeable about Airy asymptotics who wants to collaborate?

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- "Deformed binomial" and "deformed hypergeometric" series (see below).

A more general approach, continued ...

- **Example**: For Θ_0 we have

$$-x_0(y) \;=\; 1+y+2y^2+4y^3+9y^4+21y^5+52y^6+133y^7+351y^8+\ldots$$

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  - Alas, the method does *not* seem to generalize.

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- More generally, for $\widetilde{R}(x, y, q)$ it can be proven that

$$-x_0(y, q) \;=\; 1 + \sum_{n=1}^{\infty} \frac{P_n(q)}{Q_n(q)}\, y^n$$

where $Q_n(q) \;=\; \prod_{k=2}^{\infty} (1 + q + \ldots + q^{k-1})^{\lfloor n/\binom{k}{2} \rfloor}$

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Can any of this be proven???

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- Start from the Taylor series for the binomial $f(x) = (1 - \mu x)^{-1/\mu}$ and introduce factors $y^{n(n-1)/2}$ as usual:

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- Can any of this be proven for $\mu \neq 1$?

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Can any of this be proven???