Shrinking targets and recurrent behaviour for forward compositions of inner functions

Gwyneth Stallard
The Open University

Joint with Anna Miriam Benini, Vasiliki Evdoridou, Nuria Fagella and Phil Rippon

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Collaborators

Part 1: Motivation from complex dynamics

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The **Julia set** (or chaotic set) is

$$J(f) = \mathbb{C} \setminus F(f).$$

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Periodic Fatou components are well understood and there is a classification essentially due to Fatou and Cremer (1920s).

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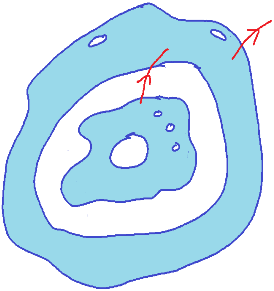

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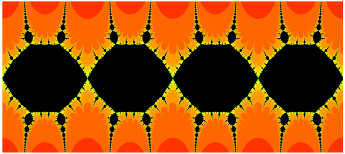

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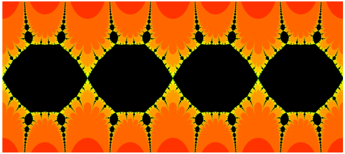

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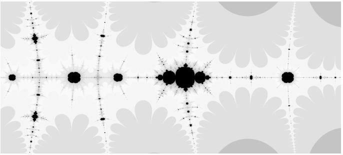

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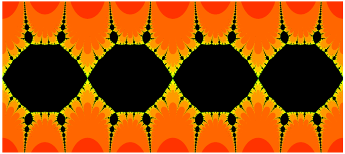

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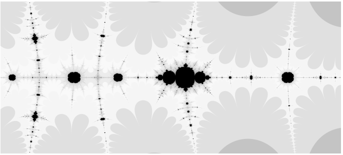

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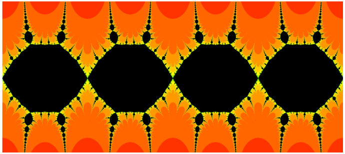

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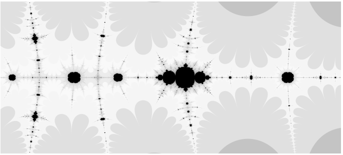

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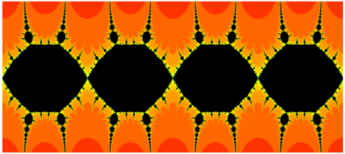

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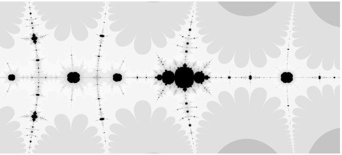

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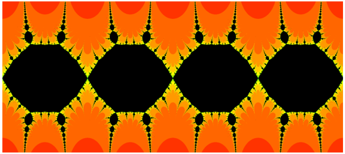

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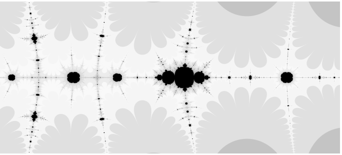

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Distance from boundary

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$$\sum_{n=0}^{\infty}(1 - |g_n'(0)|) = \infty \iff g_n(w) \to 0 \text{ as } n \to \infty$$

for all $w \in \mathbb{D}$.

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Aim: extend the classical study of iterates f^n of inner functions to non-autonomous dynamics. Let (f_n) be a sequence of inner functions and consider the sequence (F_n) where $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$.

The Denjoy-Wolff Theorem

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Question In this situation, are there boundary points whose orbits have the same limiting behaviour?

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(a) *If* $\sum_{n \geq 0}(1 - |f^n(0)|) < \infty$, *then* $\lim_{n\to\infty} f^n(w)$ *is equal to the Denjoy-Wolff point for almost all* $w \in \partial\mathbb{D}$.

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(b) *If $\sum_{n \geq 0}(1 - |f^n(0)|) = \infty$, then the iterates $f^n(w)$ are dense in $\partial\mathbb{D}$ for almost all $w \in \partial\mathbb{D}$.*

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Proof of part (a) based on Löwner's lemma and the *first* Borel-Cantelli lemma.

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Then $F_n(\zeta) - F_n(0) \to 0$ as $n \to \infty$ for almost all $\zeta \in \partial\mathbb{D}$.

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Generalising the ADM dichotomy

Part (a) generalises using a similar proof:

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then the orbits $F_n(w)$ are dense in $\partial\mathbb{D}$ for almost all $w \in \partial\mathbb{D}$.

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We have examples which show that this condition is sharp:

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2. We normalise the functions f_n by 'conjugating' with Möbius maps to give inner functions g_n with $g_n(0) = 0$, $g_n(1) = 1$ and $\lambda_n = |g_n'(0)|$. We put $G_n = g_n \circ \cdots \circ g_1$.

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3. The problem now is to show that the set of points $w \in \partial\mathbb{D}$ for which $G_n(w) \in I_n$ for infinitely many $n \in \mathbb{N}$ has full measure, the arcs (I_n) form a shrinking target with $|I_n| \approx 1 - |F_n(0)|$.

Shrinking target result

BEFRS 2024a

Let $g_n : \mathbb{D} \to \mathbb{D}$ be inner functions with $g_n(0) = 0$, $|g_n'(0)| = \lambda_n$.
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  implies $G_n(w) \in I_n$ infinitely often, for almost no $w \in \partial\mathbb{D}$.
- $$\sum_{n\geq 0}|I_n| = \infty, \text{ but } \sum_{n\geq 0}(1-\lambda_n)|I_n| < \infty,$$
  then examples show either conclusion is possible, depending on arrangement of the arcs (I_n).

Proof of shrinking target result

Theorem (BEFRS 2024a, uniform contraction)

Let $G_n = g_n \circ \cdots \circ g_1$*, with* $g_n(0) = 0$, $|g_n'(0)| \leq \lambda < 1$.

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We prove the general result when $|g_n'(0)| = \lambda_n$ and $\sum_{n\geq 0}(1 - \lambda_n)|I_n| = \infty$ by blocking together groups of g_n to give inner functions of uniform contraction.

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Application to dense orbits would depend on the geometry of the domains.

Happy Birthday Alex!

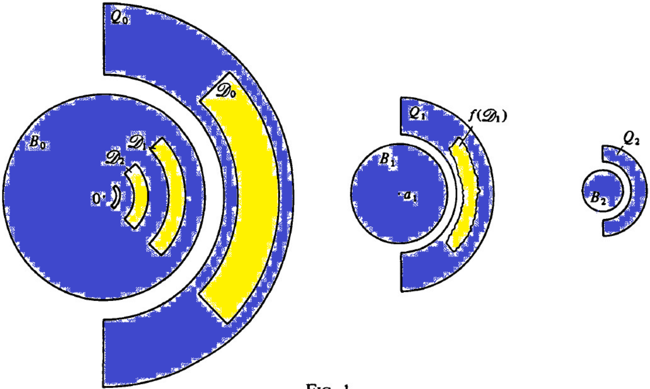

FIG. 1