Rate distortion dimension of random Brody curves

Masaki Tsukamoto

Main purpose

To propose an ergodic theoretic approach to Brody curves.

Brody curves are one-Lipschitz holomorphic maps

$$f\colon \mathbb{C} \to \mathbb{C}P^N.$$

Main result (very roughly)

Brody curves $\approx$ Axiom A diffeomorphisms.

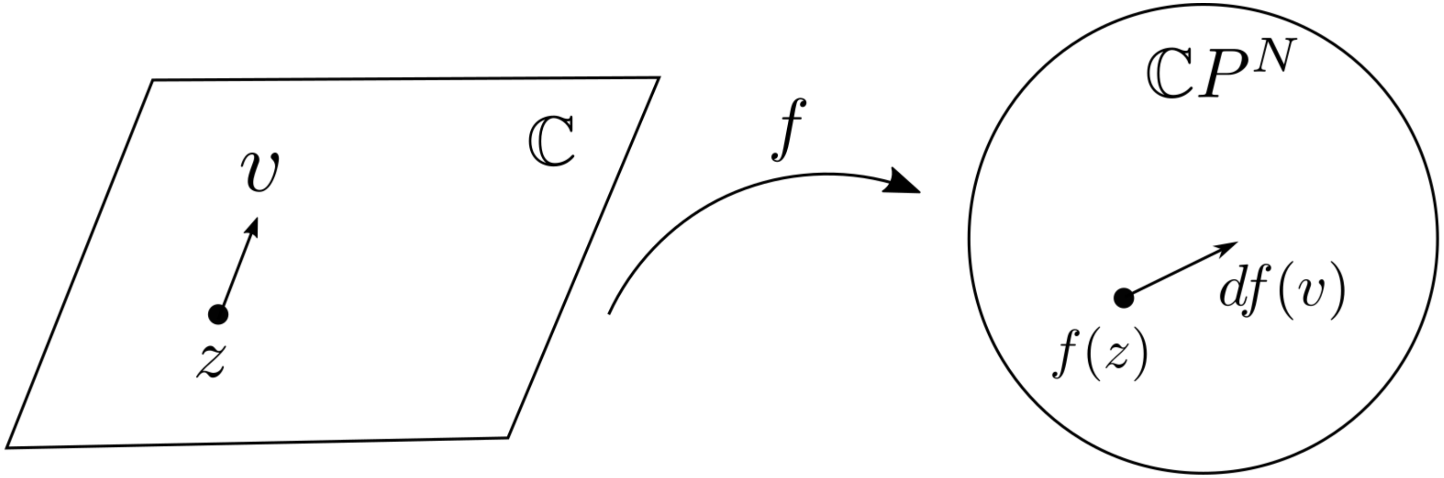
1 What are Brody curves?

$f = [f_0 : \cdots : f_N] : \mathbb{C} \to \mathbb{C}P^N$: holomorphic.

Define local Lipschitz constant $|df|(z)$ by

$$|df|^2(z) = \frac{1}{4\pi}\Delta \log\left(|f_0(z)|^2 + \cdots + |f_N(z)|^2\right).$$

Here $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.



$$|df|^2 = \frac{1}{4\pi}\Delta \log\left(|f_0|^2 + |f_1|^2 + \cdots + |f_N|^2\right).$$

Geometrically: $|df(v)| = |df|(z) \times |v|$.

$f : \mathbb{C} \to \mathbb{C}P^N$: Brody curve $\overset{\text{def}}{\Longleftrightarrow} |df| \leq 1$ all over $\mathbb{C}$.

Why is this interesting?

Brody (1978) proved that a projective variety is Kobayashi hyperbolic iff it does not contain any nonconstant Brody curve.

Shoshichi Kobayashi; from Wikipedia

$$\mathcal{B}^N := \{f\colon \mathbb{C} \to \mathbb{C}P^N \mid \text{Brody curve}\}.$$

Define a metric on it by

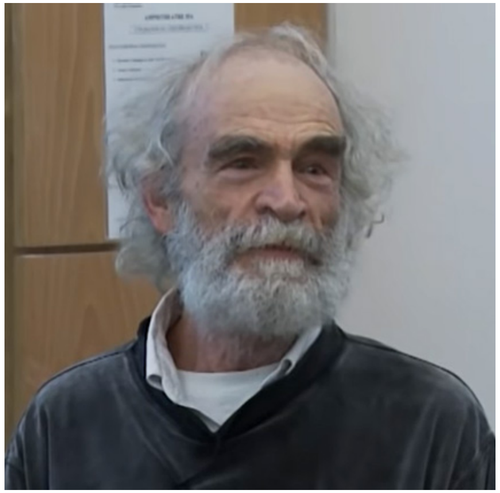
$$\mathbf{d}(f,g) = \sup_{|z|\le 1} d_{\mathbb{C}P^N}\left(f(z), g(z)\right).$$

$(\mathcal{B}^N, \mathbf{d})$: compact space with group action:

$$T\colon \mathbb{C} \times \mathcal{B}^N \to \mathcal{B}^N, \quad (a, f(z)) \mapsto f(z+a).$$

Gromov (1999) began to study mean dimension $\mathrm{mdim}\left(\mathcal{B}^N, T\right)$.

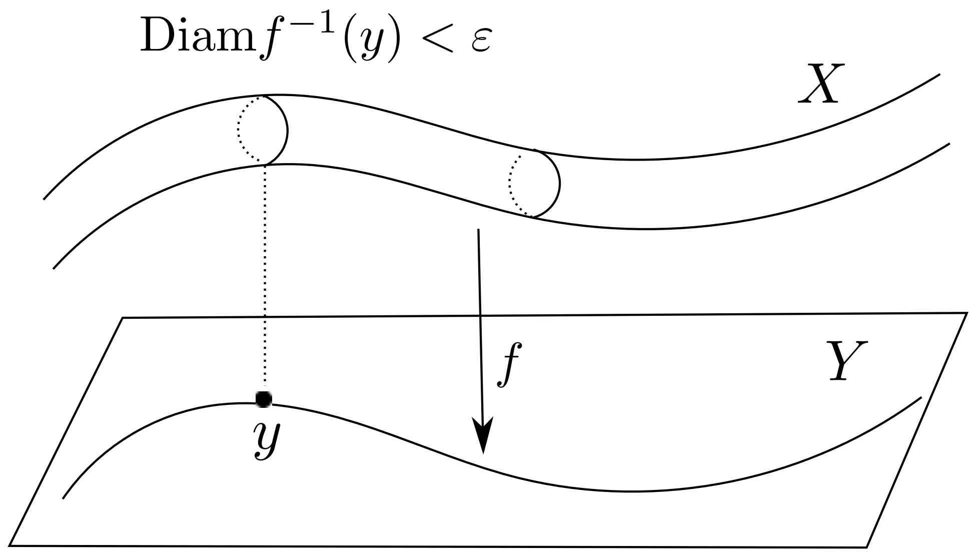
Mean dimension is the number of parameters per unit area of $\mathbb{C}$ for describing the orbits of $(\mathcal{B}^N, T)$



Mikhael Gromov; from Wikipedia

2 What is mean dimension?

(X, d): compact metric space. For $\varepsilon > 0$, a continuous $f\colon X \to Y$ is called ε-embedding if $\mathrm{Diam}\, f^{-1}(y) < \varepsilon$ for all $y \in Y$.



Define $\mathrm{Widim}_\varepsilon(X, \mathbf{d})$ as the minimum integer n for which $\exists$ an n-dimension simplicial complex P and an ε-embedding $f : X \to P$.

Pavel Urysohn; from Wikipedia

Let $T\colon \mathbb{C}\times X \to X$ be a continuous action. For $R>0$ we define a metric d_R on X by

$$d_R(x,y) = \sup_{|a|\le R} d\left(T^a x, T^a y\right).$$

We define mean dimension by

$$\mathrm{mdim}(X,T) = \lim_{\varepsilon\to 0}\left(\lim_{R\to\infty} \frac{\mathrm{Widim}_\varepsilon(X,d_R)}{\pi R^2}\right).$$

3 Brody curves and mean dimension

$\mathcal{B}^N = \{f\colon \mathbb{C} \to \mathbb{C}P^N \mid \text{Brody curve}\}$ with the group action $T\colon \mathbb{C} \times \mathcal{B}^N \to \mathcal{B}^N$.

Based on a result of Eremenko, Gromov proved

$$\operatorname{mdim}\left(\mathcal{B}^N, T\right) \leq 4N.$$

It had been an open problem to improve this estimate.

For a Brody curve $f : \mathbb{C} \to \mathbb{C}P^N$, define its energy density by

$$\rho(f) := \lim_{R\to\infty} \left(\frac{1}{\pi R^2} \sup_{a\in\mathbb{C}} \int_{|z-a|<R} |df|^2 \, dx dy \right).$$

Set

$$\rho(\mathbb{C}P^N) := \sup_{f\in\mathcal{B}_N} \rho(f).$$

It is known:

$$0 < \rho(\mathbb{C}P^N) < 1, \qquad \lim_{N\to\infty} \rho(\mathbb{C}P^N) = 1.$$

Theorem (Matsuo–T. 2015, T. 2018)

The mean dimension of the system of Brody curves is given by

$$\mathrm{mdim}\left(\mathcal{B}^N, T\right) = 2(N+1)\rho(\mathbb{C}P^N).$$

Shinichiroh Matsuo; from his homepage

Problem

Can we understand the formula

$$\mathrm{mdim}\left(\mathcal{B}^N, T\right) = 2(N+1)\rho(\mathbb{C}P^N)$$

in terms of invariant probability measures?

4 Invariant probability measures on $\mathcal{B}^N$

We study T-invariant probability measures μ on $\mathcal{B}^N$. Here μ: T-invariant if $\mu(T^{-a}A) = \mu(A)$ for all Borel sets $A \subset \mathcal{B}^N$ and $a \in \mathbb{C}$.

Example 1. Let $L \gg 1$ and $a \gg 1$. Set

$$\Lambda = \mathbb{Z}L + \mathbb{Z}\sqrt{-1}L, \quad D = \{u \in \mathbb{C} \mid |u - a| \leq 1\}.$$

For $w \in [0, L]^2$ and $u = (u_\lambda)_\lambda \in D^\Lambda$, define

$$f_{w,u}(z) := \sum_{\lambda \in \Lambda} \frac{u_\lambda}{(z - w - \lambda)^3} \in \mathcal{B}^1.$$

We independently choose w and u_λ $(\lambda \in \Lambda)$ from the uniform distributions of $[0, L]^2$ and D respectively. Then

$$f_{w,u}(z) = \sum_{\lambda \in \Lambda} \frac{u_\lambda}{(z - w - \lambda)^3}$$

becomes a random function. Its distribution is translation-invariant. So it defines a T-invariant probability measure μ on $\mathcal{B}^1$.
In general, invariant probability measures on $\mathcal{B}^N$ correspond to such random Brody curves.

Define $\mathscr{M}^T\left(\mathcal{B}^N\right)$ as the space of all T-invariant Borel probability measures on $\mathcal{B}^N$.

We try to express both sides of

$$\mathrm{mdim}\left(\mathcal{B}^N, T\right) = 2(N+1)\rho(\mathbb{C}P^N)$$

in terms of $\mu \in \mathscr{M}^T\left(\mathcal{B}^N\right)$.

Recall $\rho(\mathbb{C}P^N) = \sup_{f \in \mathcal{B}^N} \rho(f)$ where

$$\rho(f) = \lim_{R \to \infty} \frac{1}{\pi R^2} \sup_{a \in \mathbb{C}} \int_{|z-a|<R} |df|^2 \, dx dy.$$

Define $\psi \colon \mathcal{B}^N \to \mathbb{R}$ by

$$\psi(f) = 2(N+1)|df|^2(0).$$

We have:

$$2(N+1)\rho(\mathbb{C}P^N) = \sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \int_{\mathcal{B}^N} \psi \, d\mu.$$

What is the integral $\int_{\mathcal{B}^N} \psi \, d\mu$?

For $f : \mathbb{C} \to \mathbb{C}P^N$, define

$$T(R, f) = \int_1^R \left(\int_{|z|<r} |df|^2 \, dx dy \right) \frac{dr}{r}.$$

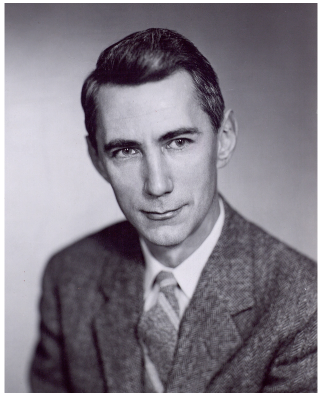
Let $\mu \in \mathscr{M}^T(\mathcal{B}^N)$ be an ergodic measure. Then for μ-a.e. $f \in \mathcal{B}^N$

$$T(R, f) = \frac{\pi R^2}{4(N+1)} \int_{\mathcal{B}^N} \psi \, d\mu + o(R^2).$$

$\mathrm{mdim}(\mathcal{B}^N, T) = 2(N+1)\rho(\mathbb{C}P^N)$ becomes

$$\mathrm{mdim}\left(\mathcal{B}^N, T\right) = \sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \int_{\mathcal{B}^N} \psi \, d\mu.$$

Next we relate L.H.S. to rate distortion theory.



Claude Shannon; from Wikipedia

5 Rate distortion dimension

$\mathcal{B}^N$ has metric $\mathbf{d}(f,g) = \sup_{|z|\leq 1} d_{\mathbb{C}P^N}(f(z), g(z))$.

Let $\mu \in \mathscr{M}^T(\mathcal{B}^N)$, and randomly choose $f \in \mathcal{B}^N$ according to μ. For $\varepsilon > 0$, we define the rate distortion function $R(\mathbf{d}, \mu, \varepsilon)$ as the minimum bits per unit area of $\mathbb{C}$ for describing f within average distortion bounded by ε.

Roughly, $R(\mathbf{d}, \mu, \varepsilon)$ is the entropy rate of the process f up to error $< \varepsilon$.

We define rate distortion dimension by

$$\operatorname{rdim}\left(\mathcal{B}^N, T, \mathbf{d}, \mu\right) = \limsup_{\varepsilon \to 0} \frac{R(\mathbf{d}, \mu, \varepsilon)}{\log(1/\varepsilon)}.$$

Tsutomu Kawabata

from homepage

Amir Dembo

from homepage

Variational principle (Lindenstrauss–T.)

$$\mathrm{mdim}(\mathcal{B}^N, T) = \sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu)$$

Elon Lindenstrauss and Benjamin Weiss and myself

Now the formula

$$\mathrm{mdim}(\mathcal{B}^N, T) = 2(N+1)\rho(\mathbb{C}P^N)$$

becomes

$$\sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu) = \sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \int_{\mathcal{B}^N} \psi \, d\mu,$$

where ψ is defined by $\psi(f) = 2(N+1)|df|^2(0)$.

6 Main results

We have

$$\sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu) = \sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \int_{\mathcal{B}^N} \psi \, d\mu.$$

Question

What is a relation between rate distortion dimension $\mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu)$ and $\int_{\mathcal{B}^N} \psi \, d\mu$ for each $\mu \in \mathscr{M}^T(\mathcal{B}^N)$?

Example 2. Let $\Lambda = \mathbb{Z}L + \mathbb{Z}L\sqrt{-1}$ with $L \gg 1$. Let $f \colon \mathbb{C} \to \mathbb{C}P^N$ be a Λ-periodic Brody curve, e.g. Weierstrass' $\wp$ function. The orbit of f is a periodic orbit in $\mathcal{B}^N$. Let μ be the uniform measure on it. Then

$$\mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu) = 0,$$

$$\int_{\mathcal{B}^N} \psi \, d\mu = \frac{2(N+1)}{L^2} \int_{[0,L]^2} |df|^2 \, dx dy.$$

Example 3. Let $L \gg 1$ and $a \gg 1$. Let $\mu \in \mathscr{M}^T(\mathcal{B}^1)$ be the distribution of the random function

$$\sum_{\lambda \in \mathbb{Z}L + \mathbb{Z}L\sqrt{-1}} \frac{u_\lambda}{(z - w - \lambda)^3} \in \mathcal{B}^1,$$

where w and u_λ are independently and uniformly chosen from $[0, L]^2$ and $\{|u - a| \le 1\}$.

$$\mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu) = \frac{2}{L^2}, \quad \int_{\mathcal{B}^1} \psi \, d\mu = \frac{12}{L^2}.$$

Main Theorem 1

For any $\mu \in \mathscr{M}^T(\mathcal{B}^N)$, we have

$$\operatorname{rdim}\left(\mathcal{B}^N, T, \mathbf{d}, \mu\right) \le \int_{\mathcal{B}^N} \psi \, d\mu.$$

We will see that this is analogous to Ruelle inequality of smooth ergodic theory. So we call this "Ruelle inequality for Brody curves" .

Main Theorem 2

For any $0 \leq c < 2(N+1)\rho(\mathbb{C}P^N)$, there exists $\mu \in \mathscr{M}^T(\mathcal{B}^N)$ satisfying

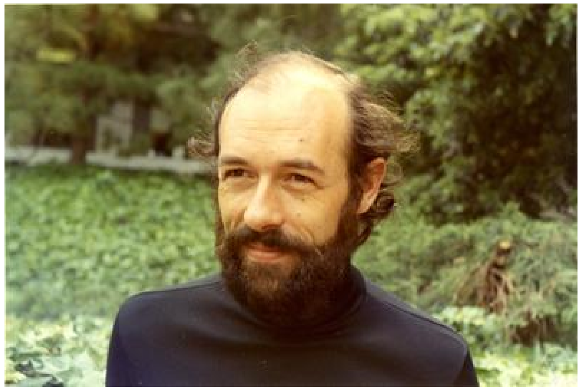
$$\mathrm{rdim}\left(\mathcal{B}^N, T, \mathbf{d}, \mu\right) = \int_{\mathcal{B}^N} \psi \, d\mu = c.$$

Main Theorems 1 and 2 immediately imply

$$\sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu) = \sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \int_{\mathcal{B}^N} \psi \, d\mu.$$

7 Axiom A diffeomorphisms

The proofs of Main Theorems 1 and 2 are motivated by the thermodynamic formalism for Axiom A diffeomorphisms. So we review it.

Yakov Sinai, David Ruelle and Rufus Bowen from Wikipedia.

M: compact Riemannian manifold with Axiom A diffeomorphism $T\colon M \to M$. (Nonwandering set is hyperbolic and periodic points are dense in it.) Let Ω be a basic set of T, and let

$$T_x M = E_x^s \oplus E_x^u \quad (x \in \Omega)$$

splitting into stable and unstable directions.

Example 4. $M = \mathbb{R}^2/\mathbb{Z}^2$ with $T(x,y) = (x+y, x)$. Then $\Omega = M$, and $\mathbb{R}^2 = \mathbb{R}\left(\frac{1-\sqrt{5}}{2}, 1\right) \oplus \mathbb{R}\left(\frac{1+\sqrt{5}}{2}, 1\right)$ provides stable and unstable directions.

Define $\phi\colon \Omega \to \mathbb{R}$ by

$$\phi(x) = \log \left|\det \left(dT_x \colon E^u_x \to E^u_{Tx}\right)\right|.$$

A fundamental result is:

$$\sup_{\mu \in \mathscr{M}^T(\Omega)} \left(h_\mu(T) - \int_\Omega \phi \, d\mu \right) = P_T(-\phi) \leq 0.$$

Then, (a special case of) Ruelle inequality follows:

$$h_\mu(T) \leq \int_\Omega \phi \, d\mu \quad (\forall \mu \in \mathscr{M}^T(\Omega)).$$

Moreover, if Ω is an attractor, then

$$\sup_{\mu \in \mathscr{M}^T(\Omega)} \left(h_\mu(T) - \int_\Omega \phi \, d\mu \right) = P_T(-\phi) = 0,$$

and $\exists \mu \in \mathscr{M}^T(\Omega)$ attaining the supremum.
This μ is called SRB measure. It satisfies

$$h_\mu(T) = \int_\Omega \phi \, d\mu.$$

8 Mean dimension with potential

(X, d): compact metric space with a continuous function $\varphi \colon X \to \mathbb{R}$. Define

$$\mathrm{Widim}_\varepsilon(X, d, \varphi)$$

$$= \inf_{\substack{P\text{:simplicial complex} \\ f\colon X \to P\text{:}\varepsilon\text{-embedding}}} \left\{ \max_{x \in X} \left(\dim_{f(x)} P + \varphi(x) \right) \right\}.$$

Here $\dim_{f(x)} P$ is the local dimension of P around $f(x)$.

Let $T\colon \mathbb{C}\times X \to X$ be a continuous actions. For $R>0$, define new metric d_R and function φ_R on X by

$$d_R(x,y) = \sup_{|a|\le R} d\left(T^a x, T^a y\right),$$

$$\varphi_R(x) = \int_{|a|\le R} \varphi(T^a x)\, da_1 da_2.$$

We define mean dimension with potential by

$$\mathrm{mdim}\,(X,T,\varphi) = \lim_{\varepsilon\to 0}\left(\lim_{R\to\infty} \frac{\mathrm{Widim}_\varepsilon\,(X, d_R, \varphi_R)}{\pi R^2}\right).$$

9 Proofs of main theorems

$\mathcal{B}^N$ is the space of Brody curves $f\colon \mathbb{C} \to \mathbb{C}P^N$ with a natural action $T\colon \mathbb{C} \times \mathcal{B}^N \to \mathcal{B}^N$. We introduced a metric $\mathbf{d}(f,g) = \sup_{|z|\le 1} d_{\mathbb{C}P^N}(f(z), g(z))$ and a function $\psi(f) = 2(N+1)|df|^2(0)$.

A fundamental equation is:

$$\sup_{\mu \in \mathscr{M}^T(\mathcal{B}^N)} \left(\mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu) - \int_{\mathcal{B}^N} \psi \, d\mu \right)$$

$$= \mathrm{mdim}(\mathcal{B}^N, T, -\psi) = 0.$$

Then we have an analogy of Ruelle inequality:

$$\mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu) \leq \int_{\mathcal{B}^N} \psi\, d\mu, \quad (\forall \mu \in \mathscr{M}^T(\mathcal{B}^N)).$$

This proves Main Theorem 1. Moreover we can construct plenty of $\mu \in \mathscr{M}^T(\mathcal{B}^N)$ attaining the supremum of the fundamental equation, i.e. satisfying

$$\mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu) = \int_{\mathcal{B}^N} \psi\, d\mu.$$

This provides Main Theorem 2.

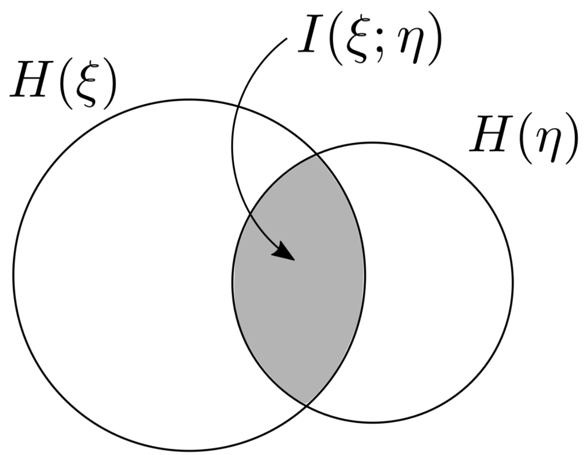
Remark 5. There is an important difference between Axiom A attactors and Brody curves. In the case of Axiom A attractors, the SRB measure is unique. However, in the case of Brody curves, there exist plenty of $\mu \in \mathscr{M}^T(\mathcal{B}^N)$ satisfying

$$\mathrm{rdim}(\mathcal{B}^N, T, \mathbf{d}, \mu) = \int_{\mathcal{B}^N} \psi \, d\mu.$$

It seems that there is no way to select one distinguished measure for Brody curves.

10 Rate distortion theory (if time permits)

$(\Omega, \mathbb{P})$: probability space, $\xi\colon \Omega \to \mathcal{X}$ and $\eta\colon \Omega \to \mathcal{Y}$: random variables. We want to define the mutual information $I(\xi; \eta)$.



Schematic picture of mutual information $I(\xi; \eta)$.

Step 1. When $\mathcal{X}$ and $\mathcal{Y}$ are finite sets,

$$I(\xi;\eta) := H(\xi) + H(\eta) - H(\xi,\eta).$$

Step 2. In general

$$I(\xi;\eta) := \sup_{\alpha,\beta} I\left(\alpha \circ \xi; \beta \circ \eta\right)$$

where α and β run over all finite measurable partitions of $\mathcal{X}$ and $\mathcal{Y}$ respectively.

(X,d): compact metric space. For $A \subset \mathbb{C}$ with $\mathbf{m}(A) < \infty$, define $L^1(A,X)$ as the space of measurable maps $f\colon A \to X$ with a metric

$$D(f,g) := \int_A d\left(f(u), g(u)\right) \, d\mathbf{m}(u).$$

Let $T\colon \mathbb{C} \times X \to X$ be a continuous action. Let $\mu \in \mathscr{M}^T(X)$ be a T-invariant measure. For $\varepsilon > 0$, we will define the rate distortion function $R(d,\mu,\varepsilon)$.

Let $A \subset \mathbb{C}$: bounded with $\mathbf{m}(A) > 0$. We define $R(\varepsilon, A)$ as the infimum of $I(\xi; \eta)$ where ξ and η are random variables

- ξ takes values in X according to μ.
- η takes values in $L^1(A, X)$ such that

$$\mathbb{E}\left(\frac{1}{\mathbf{m}(A)} \int_A d\left(T^u \xi, \eta_u\right) \, d\mathbf{m}(u)\right) < \varepsilon.$$

Define

$$R(d, \mu, \varepsilon) = \lim_{L \to \infty} \frac{R(\varepsilon, [0, L]^2)}{L^2}.$$

Finally, we define rate distortion dimension by

$$\operatorname{rdim}(X, T, d, \mu) = \limsup_{\varepsilon \to 0} \frac{R(d, \mu, \varepsilon)}{\log(1/\varepsilon)}.$$

11 Conclusion

(1) We study invariant probability measures on the space of Brody curves $\mathcal{B}^N$.

(2) They satisfy an inequality analogous to Ruelle inequality.

(3) ∃ a rich variety of measures attaining equality in this Ruelle inequality for Brody curves.

Hopefully this is just the tip of iceberg. A bigger picture is something like "a fusion of hyperbolic dynamics and geometric analysis".