A counterexample to Eremenko's Conjecture

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Joint work with David Martí-Pete and Lasse Rempe

Basic definitions

- Let $f : \mathbb{C} \to \mathbb{C}$ be analytic.
- Denote by f^n the nth iterate of f.
- What happens as $n \to \infty$?

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The **Julia set** is

$$J(f) = \mathbb{C} \setminus F(f).$$

The escaping set of a polynomial

Definition

The **escaping set** is

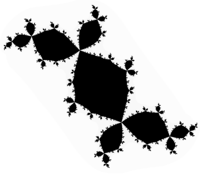
$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

- $I(f)$ is a neighborhood of ∞.
- $\partial I(f) = J(f)$.
- $I(f) \subset F(f)$.

Polynomials vs non-polynomials

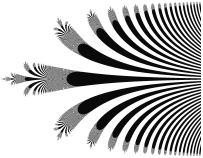
Polynomials

- $J(f)$ is always bounded
- No wandering domains
- $0 \leq \dim J(f) \leq 2$
- Description of dynamics of $J(f)$ given by $I(f)$: external rays, etc



Transcendental entire

- $J(f)$ is never bounded
- Wandering domains possible
- $\dim J(f) \geq 1$
- Description of dynamics of $J(f)$ given by $I(f)$??



The escaping set of a transcendental entire function

Definition

The **escaping set** is

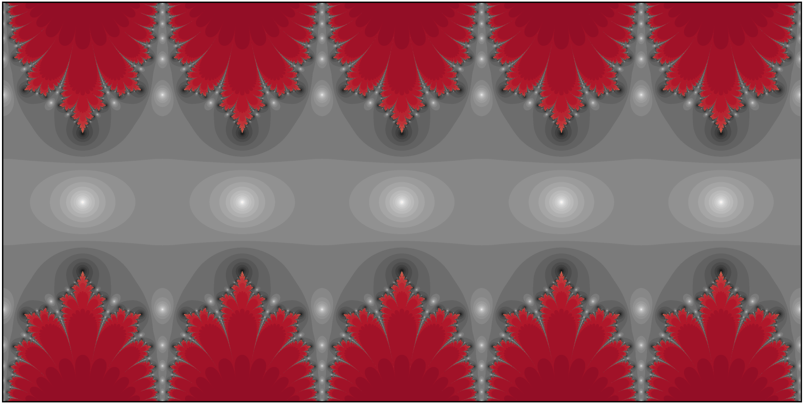
$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

- $I(f)$ is not a neighborhood of ∞.
- $I(f)$ can meet $F(f)$ and $J(f)$.
- Always points in $I(f)$ with different rates of escape.

Observation (Fatou (1926), Devaney (1980s))

$I(f)$ often contains curves tending to ∞.

Fatou's question

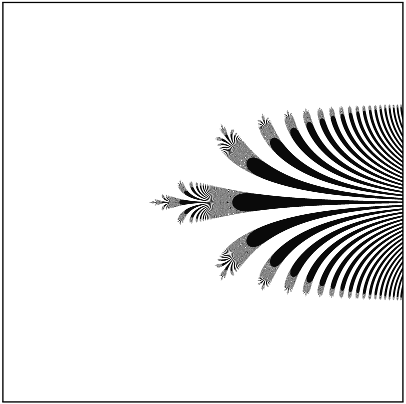


$c \sin z$

Question (Fatou, 1926)

Does $I(f)$ contain curves for "more general" functions?

Example: The exponential function



$\frac{1}{4}\exp(z)$

Investigated by Devaney and Tangerman (1986).

- $F(f)$ (white) contains a left half-plane
- $J(f)$ (black) is a "Cantor bouquet of curves"
- $I(f)$ (black) are these curves without some of the endpoints

Eremenko's conjectures

Eremenko (1989) showed $I(f)$ has the following properties:

- $I(f) \cap J(f) \neq \emptyset$,
- $\partial I(f) = J(f)$,
- $\overline{I(f)}$ has no bounded components.

The strong Eremenko conjecture (Eremenko, 1989)

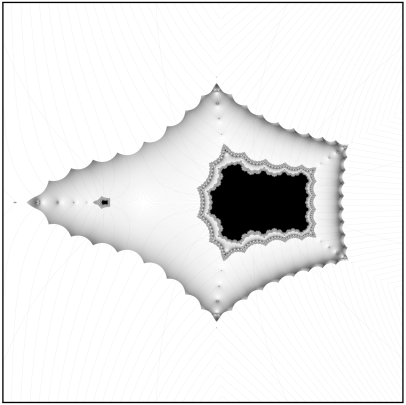
Every point of $I(f)$ can be joined to ∞ by a curve of points in $I(f)$.

Eremenko's conjecture (Eremenko, 1989)

All components of $I(f)$ are unbounded.[1]

[1] "It is plausible that the set $I(f)$ has no bounded connected components."

Examples: Spider's web



$\frac{1}{2}(\cos z^{1/4} + \cosh z^{1/4})$

- $F(f)$ is an infinite collection of bounded basins of attraction
- $J(f)$ and $I(f)$ are connected and form "spider's webs"

What is known about the structure of $I(f)$?

It can be very hard to control the components of the escaping set.

Theorem (Rippon and Stallard, 2011)

$I(f) \cup \{\infty\}$ is always connected and every bounded component of $I(f)$ meets $J(f)$.

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Theorem (Rippon and Stallard, 2005, 2014)

$I(f)$ has at least one unbounded component, and moreover $I(f)$ is connected or it has infinitely many unbounded components.

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Theorem (Rempe, Rottenfußer, Rückert, Schleicher, 2011; Barański, 2007)

There exists a transcendental entire function f such that every path-connected component of $J(f)$ is bounded. However, the strong conjecture does hold for a large class of entire functions. In particular, class $\mathcal{B}$ of finite order.

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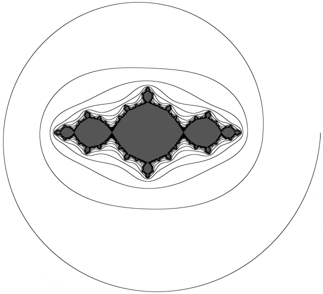
Further counterexamples:

- Bishop (2015): the strong version of Eremenko's conjecture fails for transcendental entire functions with a finite set of singular values
- Rempe (2016): arc-like continua as Julia continua
- Benitez-Rempe (2021): every Julia continuum is a pseudo-arc
- Brown (2024): counterexamples with slow growth

New counterexamples to the strong Eremenko conjecture

Theorem (Martí-Pete, Rempe, W)

Let $K \subseteq \mathbb{C}$ be a continuum with connected complement. Then there exists a transcendental entire function f such that every path-connected component of K is a path-connected component of the escaping set $I(f)$, and every path-connected component of ∂K is a path-connected component of $J(f)$. In particular, no point of K can be connected to ∞ by a curve in $I(f)$.



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Theorem (Martí-Pete, Rempe, W)

Let $X \subset \mathbb{C}$ be a non-empty connected compact set with connected complement. Then there exists a transcendental entire function f such that X is a connected component of $I(f)$.

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- How do we *realize* this structure?
- Give a new general framework for constructing entire functions having interesting sets of points with unbounded orbits.

Arakelyan's theorem

Theorem (Arakelyan, 1964)

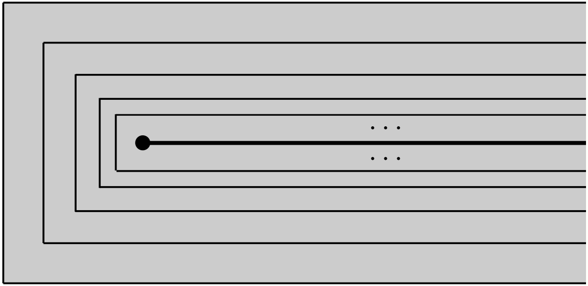
Let $A \subseteq \mathbb{C}$ be a closed set such that

1. (i) $\widehat{\mathbb{C}} \setminus A$ is connected;
2. (ii) $\widehat{\mathbb{C}} \setminus A$ is locally connected at ∞.

Suppose that $g\colon A \to \mathbb{C}$ is a continuous function that is holomorphic on $\operatorname{int}(A)$. Then for every $\varepsilon > 0$, there exists an entire function f such that

$$|f(z) - g(z)| < \varepsilon \quad \text{for all } z \in A.$$

The structure



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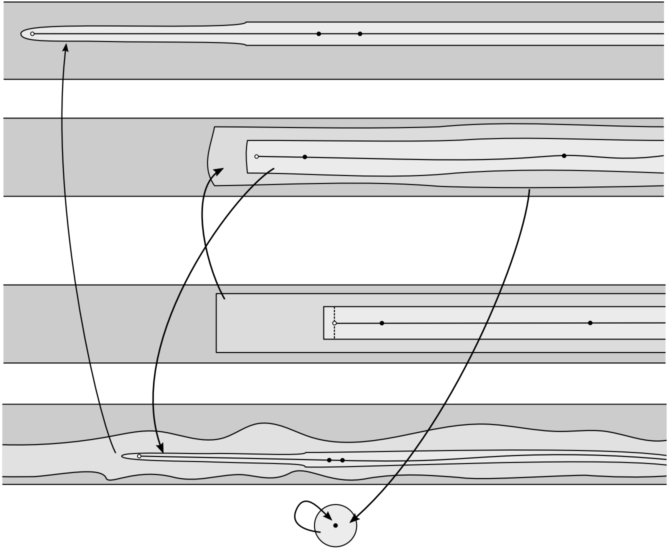

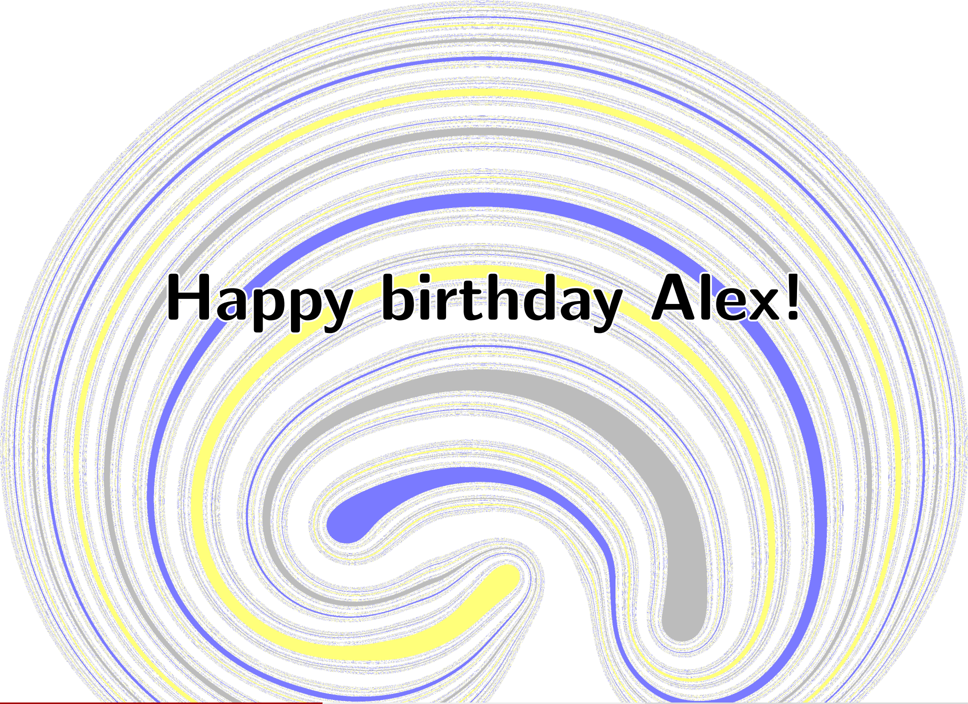

Happy birthday Alex!